

ON THE TOPOLOGICAL DIMENSION OF THE SOLUTIONS SETS FOR SOME CLASSES OF OPERATOR AND DIFFERENTIAL INCLUSIONS

RALF BADER

Technical University Muenchen

D-80290 Muenchen, Germany

e-mail: bader@appl-math.tu-muenchen.de

BORIS D. GEL'MAN¹

MIKHAIL KAMENSKII¹

VALERI OBUKHOVSKII¹

Department of Mathematics

Voronezh University

(394693) Voronezh, Russia

e-mail: gelman@func.vsu.ru

e-mail: Mikhail@kam.vsu.ru

e-mail: valeri@ob.vsu.ru

Abstract

In the present paper, we give the lower estimation for the topological dimension of the fixed points set of a condensing continuous multimap in a Banach space. The abstract result is applied to the fixed point set of the multioperator of the form $\mathcal{F} = S\mathcal{P}_F$ where \mathcal{P}_F is the superposition multioperator generated by the Carathéodory type multifunction F and S is the shift of a linear injective operator. We present sufficient conditions under which this set has the infinite topological dimension. In the last section of the paper, we consider the applications of the solutions sets for Cauchy and periodic problems for semilinear differential inclusions in a Banach space.

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1. INTRODUCTION

The investigation of topological properties of solutions sets of operator and differential inclusions in abstract spaces attracts the attention of many researchers (see, e.g. the recent monograph [8]). The topological dimension of solutions sets was studied in the papers [19, 9, 18, 11, 12] and others.

In the present paper, we develop the results of [11, 12] and give the lower estimation for the topological dimension of the fixed points set of a condensing continuous multimap in a Banach space (Theorem 2.3). The abstract result is applied to the fixed points set of the multioperator of the form $\mathcal{F} = S\mathcal{P}_F$ where \mathcal{P}_F is the superposition multioperator generated by the Carathéodory type multifunction F and S is the shift of a linear injective operator. We present sufficient conditions under which this set has the infinite topological dimension (Theorem 3.4). In the last section of the paper, we consider the applications of the solutions sets for Cauchy and periodic problems for semilinear differential inclusions in a Banach space (Theorems 4.3 and 4.7).

2. TOPOLOGICAL DIMENSION OF THE FIXED POINTS SET FOR A MULTIMAP

Let \mathcal{E} be a Banach space; by the symbol $2^{\mathcal{E}}$ we denote the collection of all subsets of \mathcal{E} ;

$$P(\mathcal{E}) = \{\Omega \in 2^{\mathcal{E}} : \Omega \text{ is nonempty}\},$$

$$Kv(\mathcal{E}) = \{\Omega \in P(\mathcal{E}) : \Omega \text{ is convex compact}\}.$$

Let us recall some notions (see, e.g. [3, 4, 13, 15, 14]).

Let (A, \geq) be a partially ordered set. A map

$$\beta : 2^{\mathcal{E}} \rightarrow A$$

is called *the measure of noncompactness* (MNC) in \mathcal{E} if

$$\beta(\overline{co}\Omega) = \beta(\Omega)$$

for every $\Omega \in 2^{\mathcal{E}}$.

A MNC β is called:

- (i) monotone if $\Omega_0, \Omega_1 \in 2^{\mathcal{E}}$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$;
- (ii) nonsingular if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in \mathcal{E}$, $\Omega \in 2^{\mathcal{E}}$.

As an example of MNC possessing all these properties we may consider the Hausdorff MNC,

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

In the sequel, we will consider only monotone nonsingular measures of noncompactness.

Definition 2.1. Let X be a closed subset of a Banach space \mathcal{E} , L a compact topological space. A multimap $\mathcal{F} : X \rightarrow Kv(\mathcal{E})$ or a family of multimaps $G : X \times L \rightarrow Kv(\mathcal{E})$ is said to be *condensing with respect to an MNC β* (or simply *β -condensing*) if

$$\beta(\mathcal{F}(\Omega)) \geq \beta(\Omega)$$

or, respectively,

$$\beta(G(\Omega \times L)) \geq \beta(\Omega)$$

imply the relative compactness of Ω for every $\Omega \subset X$.

Below we deal with the classes of upper semicontinuous (u.s.c.), lower semicontinuous (l.s.c.), and continuous multimaps (necessary definitions and details may be found in the same sources).

