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EXISTENCE OF THREE ANTI-PERIODIC SOLUTIONS FOR SECOND-ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS WITH TWO PARAMETERS

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Abstract

Applying two three critical points theorems, we prove the existence of at least three anti-periodic solutions for a second-order impulsive differential inclusion with a perturbed nonlinearity and two parameters.

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1. Introduction

The aim of this paper is to investigate the existence of at least three solutions for the following two parameters second-order impulsive differential inclusion subject to anti-periodic boundary conditions

$$(1.1)\quad \begin{cases} -(\phi_p(u'(x)))' + M\phi_p(u(x)) \in \lambda F(u(x)) + \mu G(x,u(x)) & \text{in } [0,T]\setminus Q, \\ -\Delta\phi_p(u'(x_k)) = I_k(u(x_k)), \qquad k = 1,2,\dots,m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T), \end{cases}$$

where $p > 1$, $Q = \{x_1, x_2, \dots, x_m\}$, $T > 0$, $M \geq 0$, $\phi_p(x) := |x|^{p-2}x$, $0 = x_0 < x_1 < \dots < x_m < x_{m+1} = T$, $\Delta\phi_p(u'(x_k)) := \phi_p(u'(x_k^+)) - \phi_p(u'(x_k^-))$, with $u'(x_k^+)$ and $u'(x_k^-)$ denoting the right and left limits, respectively, of $u'(x)$ at $x = x_k$, $I_k \in C(\mathbb{R},\mathbb{R})$, $k = 1,2,\dots,m$, λ is a positive parameter, μ is a nonnegative parameter, and F is a multifunction defined on $\mathbb{R}$, satisfying

(F_1) $F : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous with compact convex values;

(F_2) $\min F$, $\max F : \mathbb{R} \to \mathbb{R}$ are Borel measurable;

(F_3) $|\xi| \leq a(1 + |s|^{r-1})$ for all $s \in \mathbb{R}$, $\xi \in F(s)$, $r > 1\,(a > 0)$.

G is a multifunction defined on $[0,T] \times \mathbb{R}$, satisfying

(G_1) $G(x,\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous with compact convex values for a.e. $x \in [0,T]\setminus Q$;

(G_2) $\min G$, $\max G : ([0,T]\setminus Q) \times \mathbb{R} \to \mathbb{R}$ are Borel measurable;

(G_3) $|\xi| \leq a(1 + |s|^{r-1})$ for a.e. $x \in [0,T]$, $s \in \mathbb{R}$, $\xi \in G(x,s)$, $r > 1\,(a > 0)$.

Impulsive differential equations describe various processes of the real world described by models that are subject to sudden changes in their states. These models are studied in physics, population dynamics, ecology, industrial robotics, biotechnology, economics, optimal control, and so forth. Associated with this development, a theory of impulsive differential equations has been given extensive attention. Differential inclusions arise in models for control systems, mechanical systems, economical systems, game theory, and biological systems to name a few. It is very important to study anti-periodic boundary value problems because they can be applied to interpolation problems [5], antiperiodic wavelets [3], the Hill differential operator [6], and so on. It is natural from both a physical standpoint as well as a theoretical view to give considerable attention to a synthesis involving problems for impulsive differential inclusion with anti-periodic boundary conditions.

Recently, multiplicity of solutions for differential inclusions via non-smooth variational methods and critical point theory has been considered in the papers [9, 10, 11, 12, 16]. For instance, in [11], the author, employing a non-smooth Ricceri-type variational principle [15] developed by Marano and Motreanu [13], has established the existence of infinitely many, radially symmetric solutions for a differential inclusion problem in $\mathbb{R}^N$. Also, in [12], the authors extended a recent result of Ricceri concerning the existence of three critical points of certain non-smooth functionals. Two applications have been given, both in the theory of differential inclusions. The first one concerns a non-homogeneous Neumann boundary value problem, the second one treats a quasilinear elliptic inclusion problem in the whole $\mathbb{R}^N$. In [9], the author, under convenient assumptions, has investigated the existence of at least three positive solutions for a differential inclusion involving the p-Laplacian operator on a bounded domain, with homogeneous Dirichlet boundary conditions and a perturbed nonlinearity depending on two positive parameters. His result also ensured an estimate on norms of solutions independent of both perturbations and parameters. Very recently, Tian and Henderson in [16], based on a non-smooth version of critical point theory of Ricceri due to Iannizzotto [9], have established the existence of at least three solutions for the problem (1.1) whenever λ is large enough and μ is small enough.

In the present paper, motivated by [16], employing two kinds of three-critical-point theorems obtained in [1] and [2] (see Theorems 2.6 and 2.7 below), we are interested in ensuring the existence of at least three anti-periodic solutions for the problem (1.1); see Theorems 3.1 and 3.4 below.

A special case of Theorem 3.1 is the following theorem.