Now, let $U \subset \mathcal{E}$ be a bounded open set and $\mathcal{F} : \overline{U} \rightarrow Kv(\mathcal{E})$ be an u.s.c. β -condensing multimap. A multimap $\Phi = i - \mathcal{F} : \overline{U} \rightarrow Kv(\mathcal{E})$ defined by the formula $\Phi(x) = x - \mathcal{F}(x)$ will be called a *β -condensing multifield* generated by \mathcal{F} .

We will say that the multifield Φ is *nonsingular* on the boundary ∂U provided $0 \notin \Phi(x)$ for all $x \in \partial U$. It is clear that this condition is equivalent to the absence of fixed points of \mathcal{F} , $x \in \mathcal{F}(x)$ on ∂U . By the symbol $\mathcal{D}(\overline{U}, \partial U)$ we will denote the collection of all condensing multifields nonsingular on ∂U . A subcollection of $\mathcal{D}(\overline{U}, \partial U)$ consisting of multifields generated by continuous multimaps will be denoted by $\mathcal{D}_C(\overline{U}, \partial U)$.

It is known (see, e.g. [3, 4, 14]) that for every multifield $\Phi = i - \mathcal{F} \in \mathcal{D}(\overline{U}, \partial U)$ the topological degree $\deg(\Phi, \overline{U})$ is defined. This integer characteristic possesses properties which are standard for the topological degree theory.

Let us denote by $N(\Phi, \overline{U})$ the set of all nonsingular points of Φ which coincides with the fixed points set of \mathcal{F} , i.e.

$$N(\Phi, \overline{U}) = \{x \in \overline{U} : x \in \mathcal{F}(x)\} = \{x \in \overline{U} : 0 \in \Phi(x)\}.$$

It is easy to prove that the set $N(\Phi, \overline{U})$ is compact. We will study the topological dimension \dim (see, e.g., [1, 10]) of this set.

The following statement will play an important role in our constructions.

Lemma 2.2 ([19]). *Let X be a paracompact topological space, $\dim(X) \leq n - 1$ where $n \geq 1$. Let \mathcal{E} be a Banach space and $T : X \rightarrow Kv(\mathcal{E})$ a l.s.c. multimap satisfying the following conditions:*

- (i) $0 \in T(x)$ for all $x \in X$;
- (ii) $\dim(T(x)) \geq n$ for all $x \in X$.

Then there exists a continuous selection $f : X \rightarrow \mathcal{E}$ of T such that $f(x) \neq 0$ for all $x \in X$.

Using this result we may give the following "condensing" version of Theorem 2.2 of [12] giving the lower estimate for the dimension of the set $N(\Phi, \overline{U})$.

Theorem 2.3. *Let $\Phi = i - \mathcal{F} \in \mathcal{D}_C(\overline{U}, \partial U)$ satisfy the following conditions:*

- (i) $\deg(\Phi, \overline{U}) \neq 0$;
- (ii) $\dim(\mathcal{F}(x)) \geq n$ for all $x \in U$ where $n \geq 1$.

Then $\dim(N(\Phi, \overline{U})) \geq n$.

Proof. From the assumption (i) it follows that $\emptyset \neq N(\Phi, \overline{U}) \subset U$. Supposing the contrary to the conclusion we will have that $\dim(N(\Phi, \overline{U})) \leq n - 1$. Then the restriction $\hat{\Phi} = \Phi|_{N(\Phi, \overline{U})}$ satisfies the conditions of Lemma 2.2 and hence there exists a continuous selection $\hat{\varphi} : N(\Phi, \overline{U}) \rightarrow \mathcal{E}$ of a multifield $\hat{\Phi}$ such that $0 \neq \hat{\varphi}(x)$ for all $x \in N(\Phi, \overline{U})$. By virtue of the Michael continuous selection theorem ([17]) there exists a continuous selection $\varphi : \overline{U} \rightarrow \mathcal{E}$ of Φ which is the extension of $\hat{\varphi}$: $\hat{\varphi} = \varphi|_{N(\Phi, \overline{U})}$. It is clear that $0 \neq \varphi(x)$ for all $x \in \overline{U}$ since $0 \notin \Phi(x)$ if $x \notin N(\Phi, \overline{U})$.

The map $f : \bar{U} \rightarrow \mathcal{E}$ defined by $\varphi(x) = x - f(x)$ is a continuous selection of the multimap \mathcal{F} and hence is condensing. It is easy to see that

$$\deg(\varphi, \bar{U}) = \deg(\Phi, \bar{U}) \neq 0$$

and hence $N(\varphi, \bar{U}) \neq \emptyset$ giving the contradiction. \blacksquare

Corollary 2.4. *Let B be a closed ball in \mathcal{E} ; $\Phi = i - \mathcal{F} \in \mathcal{D}_C(B, S)$ where $S = \partial B$. If \mathcal{F} satisfies the following conditions:*

- (i) $\mathcal{F}(x) \cap B \neq \emptyset$ for every $x \in S$;
- (ii) $\dim(\mathcal{F}(x)) \geq n$ for all $x \in \text{Int}(B)$ where $n \geq 1$.

Then $\dim(N(\Phi, B)) \geq n$.

Proof. Condition (i) implies that $\deg(\Phi, B) = 1$ ([3], Theorem 1.2.70; [14], Theorem 3.3.2). \blacksquare

3. TOPOLOGICAL DIMENSION OF THE SOLUTIONS SET FOR SOME OPERATOR INCLUSIONS

Let the interval $[a, b]$ be endowed with a Lebesgue measure μ and E be a separable Banach space.

We will need the following property of a measurable multifunction which is the infinite-dimensional version of Lemma 2.6 in [9].

Lemma 3.1. *Let $\Gamma : [a, b] \rightarrow Kv(E)$ be a measurable multifunction, and suppose that there exists a measurable subset $\Delta \subseteq [a, b]$, $\mu(\Delta) > 0$ such that $\dim(\Gamma(t)) \geq 1$ for every $t \in \Delta$. Then for every positive integer m there exists a collection $\{\gamma_i\}_{i=1}^m$ of measurable selections of Γ which are linearly independent on $[a, b]$.*

Now, let $F : [0, d] \times E \rightarrow Kv(E)$ be a multimap satisfying the following properties:

F1) the multifunction $F(\cdot, x) : [0, d] \rightarrow Kv(E)$ is measurable for every $x \in E$;

F2) the multimap $F(t, \cdot) : E \rightarrow Kv(E)$ is continuous;

F3) there exists a function $\nu(\cdot) \in L_+^1([0, d])$ such that for all $x \in E$:

$$\|F(t, x)\| := \sup\{\|y\| : y \in F(t, x)\} \leq \nu(t)(1 + \|x\|)$$

for a.e. $t \in [0, d]$;

F4) there exists a function $k(\cdot) \in L_+^1[0, d]$ such that for every bounded set $D \subset E$ we have that

$$\chi(F(t, D)) \leq k(t) \cdot \chi(D)$$

for a.e. $t \in [0, d]$, where χ is the Hausdorff MNC in E .

It is known (see, e.g. [14], Theorem 1.3.4) that for every function $x(\cdot) \in C([0, d]; E)$ the multifunction $F(t, x(t))$ is measurable and hence the superposition multioperator

$$\mathcal{P}_F : C([0, d]; E) \rightarrow P(L^1([0, d]; E))$$

may be defined in the following way:

$$\mathcal{P}_F(x) = \{f \in L^1([0, d]; E) : f(t) \in F(t, x(t)) \text{ a.e. } t \in [0, d]\}.$$

Let us note that the multioperator \mathcal{P}_F has the following continuity properties (see, e.g. [2, 14]).

Lemma 3.2. (i) \mathcal{P}_F is weakly closed in the following sense: assume the sequences $\{x_n\}_{n=1}^\infty \subset C([0, d]; E)$, $\{f_n\}_{n=1}^\infty \subset L^1([0, d]; E)$, $f_n \in \mathcal{P}_F(x_n)$, $n \geq 1$ are such that $x_n \rightarrow x_0$, $f_n \xrightarrow{w} f_0$. Then $f_0 \in \mathcal{P}_F(x_0)$;

(ii) \mathcal{P}_F is l.s.c.

Further, consider the operator $S : L^1([0, d]; E) \rightarrow C([0, d]; E)$ having the form:

$$S(f) = z_0 + G(f)$$

where $z_0 \in C([0, d]; E)$ is a constant function and $G : L^1([0, d]; E) \rightarrow C([0, d]; E)$ is linear and injective.