Theorem 1.1. *Let F be a multifunction defined on $\mathbb{R}$, satisfying* (F_1)–(F_3) *and let $I_k \in C(\mathbb{R},\mathbb{R})$, satisfying $I_k(0)=0$, $I_k(s)s<0$, $s\in\mathbb{R}$, $k=1,2,\dots,m$, be such that*

$$\liminf_{\xi\to 0}\frac{\sup_{|u|\le\xi}\min\int_0^u F(s)ds}{\xi^2}=\limsup_{|\xi|\to+\infty}\frac{\int_0^T\min\int_0^{\xi(\frac{T}{2}-x)}F(s)dsdx}{\frac{1}{2}\xi^2T-\sum_{i=1}^m\int_0^{\xi(\frac{T}{2}-x_i)}I_i(s)ds}=0.$$

Then, there is $\lambda^*>0$ such that for every $\lambda>\lambda^*$ and for every multifunction G satisfying (G_1)-(G_3) *and the asymptotical condition*

$$\limsup_{|\xi|\to+\infty}\frac{\sup\limits_{x\in[0,T]}\min\int_0^\xi G(x,s)ds}{\xi^2}<+\infty,$$

there exists $\delta^*_{\lambda,G} > 0$ *such that, for each* $\mu \in [0, \delta^*_{\lambda,G}[$, *the problem*

$$\begin{cases} -u''(x) \in \lambda F(u(x)) + \mu G(x, u(x)) & \text{in } [0,T] \setminus Q, \\ -(u'(x_k^+) - u'(x_k^-)) = I_k(u(x_k)), & k = 1, 2, \ldots, m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T) \end{cases}$$

admits at least three solutions in the space $\{u \in W^{1,2}([0,T]) : u(0) = -u(T)\}$.

For a couple of references on impulsive differential inclusions, we refer to [7] and [8].

2. Basic definitions and preliminary results

Let $(X, \|\cdot\|_X)$ be a real Banach space. We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A function $\varphi : X \to \mathbb{R}$ is called locally Lipschitz if, for all $u \in X$, there exist a neighborhood U of u and a real number $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \leq L\|v - w\|_X \qquad \text{for all } \ v, w \in U.$$

If φ is locally Lipschitz and $u \in X$, the generalized directional derivative of φ at u along the direction $v \in X$ is

$$\varphi^\circ(u; v) := \limsup_{w \to u,\, \tau \to 0^+} \frac{\varphi(w + \tau v) - \varphi(w)}{\tau}.$$

The generalized gradient of φ at u is the set

$$\partial\varphi(u) := \{u^* \in X^* : \langle u^*, v \rangle \leq \varphi^\circ(u; v) \text{ for all } v \in X\}.$$

So $\partial\varphi : X \to 2^{X^*}$ is a multifunction. We say that φ has compact gradient if $\partial\varphi$ maps bounded subsets of X into relatively compact subsets of X^*.

Lemma 2.1 [14, Proposition 1.1]. *Let* $\varphi \in C^1(X)$ *be a functional. Then* φ *is locally Lipschitz and*

$$\varphi^\circ(u; v) = \langle \varphi'(u), v \rangle \quad \text{for all } \ u, v \in X;$$

$$\partial\varphi(u) = \{\varphi'(u)\} \quad \text{for all } \ u \in X.$$

Lemma 2.2 [14, Proposition 1.3]. *Let* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz functional. Then*

$\varphi^\circ(u;\cdot)$ *is subadditive and positively homogeneous for all* $u \in X$*, and*

$\varphi^\circ(u;v) \le L\|v\|$ *for all* $u, v \in X$*, with* $L > 0$ *being a Lipschitz constant for* φ *around* u*.*

Lemma 2.3 [4]**.** *Let* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz functional. Then* $\varphi^\circ : X \times X \to \mathbb{R}$ *is upper semicontinuous and for all* $\lambda \ge 0$*,* $u, v \in X$*, one has*

$$(\lambda\varphi)^\circ(u;v) = \lambda\varphi^\circ(u;v).$$

Moreover, if $\varphi, \psi : X \to \mathbb{R}$ *are locally Lipschitz functionals, then*

$$(\varphi+\psi)^\circ(u;v) \le \varphi^\circ(u;v) + \psi^\circ(u;v) \quad \text{for all } u, v \in X.$$

Lemma 2.4 [14, Proposition 1.6]**.** *Let* $\varphi, \psi : X \to \mathbb{R}$ *be locally Lipschitz functionals. Then*

$$\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u) \text{ for all } u \in X,\ \lambda \in \mathbb{R}, \text{ and}$$

$$\partial(\varphi+\psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u) \text{ for all } u \in X.$$

Lemma 2.5 [9, Proposition 1.6]**.** *Let* $\varphi : X \to \mathbb{R}$ *be a locally Lipschitz functional with a compact gradient. Then* φ *is sequentially weakly continuous.*

We say that $u \in X$ is a (generalized) critical point of locally Lipschitz functional φ if $0 \in \partial\varphi(u)$, i.e.,

$$\varphi^\circ(u;v) \ge 0 \quad \text{for all } v \in X.$$

Our main tools are three-critical-point theorems that we recall here in a convenient form. The first one was obtained in [2] and it is a more precise version of Theorem 3.2 of [1]. The second one was established in [1]. In the first one the coercivity of the functional $\mathcal{N} - \lambda\mathcal{M}$ is required, in the second one a suitable sign hypothesis is assumed.