It will be supposed that the operator G satisfies the following conditions.

G1) there exists $D \geq 0$ such that

$$\|G(f)(t) - G(g)(t)\| \leq D \int_0^t \|f(s) - g(s)\| ds$$

for every $f, g \in L^1([0, d]; E)$, $0 \leq t \leq d$;

G2) for any compact $K \subset E$ and sequence $\{f_n\}_{n=1}^\infty \subset L^1([0, d]; E)$ such that $\{f_n(t)\}_{n=1}^\infty \subset K$ for a.e. $t \in [0, d]$ the weak convergence $f_n \xrightarrow{w} f_0$ implies $G(f_n) \rightarrow G(f_0)$;

Note that condition (G1) implies that the operator G satisfies the Lipschitz condition

$$G1') \quad \|G(f) - G(g)\|_C \leq D\|f - g\|_{L^1}.$$

We will study the multimap

$$\mathcal{F} = S\mathcal{P}_F : C([0, d]; E) \rightarrow P(C([0, d]; E))$$

and the following operator inclusion:

$$(1) \quad x \in \mathcal{F}(x).$$

Define on bounded subsets of $C([0, d]; E)$ the following MNC ψ with values in the partially ordered set (\mathbf{R}^2, \geq) where the order is induced by the cone \mathbf{R}_+^2 of nonnegative pairs:

$$\psi(\Omega) = (\sigma(\Omega), mod_C(\Omega)).$$

Here

$$\sigma(\Omega) = \sup_{t \in [0, d]} \{e^{-Lt} \chi(\Omega(t))\},$$

$L > 0$ is large enough, χ is the Hausdorff MNC in E and

$$\Omega(t) = \{y(t) : y \in \Omega\};$$

$$mod_C(\Omega) = \limsup_{\delta \rightarrow 0} \max_{x \in \Omega} \max_{|t_1 - t_2| < \delta} \|x(t_1) - x(t_2)\|$$

is the modulus of equicontinuity of the set Ω . It is easy to see that the MNC ψ is monotone and nonsingular.

We may summarize the facts known from [6] and [14] about the multimap \mathcal{F} in the following statement.

Proposition 3.3. *Under conditions (F1) – (F4) and (G1) – (G2) the multimap \mathcal{F} has compact convex values, it is continuous and ψ -condensing on bounded subsets.*

Let us denote by $\Sigma_{\mathcal{F}}$ the set of all solutions to the operator inclusion (1). Now we are in position to prove the main result of this section.

Denote $a = \int_0^d \nu(s)ds$ and $r_0 = (\|z_0\| + Da) \cdot e^{Da}$ where ν satisfies condition (F3) while D satisfies from condition (G1) and $z_0 = S(0)$.

Theorem 3.4. *Under conditions (F1) – (F4) and (G1) – (G2) suppose (F5) : for some $r > r_0$ the set*

$$\Delta = \{t \in [0, d] : \dim(F(t, x)) \geq 1 \text{ for every } x \in E, \|x\| < r\}$$

is measurable and $\mu(\Delta) > 0$. Then the set $\Sigma_{\mathcal{F}}$ is nonempty compact and $\dim(\Sigma_{\mathcal{F}}) = \infty$.

Proof. Using estimates (G1) and (F3) and applying the standard technique based on the Gronwall type inequality one can see that the set

$$\{x \in C([0, d]; E) : x \in \lambda \cdot \mathcal{F}(x) \text{ for some } \lambda \in (0, 1]\}$$

is a priori bounded in norm by the constant r_0 .

Now take a closed ball $B = B_r(0) \subset C([0, d]; E)$ and consider a family $\mathcal{G} : B \times [0, 1] \rightarrow Kv(C([0, d]; E))$,

$$\mathcal{G}(x, \lambda) = \lambda \cdot \mathcal{F}(x).$$

It is easy to verify that \mathcal{G} is ψ -condensing and, as $r > r_0$, it follows that $x \notin \mathcal{G}(x, \lambda)$ for all $(x, \lambda) \in \partial B \times [0, 1]$. Now using the property of the homotopy invariance of the topological degree we obtain that

$$\deg(i - \mathcal{F}, B) = \deg(i - \mathcal{G}(\cdot, 1), B) = \deg(i - \mathcal{G}(\cdot, 0), B) = 1$$

and we conclude that the set $\Sigma_{\mathcal{F}}$ is nonempty and compact.