Let $\mathcal{N}$ and $\mathcal{M}$ be locally Lipschitz functionals and write $J_\lambda := \mathcal{N} - \lambda\mathcal{M}$.

Theorem 2.6 [2, Theorem 3.6]**.** *Let* X *be a reflexive real Banach space,* $\mathcal{N} : X \to \mathbb{R}$ *be a coercive and sequentially weakly lower semicontinuous functional, and* $\mathcal{M} : X \to \mathbb{R}$ *be a sequentially weakly upper semicontinuous functional such that*

$$\mathcal{N}(0) = \mathcal{M}(0) = 0.$$

Assume that there exist $r \in \mathbb{R}$ *and* $\overline{x} \in X$*, with* $0 < r < \mathcal{N}(\overline{x})$*, such that*

(a_1) $\frac{\sup_{\mathcal{N}(x)\le r}\mathcal{M}(x)}{r} < \frac{\mathcal{M}(\overline{x})}{\mathcal{N}(\overline{x})}$;

(a_2) *for each* $\lambda \in \Lambda_r := \left(\frac{\mathcal{N}(\overline{x})}{\mathcal{M}(\overline{x})}, \frac{r}{\sup_{\mathcal{N}(x)\le r} \mathcal{M}(x)} \right)$*, the functional* J_λ *is coercive.*

Then, for each $\lambda \in \Lambda_r$*, the functional* J_λ *has at least three distinct critical points in* X*.*

Theorem 2.7 [1, Corollary 3.1]**.** *Let* X *be a reflexive real Banach space,* $\mathcal{N} : X \to \mathbb{R}$ *be a convex, coercive and sequentially weakly lower semicontinuous functional, and* $\mathcal{M} : X \to \mathbb{R}$ *be a sequentially weakly upper semicontinuous functional such that*

$$\inf_X \mathcal{N} = \mathcal{N}(0) = \mathcal{M}(0) = 0.$$

Assume that there exist two positive constants r_1, r_2 *and* $\overline{x} \in X$ *with* $2r_1 < \mathcal{N}(\overline{x}) < \frac{r_2}{2}$ *such that*

(b_1) $\frac{\sup_{\mathcal{N}(x)<r_1} \mathcal{M}(x)}{r_1} < \frac{2}{3}\frac{\mathcal{M}(\overline{x})}{\mathcal{N}(\overline{x})}$;

(b_2) $\frac{\sup_{\mathcal{N}(x)<r_2} \mathcal{M}(x)}{r_2} < \frac{1}{3}\frac{\mathcal{M}(\overline{x})}{\mathcal{N}(\overline{x})}$;

(b_3) *for each* $\lambda \in \Lambda'_{r_1,r_2} := \left(\frac{3}{2}\frac{\mathcal{N}(\overline{x})}{\mathcal{M}(\overline{x})}, \min\left\{ \frac{r_1}{\sup_{\mathcal{N}(x)<r_1} \mathcal{M}(x)}, \frac{r_2}{2\sup_{\mathcal{N}(x)<r_2} \mathcal{M}(x)} \right\} \right)$ *and for every* $x_1, x_2 \in X$ *which are local minima for the functional* J_λ *and such that* $\mathcal{M}(x_1) \ge 0$ *and* $\mathcal{M}(x_2) \ge 0$*, one has* $\inf_{s\in[0,1]} \mathcal{M}(sx_1 + (1-s)x_2) \ge 0$*.*

Then, for each $\lambda \in \Lambda'_{r_1,r_2}$*, the functional* J_λ *admits at least three critical points which lie in* $\mathcal{N}^{-1}(-\infty, r_2)$*.*

We recall here some basic definitions and notations. On the reflexive Banach space
$X := \{u \in W^{1,p}([0,T]) : u(0) = -u(T)\}$ we consider the norm

$$\|u\|_X := \left(\int_0^T \left(|u'(x)|^p + M|u(x)|^p\right) dx \right)^{1/p}$$

for all $u \in X$, which is equivalent to the usual norm (note that $M \ge 0$).

Since $p > 1$, X is compactly embedded into the space $C^0([0,T])$ endowed with the maximum norm $\|\cdot\|_{C^0}$.

Lemma 2.8 [16, Lemma 3.3]**.** *Let* $u \in X$*. Then*

$$\|u\|_{C^0} \le \frac{1}{2} T^{1/q} \|u\|_X, \tag{2.1}$$

where $1/p + 1/q = 1$*.*

Obviously, X is compactly embedded into $L^\gamma([0,T])$ endowed with the usual norm $\|\cdot\|_{L^\gamma}$, for all $\gamma \geq 1$.

Definition 2.9. A function $u \in X$ is a weak solution of the problem (1.1) if there exists $u^* \in L^\gamma([0,T])$ (for some $\gamma > 1$) such that

$$\int_0^T \Big[\phi_p(u'(x))v'(x) + M\phi_p(u(x))v(x) - u^*(x)v(x)\Big]dx - \sum_{i=1}^m I_i(u(x_i))v(x_i) = 0$$

for all $v \in X$ and $u^* \in \lambda F(u(x)) + \mu G(x,u(x))$ for a.e. $x \in [0,T]$.

Definition 2.10. By a solution of the impulsive differential inclusion (1.1) we will understand a function $u : [0,T] \setminus Q \to \mathbb{R}$ is of class C^1 with $\phi_p(u')$ absolutely continuous, satisfying

$$\begin{cases} -(\phi_p(u'(x)))' + M\phi_p(u(x)) = u^* & \text{in } [0,T] \setminus Q, \\ -\Delta\phi_p(u'(x_k)) = I_k(u(x_k)), & k = 1,2,\dots,m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T), \end{cases}$$

where $u^* \in \lambda F(u(x)) + \mu G(x,u(x))$ and $u^* \in L^\gamma([0,T])$ (for some $\gamma > 1$).

Lemma 2.11 [16, Lemma 3.5]**.** *If a function $u \in X$ is a weak solution of* (1.1)*, then u is a solution of* (1.1)*.*

We introduce for a.e. $x \in [0,T]$ and all $s \in \mathbb{R}$, the Aumann-type set-valued integral

$$\int_0^s F(t)dt = \left\{ \int_0^s f(t)dt \,:\, f : \mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } F \right\}$$

and set $\mathcal{F}(u) = \int_0^T \min \int_0^u F(s)dsdx$ for all $u \in L^p([0,T])$.

The Aumann-type set-valued integral

$$\int_0^s G(x,t)dt = \left\{ \int_0^s g(x,t)dt \,:\, g : [0,T]\times\mathbb{R} \to \mathbb{R} \text{ is a measurable selection of } G \right\}$$

and set $\mathcal{G}(u) = \int_0^T \min \int_0^u G(x,s)dsdx$ for all $u \in L^p([0,T])$.

Lemma 2.12 [10, Lemma 3.1]**.** *The functionals $\mathcal{F}, \mathcal{G} : L^p([0,T]) \to \mathbb{R}$ are well defined and Lipschitz on any bounded subset of $L^p([0,T])$. Moreover, for all $u \in L^p([0,T])$ and all $u^* \in \partial(\mathcal{F}(u) + \mathcal{G}(u))$,*

$$u^*(x) \in F(u(x)) + G(x, u(x)) \qquad \text{for a.e. } x \in [0,T].$$

We define an energy functional for the problem (1.1) by setting

$$J_\lambda(u) = \frac{1}{p}\|u\|_X^p - \lambda\mathcal{F}(u) - \mu\mathcal{G}(u) - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds$$

for all $u \in X$.

Lemma 2.13 [16, Lemma 4.4]**.** *The functional $J_\lambda : X \to \mathbb{R}$ is locally Lipschitz. Moreover, for each critical point $u \in X$ of J_λ, u is a weak solution of* (1.1).

3. Main results

Fixing $c, d > 0$ such that

$$\frac{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum\limits_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx} < \frac{\frac{1}{p}\left(\frac{2c}{T}\right)^p}{\sup\limits_{|u|\le c} \min \int_0^u F(s)ds}$$

and picking

(3.1)
$$\lambda \in \Lambda_1 := \left(\frac{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum\limits_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}, \frac{\frac{1}{p}\left(\frac{2c}{T}\right)^p}{\sup\limits_{|u|\le c} \min \int_0^u F(s)ds} \right),$$

put

(3.2)
$$\delta_{\lambda,G} := \min\left\{ \frac{\frac{1}{p}\left(\frac{2c}{T}\right)^p - \lambda \sup\limits_{|u|\le c} \min \int_0^u F(s)ds}{\sup\limits_{|u|\le c} \min \int_0^u G(x,s)ds}, \right.$$
$$\left. \frac{\lambda \int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx - \frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) + \sum\limits_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}{-\int_0^T \min \int_0^{d(\frac{T}{2}-x)} G(x,s)dsdx} \right\}$$

and

$$(3.3)\qquad \overline{\delta}_{\lambda,G} := \min\left\{\delta_{\lambda,G},\ \frac{\frac{1}{p}\left(\frac{2}{T}\right)^p}{\max\left\{0, \limsup\limits_{|\xi|\to+\infty}\frac{\sup\limits_{x\in[0,T]}\min\int_0^\xi G(x,s)ds}{\xi^p}\right\}}\right\},$$

where we read $\frac{r}{0} = +\infty$.

We formulate our main result as follows.

Theorem 3.1. *Assume that* (F_1)–(F_3) *hold. Furthermore, suppose that there exist two positive constants* c *and* d *with*

$$(3.4)\qquad \left(\frac{2c}{T}\right)^p < d^p\left(1+\frac{M}{p+1}\left(\frac{T}{2}\right)^p\right)$$

and such that

(F_4) $\dfrac{\sup_{|u|\le c}\min\int_0^u F(s)ds}{\frac{1}{p}\left(\frac{2c}{T}\right)^p} < \dfrac{\int_0^T \min\int_0^{d(\frac{T}{2}-x)} F(s)dsdx}{\frac{d^p}{p}\left(T+\frac{2M}{p+1}(\frac{T}{2})^{p+1}\right)-\sum_{i=1}^m\int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}$;

(F_5) $\limsup_{|\xi|\to+\infty}\dfrac{\min\int_0^\xi F(s)ds}{\xi^p} \le 0$;

(I) $I_i(0) = 0$, $I_i(s)s < 0$, $s \in \mathbb{R}$, $i = 1, 2, \dots, m$.