From the assumption (F5) it follows that, taking any $x \in \text{int}(B)$ and a positive integer n , we have a collection $\{\gamma_i(\cdot)\}_{i=1}^{n+1}$ of measurable selections of $F(t, x(t))$ which are linearly independent on $[0, d]$ (Lemma 3.1). Since the linear operator G is injective, we obtain linearly independent functions $\{S(\gamma_i)\}_{i=1}^{n+1}$. Now we see that the multimap \mathcal{F} satisfies the conditions of Theorem 2.3 and the conclusion $\dim(\Sigma_{\mathcal{F}}) = \infty$ follows from the arbitrariness of n . ■

4. APPLICATIONS: SOLUTIONS SETS OF SEMILINEAR DIFFERENTIAL INCLUSIONS

(a) Cauchy problem

As an application of the above developed abstract theory we will consider in a separable Banach space E the Cauchy problem for a differential inclusion of the form

$$(2) \quad x'(t) \in Ax(t) + F(t, x(t)), \quad t \in [0, d],$$

$$(3) \quad x(0) = x_0$$

under the suppositions that the multimap $F : [0, d] \times E \rightarrow Kv(E)$ satisfies conditions (F1) – (F4) of the previous section and it is assumed that

(A) the linear part $A : D(A) \subset E \rightarrow E$ is the densely defined infinitesimal generator of a C_0 -semigroup $\exp\{At\}$.

Recall that the function $x(\cdot) \in C([0, d]; E)$, is a *mild solution* to the problem (2), (3) on the interval $[0, d]$ if it has the following representation

$$x(t) = \exp\{At\} x_0 + \int_0^t \exp\{A(t-s)\} f(s) ds, \quad f \in \mathcal{P}_F(x).$$

Denote by Σ the set of all mild solutions to the problem (2), (3). It is known (see, e.g. [14]) that under assumptions (A) and (F1) – (F4) the set $\Sigma \subset C([0, d]; E)$ is nonempty and compact.

The linear operator $G : L^1([0, d]; E) \rightarrow C([0, d]; E)$ defined as

$$G(f) = \int_0^t \exp\{A(t-s)\} f(s) ds$$

is said to be *the Cauchy operator* corresponding to the problem (2), (3). It is clear that the set of mild solutions Σ coincides with the fixed points set $\text{Fix}\mathcal{F}$ where $\mathcal{F} = S \circ \mathcal{P}_F$ and $S(f) = z_0 + G(f)$, $z_0(t) = \exp\{At\} x_0$.

Lemma 4.1. *The operator G satisfies conditions (G1) and (G2).*

Proof. See [14], Lemma 4.2.1. ■

Lemma 4.2. *The Cauchy operator G is injective.*

Proof. Suppose that for certain $f \in L^1([0, d]; E)$ we have that

$$(4) \quad v(t) = \int_0^t \exp\{A(t-s)\} f(s) ds = 0 \quad \text{for all } t \in [0, d].$$

Then for any $t \in [0, d]$, $h > 0$ and $t+h \in [0, d]$ we have that

$$\begin{aligned} 0 &= \frac{v(t+h) - v(t)}{h} \\ &= \frac{1}{h} \left[\int_0^{t+h} \exp\{A(t+h-s)\} f(s) ds - \int_0^t \exp\{A(t-s)\} f(s) ds \right] \\ &= \frac{1}{h} \left[\int_0^t \exp\{Ah\} \exp\{A(t-s)\} f(s) ds - \int_0^t \exp\{A(t-s)\} f(s) ds \right] \\ &\quad + \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} f(s) ds \\ &= \frac{1}{h} (\exp\{Ah\} - I) \int_0^t \exp\{A(t-s)\} f(s) ds \\ &\quad + \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} f(s) ds \end{aligned}$$

and therefore, by virtue of condition (4) we have

$$(5) \quad \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} f(s) ds = 0 \quad \text{for all } t \in [0, d]$$

Note now that for every $x \in E$ we have that

$$(6) \quad \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} x ds \rightarrow x, \quad \text{when } h \rightarrow 0$$

(see, e.g. [16]). Further, we have the following estimation

$$\left\| \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} (f(s) - x) ds \right\| \leq \frac{M}{h} \int_t^{t+h} \|f(s) - x\| ds,$$

where $M = \max_{t \in [0, d]} \|\exp\{At\}\|$.

By virtue of the classical Lebesgue theorem we get

$$(7) \quad \frac{1}{h} \int_t^{t+h} \|f(s) - x\| ds \rightarrow \|f(t) - x\| \quad \text{when } h \rightarrow 0$$

for a.e. $t \in [0, d]$. Note that if x belongs to some countable dense subset Δ of E we may assume without loss of generality that (7) holds for every $x \in \Delta$ and $t \in m$ where $m \subseteq [0, d]$ is the set of a full measure. Now take $t \in m$ and $x \in \Delta$ such that,

$$\|f(t) - x\| < \frac{\epsilon}{4M}$$

and choose $h_1 > 0$ such that

$$\left| \frac{1}{h} \int_t^{t+h} \|f(s) - x\| ds - \|f(t) - x\| \right| < \frac{\epsilon}{4M}$$

for all $h \in (0, h_1)$. And further choose $h_2 > 0$ such that

$$\left\| \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} x ds - x \right\| < \frac{\epsilon}{4}.$$

Now, for $h_0 = \min\{h_1, h_2\}$ and $h \in (0, h_0)$ we have that

$$\begin{aligned} & \left\| \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} f(s) ds - f(t) \right\| \\ & \leq \left\| \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} (f(s) - x) ds \right\| \\ & \quad + \left\| \frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} x ds - x \right\| \\ & \quad + \|f(t) - x\| \leq \frac{M}{h} \int_t^{t+h} \|f(s) - x\| ds + \frac{\epsilon}{4} + \frac{\epsilon}{4M} \\ & \leq M \|f(t) - x\| + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4M} < \epsilon \end{aligned}$$

since $M \geq 1$. So

$$\frac{1}{h} \int_t^{t+h} \exp\{A(t+h-s)\} f(s) ds \rightarrow f(t) \text{ when } h \rightarrow 0 \text{ for a.e. } t \in [0, d].$$

Taking into account the equality (5) we obtain that

$$f(t) = 0 \text{ for a.e. } t \in [0, d]$$

proving the lemma. ■

Now we may apply Theorem 3.4 to evaluate the topological dimension of Σ .

Theorem 4.3. *Under conditions (A) and (F1)–(F4) suppose that condition (F5) holds for $r_0 = M(\|x_0\| + a)e^{Ma}$ where $M = \sup_{t \in [0, d]} \|\exp\{At\}\|$ and $a = \int_0^d \nu(s) ds$. Then $\dim(\Sigma) = \infty$.*

(b) Periodic problem

Now we will study the topological dimension of the set of periodic solutions to a semilinear differential inclusion. Let, as before, E be a separable Banach space. For $T > 0$, let $C_T(E)$ denote the space of all continuous T -periodic functions from \mathbf{R}_+ into E with the usual sup-norm.

Suppose that

A') $A : D(A) \subset E \rightarrow E$ is a linear operator generating a C_0 -semigroup $\exp\{At\}$ possessing the property that 1 does not belong to the spectrum $\text{sp}(\exp\{AT\})$.

Note that from the above condition it follows that the operator $[I - \exp\{AT\}]^{-1}$ is well-defined.

It will also be assumed that

F_T) the multioperator $F : \mathbf{R}_+ \times E \rightarrow Kv(E)$ is T -periodic in the first argument:

$$F(t+T, x) = F(t, x)$$

for all $t \in \mathbf{R}_+, x \in E$.

We will denote the set of all T -periodic mild solutions to the differential inclusion (2) by the symbol Σ_T . It will be supposed that the restriction

$F : [0, T] \times E \rightarrow Kv(E)$, which will be denoted by the same symbol, possesses the properties (F1) – (F4) of Section 3 (with the change of d on T). Therefore, the superposition multioperator $\mathcal{P}_F : C([0, T]; E) \rightarrow P(L_1([0, T]; E))$ generated by F is well-defined. Note that for $x \in C_T(E)$, every $f \in \mathcal{P}_F(x)$ will be considered as T -periodically extended on \mathbf{R}_+ .