Then, for every $\lambda \in \Lambda_1$, *where* Λ_1 *is given by* (3.1), *and for every multifunction* G *satisfying* (G_1)–(G_3) *and*

(G_4) $\limsup_{|\xi|\to+\infty}\dfrac{\sup\limits_{x\in[0,T]}\min\int_0^\xi G(x,s)ds}{\xi^p} < +\infty$,

there exists $\overline{\delta}_{\lambda,G} > 0$ *given by* (3.3) *such that, for each* $\mu \in [0, \overline{\delta}_{\lambda,G}[$, *the problem* (1.1) *admits at least three solutions in* X.

Proof. Fix λ, G and μ as in the Theorem. For each $u \in X$, put

$$\begin{aligned}\mathcal{N}(u) &:= \frac{1}{p}\|u\|_X^p - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds,\\ \mathcal{M}(u) &:= \int_0^T \min\int_0^u F(s)dsdx + \frac{\mu}{\lambda}\int_0^T \min\int_0^u G(x,s)dsdx.\end{aligned}$$

It is a simple matter to verify that $\mathcal{N}$ is sequentially weakly lower semicontinuous on X. Clearly $\mathcal{N} \in C^1(X)$. By Lemma 2.1, $\mathcal{N}$ is locally Lipschitz on X. By Lemma 2.12, $\mathcal{F}$ and $\mathcal{G}$ are locally Lipschitz on $L^p([0,T])$. So, $\mathcal{M}$ is locally Lipschitz on $L^p([0,T])$. Moreover, X is compactly embedded into $L^p([0,T])$. So $\mathcal{M}$ is locally Lipschitz on X. Therefore, $\mathcal{M}$ is sequentially weakly upper semicontinuous. For all $u \in X$, by (I),

$$\int_0^{u(x_i)} I_i(s)ds < 0, \qquad i = 1, 2, \dots, m.$$

So, we have

$$\mathcal{N}(u) = \frac{1}{p}\|u\|_X^p - \sum_{i=1}^{m} \int_0^{u(x_i)} I_i(s)ds > \frac{1}{p}\|u\|_X^p$$

for all $u \in X$. Hence, $\mathcal{N}$ is coercive and

$$\inf_X \mathcal{N} = \mathcal{N}(0) = \mathcal{M}(0) = 0.$$

Thus, the regularity assumptions on $\mathcal{N}$ and $\mathcal{M}$ are satisfied. We will verify (a_1) and (a_2) of Theorem 2.6. Let w be the function defined by

$$w(x) := d\Big(\frac{T}{2} - x\Big), \qquad x \in [0,T], \tag{3.5}$$

and put

$$r := \frac{1}{p}\left(\frac{2c}{T^{1/q}}\right)^p.$$

Clearly, $w \in X$ and, in particular, one has

$$\|w\|_X^p = d^p\left(T + \frac{2M}{p+1}\Big(\frac{T}{2}\Big)^{p+1}\right),$$

and so

$$\mathcal{N}(w) = \frac{d^p}{p}\left(T + \frac{2M}{p+1}\Big(\frac{T}{2}\Big)^{p+1}\right) - \sum_{i=1}^{m} \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds.$$

From condition (3.4), one has $\mathcal{N}(w) > r$.

Also, we have

$$\mathcal{M}(w) := \int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx + \frac{\mu}{\lambda}\int_0^T \min \int_0^{d(\frac{T}{2}-x)} G(x,s)dsdx.$$

For all $u \in X$ with $\mathcal{N}(u) \leq r$, taking into account $\|u\|_X^p < pr$ and $\|u\|_{C^0} \leq \frac{1}{2}T^{1/q}\|u\|_X$, one has $|u(x)| \leq c$ for all $x \in [0,T]$. Therefore,

$$\frac{\sup\limits_{\mathcal{N}(u)\leq r} \mathcal{M}(u)}{r} = \frac{\sup\limits_{\mathcal{N}(u)\leq r} \left[\int_0^T \min \int_0^u F(s)dsdx + \frac{\mu}{\lambda}\int_0^T \min \int_0^u G(x,s)dsdx\right]}{r}$$
$$\leq p\Big(\frac{T}{2c}\Big)^p \sup_{|u|\leq c} \min \int_0^u F(s)ds + \frac{\mu}{\lambda} p\Big(\frac{T}{2c}\Big)^p \sup_{|u|\leq c} \min \int_0^u G(x,s)ds. \tag{3.6}$$

On the other hand, we have

$$\frac{\mathcal{M}(w)}{\mathcal{N}(w)} = \frac{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}{\frac{d^p}{p}\Big(T + \frac{2M}{p+1}\big(\frac{T}{2}\big)^{p+1}\Big) - \sum\limits_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}$$
$$+ \frac{\mu}{\lambda}\frac{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} G(x,s)dsdx}{\frac{d^p}{p}\Big(T + \frac{2M}{p+1}\big(\frac{T}{2}\big)^{p+1}\Big) - \sum\limits_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}. \tag{3.7}$$