Let us introduce the multioperator $\mathcal{F}_T : C_T(E) \rightarrow P(C(\mathbf{R}_+; E))$ defined in the following way:

$$\begin{aligned} \mathcal{F}_T x = \{y : y(t) = \exp\{At\} [I - \exp\{AT\}]^{-1} \int_0^T \exp\{A(T-s)\} f(s) ds \\ + \int_0^t \exp\{A(t-s)\} f(s) ds, \quad f \in \mathcal{P}_F(x)\}. \end{aligned}$$

From [14], Propositions 6.1.1, 6.1.2 and Theorem 6.1.1 we may easily deduce the following properties of the integral multioperator \mathcal{F}_T .

- Lemma 4.4.** (i) *The range $\mathcal{F}_T(C_T(E))$ is contained in $C_T(E)$;*
(ii) $\Sigma_T = \text{Fix } \mathcal{F}_T$;
(iii) *The multioperator \mathcal{F}_T has compact convex values, it is continuous and can be represented in the form*

$$\mathcal{F}_T = \tilde{G} \mathcal{P}_F$$

where the linear operator

$$\begin{aligned} \tilde{G}(f)(t) = \exp\{At\} [I - \exp\{AT\}]^{-1} \int_0^T \exp\{A(T-s)\} f(s) ds \\ + \int_0^t \exp\{A(t-s)\} f(s) ds, \end{aligned} \tag{8}$$

satisfies conditions $(G1')$ and $(G2)$.

To obtain the condensivity of \mathcal{F}_T we need some additional assumptions. For a linear operator $L : E \rightarrow E$ denote by $\|L\|^{(\chi)}$ its χ -norm:

$$\|L\|^{(\chi)} := \chi(LB)$$

where χ is the Hausdorff MNC in E and B is the unit ball. Let us suppose that

(AF) the semigroup $\exp\{At\}$ is χ -strongly contractive in the sense that

$$\|\exp\{At\}\|^{(\chi)} \leq e^{-\gamma t}$$

and the coefficient γ satisfies the estimation

$$\gamma > \frac{1}{T} \int_0^T k(s) ds$$

where $k(\cdot)$ is the function from the condition (F4).

On the space $C_T(E)$ consider the MNC

$$\phi(\Omega) = (\chi(\Omega(t)), \text{mod}_C(\Omega))$$

with the values in $L_T^\infty \times \mathbf{R}$ ordered by the cone $\mathbf{K} \times \mathbf{R}_+$ where \mathbf{K} is the cone of a.e. nonnegative functions. It is easy to see that the MNC ϕ is monotone and nonsingular.

Lemma 4.5. *Under conditions (A'), (F_T), (F1) – (F4) and (AF) the integral multioperator \mathcal{F}_T is ϕ -condensing on bounded subsets of $C_T(E)$.*

Proof. See [14], Theorem 6.1.2. ■

Lemma 4.6. *The linear operator \tilde{G} is injective.*

Proof. Let $\tilde{G}(f)(t) \equiv 0$. Taking $t = 0$ in the formula (8) we obtain that

$$[I - \exp\{AT\}]^{-1} \int_0^T \exp\{A(T-s)\} f(s) ds = 0$$

and hence

$$\tilde{G}(f)(t) = \int_0^t \exp\{A(t-s)\} f(s) ds \equiv 0$$

and Lemma 4.2 yields $f(t) = 0$ a.e. ■

Now the same reasonings as in Theorem 4.3 can be used to verify the following statement.

Theorem 4.7. *Under conditions (A') , (F_T) , $(F1) - (F4)$ and (AF) assume that there exists an a priori bound q_0 for mild T -periodic trajectories of the family of semilinear inclusions*

$$(9) \quad x'(t) \in Ax(t) + \lambda F(t, x(t)), \quad \lambda \in [0, 1]$$

and suppose that condition $(F5)$ holds for $r_0 = q_0$. Then $\dim(\Sigma_T) = \infty$.

Corollary 4.8. *Under conditions (A') , (F_T) , $(F1) - (F4)$ and (AF) assume that F satisfies the stronger global estimate:*

$F3'$) there exists a function $\nu(\cdot) \in L^1_+([0, d])$ such that for all $x \in E$:

$$\|F(t, x)\| \leq \nu(t) \text{ for a.e. } t \in [0, T]$$

(instead of condition $(F3)$) and suppose that condition $(F5)$ holds for $r_0 = MaT(M\|I - \exp\{At\}\|^{-1} + 1)$ where $M = \max_{t \in [0, T]} \|\exp\{At\}\|$ and $a = \int_0^T \nu(s)ds$. Then $\dim(\Sigma_T) = \infty$.

Proof. It is sufficient to verify that the number r_0 gives the a priori bound for T -periodic solutions of the family (9). ■

REFERENCES

- [1] P.S. Aleksandrov and B.A. Pasyukov, Introduction to Dimension Theory, Nauka, Moscow 1973 (in Russian).
- [2] J.P. Aubin and A. Cellina, Differential Inclusions. Set-Valued Maps and Viability Theory, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo 1984.
- [3] Yu.G. Borisovich, B.D. Gelman, A.D. Myshkis and V.V. Obukhovskii, *Topological methods in the fixed-point theory of multivalued maps*, Russian Math. Surveys **35** (1980), 65–143.
- [4] Yu.G. Borisovich, B.D. Gelman, A.D. Myshkis and V.V. Obukhovskii, *Multivalued Mappings*, J. Sov. Math. **24** (1984), 719–791.
- [5] Yu.G. Borisovich, B.D. Gelman, A.D. Myshkis and V.V. Obukhovskii, Introduction to the Theory of Multivalued Maps, Voronezh Gos. Univ., Voronezh 1986 (in Russian).
- [6] J.F. Couchouron, M. Kamenski, *A Unified Topological Point of View for Integro-Differential Inclusions*, Differential Inclusions and Opt. Control (J. Andres, L. Górniewicz, and P. Nistri eds.), Lecture Notes in Nonlinear Anal. **2** (1998), 123–137.

- [7] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lect. Notes in Math. 580, Springer-Verlag, Berlin-Heidelberg-New York 1977.
- [8] R. Dragoni, J.W. Macki, P. Nistri and P. Zecca, *Solution Sets of Differential Equations in Abstract Spaces*, Pitman Res. Notes in Math., 342, Longman 1996.
- [9] Z. Dzedzej and B. Gelman, *Dimension of a set of solutions for differential inclusions*, Demonstratio Math. **26** (1993), 149–158.
- [10] R. Engelking, *Dimension Theory*, PWN, Warszawa 1978.
- [11] B.D. Gelman, *Topological properties of the set of fixed points of a multivalued map*, Mat. Sbornik **188** (1997), 33–56 (in Russian); English transl.: Sbornik: Mathem. **188** (1997), 1761–1782.
- [12] B.D. Gelman, *On topological dimension of a set of solutions of functional inclusions*, Differential Inclusions and Opt. Control (J. Andres, L. Górniewicz and P. Nistri eds.), Lecture Notes in Nonlinear Anal. **2** (1998), 163–178.
- [13] S. Hu and N.S. Papageorgiou, *Handbook of Multivalued Analysis, Vol. I: Theory*, Kluwer Acad. Publ. Dordrecht-Boston-London 1997.
- [14] M. Kamenskii, V. Obukhovskii and P. Zecca, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, De Gruyter Ser. in Nonlinear Analysis and Appl. 7, Walter de Gruyter, Berlin-New York 2001.
- [15] M. Kisielewicz, *Differential Inclusions and Optimal Control*, PWN, Warszawa, Kluwer Academic Publishers, Dordrecht-Boston-London 1991.
- [16] M.A. Krasnoselskii, P.P. Zabreiko, E.I. Pustyl'nik and P.E. Sobolevskii, *Integral Operators in Spaces of Summable Functions*, Noordhoff International Publishing, Leyden 1976.
- [17] E. Michael, *Continuous selections I*, Ann. Math. **63** (1956), 361–382.
- [18] B. Ricceri, *On the topological dimension of the solution set of a class of nonlinear equations*, C.R. Acad. Sci. Paris Ser. I Math. **325** (1997), 65–70.
- [19] J. Saint Raymond, *Points fixes des multiapplications a valeurs convex*, C.R. Acad. Sci. Paris Ser. I Math. **298** (1984), 71–74.

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