Since $\mu < \delta_{\lambda,G}$, one has

$$\mu < \frac{\frac{1}{p}\left(\frac{2c}{T}\right)^p - \lambda \sup\limits_{|u|\leq c} \min \int_0^u F(s)ds}{\sup\limits_{|u|\leq c} \min \int_0^u G(x,s)ds},$$

and so

$$p\Big(\frac{T}{2c}\Big)^p \sup_{|u|\leq c} \min \int_0^u F(s)ds + \frac{\mu}{\lambda} p\Big(\frac{T}{2c}\Big)^p \sup_{|u|\leq c} \min \int_0^u G(x,s)ds < \frac{1}{\lambda}. \tag{3.8}$$

Similarly,

$$\mu < \frac{\lambda \int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx - \frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) + \sum\limits_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}{-\int_0^T \min \int_0^{d(\frac{T}{2}-x)} G(x,s)dsdx},$$

and so

$$\frac{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum\limits_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}$$
$$+ \frac{\mu}{\lambda}\frac{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} G(x,s)dsdx}{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum\limits_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds} > \frac{1}{\lambda}. \tag{3.9}$$

Hence, from (3.6)–(3.9), condition (a_1) of Theorem 2.6 is verified.

Since $\mu < \overline{\delta}_{\lambda,G}$, we can fix $l > 0$ such that

$$\limsup_{|\xi|\to+\infty} \frac{\sup\limits_{x\in[0,T]} \min \int_0^\xi G(x,s)ds}{\xi^p} < l$$

and $\mu l < \frac{1}{p}\left(\frac{2}{T}\right)^p$. Therefore, there exists a positive constant k such that

$$\min \int_0^\xi G(x,s)ds \leq l\xi^p + k$$

for each $(x,\xi) \in [0,T] \times \mathbb{R}$. Now, fix $0 < \varepsilon < \frac{\frac{1}{p}\left(\frac{2}{T}\right)^p - \mu l}{\lambda}$. From ($F_5$) there is a positive constant k_ε such that

$$\min \int_0^\xi F(s)ds \leq \varepsilon\xi^p + k_\varepsilon$$

for each $\xi \in \mathbb{R}$. So, for each $u \in X$,

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\|u\|_X^p - \sum_{i=1}^m \int_0^{u(x_i)} I_i(s)ds - \lambda \int_0^T \min \int_0^u F(s)dsdx \\
&\quad - \mu \int_0^T \min \int_0^u G(x,s)dsdx \\
&> \frac{1}{p}\|u\|_X^p - \lambda \int_0^T \big(\varepsilon u^p + k_\varepsilon\big)dx - \mu \int_0^T \big(lu^p + k\big)dx \\
&\geq \Big(\frac{1}{p} - \lambda\varepsilon\big(\frac{T}{2}\big)^p - \mu l\big(\frac{T}{2}\big)^p\Big)\|u\|_X^p - \lambda T k_\varepsilon - \mu T k.
\end{aligned}$$

This leads to coercivity of J_λ and condition (a_2) of Theorem 2.6 is verified.

Since, from (3.6)–(3.9),

$$\lambda \in \Lambda_1 \subseteq \left(\frac{\mathcal{N}(w)}{\mathcal{M}(w)}, \frac{r}{\sup_{\mathcal{N}(u)\leq r} \mathcal{M}(u)}\right),$$

Theorem 2.6 ensures the existence of at least three critical points for the functional J_λ. Finally, by Lemma 2.13, the critical points of J_λ are weak solutions for the problem (1.1), and by Lemma 2.11, every weak solution of (1.1) is a solution of (1.1). Hence, the proof is complete. ∎

Remark 3.2. Theorem 3.1 ensures a more precise conclusion than [16, Theorem 2.7]. In fact, Theorem 2.7 of [16] establishes that there exists a non-degenerate interval $[\alpha, \beta] \subset (0, +\infty)$ such that, for every $\lambda \in [\alpha, \beta]$ and any multifunction G satisfying (G_1)–(G_3), there exists $\delta > 0$ such that, for all $\mu \in [0, \delta]$, the problem (1.1) admits at least three solutions. Hence, neither a location of the interval $[\alpha, \beta]$ in $(0, +\infty)$ nor an estimate of δ is established.

The following result is a special case of Theorem 3.1 with $\mu = 0$.

Theorem 3.3. *Assume that* (F_1)–(F_3), (F_5) *and* (I) *hold. Furthermore, suppose that there exist two positive constants* c *and* d *such that condition* (3.4) *and the assumption* (F_4) *hold. Then, for each* $\lambda \in \Lambda_1$, *where* Λ_1 *is given by* (3.1), *the problem*

$$\begin{cases} -(\phi_p(u'(x)))' + M\phi_p(u(x)) \in \lambda F(u(x)) & \text{in } [0,T] \setminus Q, \\ -\Delta\phi_p(u'(x_k)) = I_k(u(x_k)), \qquad k = 1, 2, \ldots, m, \\ u(0) = -u(T), \qquad u'(0) = -u'(T) \end{cases} \tag{3.10}$$

has at least three solutions in X.

Now, a variant of Theorem 3.1 where no asymptotic condition on G as (G_4) is required, is pointed out.

Fixing $c_1, c_2, d > 0$ such that

$$\frac{3\left(\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum\limits_{i=1}^{m} \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds\right)}{2\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}$$
$$< \min\left\{\frac{\frac{1}{p}\left(\frac{2c_1}{T}\right)^p}{\sup\limits_{|u|\le c_1} \min \int_0^u F(s)ds}, \frac{\frac{1}{2p}\left(\frac{2c_2}{T}\right)^p}{\sup\limits_{|u|\le c_2} \min \int_0^u F(s)ds}\right\},$$

and picking

$$\lambda \in \Lambda_2 := \left(\frac{3\left(\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum_{i=1}^{m} \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds\right)}{2\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx},\right.$$
$$\left. \min\left\{\frac{\frac{1}{p}\left(\frac{2c_1}{T}\right)^p}{\sup\limits_{|u|\le c_1} \min \int_0^u F(s)ds}, \frac{\frac{1}{2p}\left(\frac{2c_2}{T}\right)^p}{\sup_{|u|\le c_2} \min \int_0^u F(s)ds}\right\}\right), \tag{3.11}$$

put

$$
\delta^*_{\lambda,G} := \min\left\{ \frac{\frac{1}{p}\left(\frac{2c_1}{T}\right)^p - \lambda \sup\limits_{|u|\le c_1} \min \int_0^u F(s)ds}{\sup\limits_{|u|\le c_1} \min \int_0^u G(x,s)ds}, \frac{\frac{1}{p}\left(\frac{2c_2}{T}\right)^p - \lambda \sup\limits_{|u|\le c_2} \min \int_0^u F(s)ds}{\sup\limits_{|u|\le c_2} \min \int_0^u G(x,s)ds} \right\}. \tag{3.12}
$$

Theorem 3.4. *Assume that* (F_1)-(F_3) *and* (I) *hold. Furthermore, suppose that there exist three positive constants* c_1, c_2 *and* d *with*

$$
\left(\frac{2c_1}{T}\right)^p < \frac{d^p}{2}\left(1 + \frac{M}{p+1}\left(\frac{T}{2}\right)^p\right) \tag{3.13}
$$

and

$$
\frac{1}{2p}\left(\frac{2c_2}{T^{1/q}}\right)^p > \frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds \tag{3.14}
$$

and such that

(F_6) $\min \int_0^t F(s)ds \ge 0$ *for each* $t \in [0, c_2]$*;*

(F_7) $\dfrac{\sup_{|u|\le c_1} \min \int_0^u F(s)ds}{\frac{1}{p}\left(\frac{2c_1}{T}\right)^p} < \dfrac{2}{3} \dfrac{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1} - \sum_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds\right)}$*;*

(F_8) $\dfrac{\sup_{|u|\le c_2} \min \int_0^u F(s)ds}{\frac{1}{p}\left(\frac{2c_2}{T}\right)^p} < \dfrac{1}{3} \dfrac{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1} - \sum_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds\right)}$.

Then, for each $\lambda \in \Lambda_2$*, where* Λ_2 *is given by* (3.11)*, and for every multifunction* G *satisfying* (G_1)-(G_3) *and*

(G_5) $\min \int_0^t G(x,s)ds \ge 0$ *for each* $x \in [0,T]$ *and* $t \in \mathbb{R}$*,*

there exists $\delta^*_{\lambda,G}$ *given by* (3.12) *and such that, for each* $\mu \in [0, \delta^*_{\lambda,G}[$*, the problem* (1.1) *has at least three solutions* u_i*,* $i = 1,2,3$*, such that*

$$
0 < u_i(x) < c_2, \qquad \forall x \in [0,T],\ i = 1,2,3.
$$

Proof. Fix λ, G and μ as in the Theorem and take $\mathcal{N}$ and $\mathcal{M}$ as in the proof of Theorem 3.1. The regularity assumption of Theorem 2.7 on $\mathcal{N}$ and $\mathcal{M}$ are satisfied. Put w as in (3.5),

$$r_1 := \frac{1}{p}\left(\frac{2c_1}{T^{1/q}}\right)^p \qquad \text{and} \qquad r_2 := \frac{1}{p}\left(\frac{2c_2}{T^{1/q}}\right)^p.$$

From the conditions (3.13) and (3.14), we observe $2r_1 < \mathcal{N}(w) < \frac{r_2}{2}$. Since $\mu < \delta^*_{\lambda,G}$, we have

$$\begin{aligned}
\frac{\sup_{\mathcal{N}(u)<r_1}\mathcal{M}(u)}{r_1} &\leq p\Big(\frac{T}{2c_1}\Big)^p \sup_{|u|\leq c_1} \min \int_0^u F(s)ds \\
&\quad + \frac{\mu}{\lambda} p\Big(\frac{T}{2c_1}\Big)^p \sup_{|u|\leq c_1} \min \int_0^u G(x,s)ds \\
&< \frac{1}{\lambda} \\
&< \frac{2}{3}\Bigg(\frac{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds} \\
&\quad + \frac{\mu}{\lambda}\frac{2\int_0^T \min \int_0^{d(\frac{T}{2}-x)} G(x,s)dsdx}{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}\Bigg) \\
&= \frac{2}{3}\frac{\mathcal{M}(w)}{\mathcal{N}(w)},
\end{aligned}$$

and similarly,

$$\begin{aligned}
2\frac{\sup_{\mathcal{N}(u)<r_2}\mathcal{M}(u)}{r_2} &\leq 2p\Big(\frac{T}{2c_2}\Big)^p \sup_{|u|\leq c_2} \min \int_0^u F(s)ds \\
&\quad + \frac{\mu}{\lambda} p\Big(\frac{T}{2c_2}\Big)^p \sup_{|u|\leq c_2} \min \int_0^u G(x,s)ds \\
&< \frac{1}{\lambda} \\
&< \frac{2}{3}\Bigg(\frac{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds} \\
&\quad + \frac{\mu}{\lambda}\frac{2\int_0^T \min \int_0^{d(\frac{T}{2}-x)} G(x,s)dsdx}{\frac{d^p}{p}\left(T + \frac{2M}{p+1}\left(\frac{T}{2}\right)^{p+1}\right) - \sum_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}\Bigg) \\
&= \frac{2}{3}\frac{\mathcal{M}(w)}{\mathcal{N}(w)}.
\end{aligned}$$

Therefore, assumptions (b_1) and (b_2) of Theorem 2.7 are verified. Now, we show that the functional J_λ satisfies assumption (b_3) of Theorem 2.7. Let u^* and u^{**} be two local minima for J_λ. Then u^* and u^{**} are critical points for J_λ, and so by Lemma 2.12, they are weak solutions for problem (1.1). Due to (F_6) and (G_5), from the Maximum Principle we have $u^*(x) \geq 0$ and $u^{**}(x) \geq 0$ for every $x \in [0,T]$. So, it follows that $su^*+(1-s)u^{**} \geq 0$ for all $s \in [0,1]$, and consequently $\mathcal{M}\big(su^*+(1-s)u^{**}\big) \geq 0$ for all $s \in [0,1]$. Thus, Theorem 2.7 ensures at least three solutions whose norm in X is less than $2c_2$. Hence, the Strong Maximum Principle and Lemma 2.8 ensure the conclusion. ■

Now, we present the following example to illustrate Theorem 3.4.

Example 3.5. Consider the problem

$$\begin{cases} -(\phi_3(u'(x)))' + \phi_3(u(x)) \in \lambda F(u(x)) + \mu G(x,u(x)) & \text{in } [0,2] \setminus \{1\}, \\ -\Delta\phi_3(u'(x_1)) = I_1(u(x_1)), \qquad x_1 = 1, \\ u(0) = -u(2), \qquad u'(0) = -u'(2), \end{cases} \tag{3.15}$$

where for all $s \in \mathbb{R}$,

$$F(s) = \begin{cases} \{0\}, & \text{if } |s| < 2^{-1/3}, \\ [0,1], & \text{if } |s| = 2^{-1/3}, \\ \{s - 2^{-1/3} + 1\}, & \text{if } s > 2^{-1/3}, \\ \{s + 2^{-1/3} + 1\}, & \text{if } s < -2^{-1/3}. \end{cases}$$

For any multifunction G satisfying (G_1)-(G_3), the problem (3.15) admits at least three solutions u_i, $i = 1,2,3$, such that

$$0 < u_i(x) < 8, \qquad \forall x \in [0,2],\ i = 1,2,3,$$

for λ and μ lying in convenient intervals. In fact, contrast to the problem (1.1), $p = 3$, $M = 1$, $T = 2$ and $x_1 = 1$. Clearly the assumptions (F_1)–(F_3) and (I) are satisfied. By choosing $c_1 = 2^{-\frac{1}{3}}$, $c_2 = 8$ and $d = 2$, we see that conditions (3.13) and (3.14) and the assumption (F_6) are easily verified. Moreover, simple calculations show that

$$\sup_{|u| \leq 2^{-\frac{1}{3}}} \min \int_0^u F(s)ds = \sup_{|u| \leq 8} \min \int_0^u F(s)ds = 0$$

and

$$\frac{\int_0^2 \min \int_0^{2(1-x)} F(s)dsdx}{\frac{20}{3} - \int_0^{2(1-x_1)} I_1(s)ds}$$
$$= \frac{3}{20}\int_{-1}^{1} \min \int_0^{2x} F(s)dsdx$$
$$= \frac{3}{20}\Bigg(\int_{-1}^{-2^{-1/3}} \int_0^{2x} \max F(s)dsdx + \int_{-2^{-1/3}}^{0} \int_0^{2x} \max F(s)dsdx$$
$$+ \int_0^{2^{-1/3}} \int_0^{2x} \max F(s)dsdx + \int_{2^{-1/3}}^{1} \int_0^{2x} \max F(s)dsdx\Bigg) > 0.$$

So, the assumptions (F_7) and (F_8) are fulfilled. Hence, using Theorem 3.4, the problem (3.15) admits at least three solutions u_i, $i = 1, 2, 3$, in $X := \{u \in W^{1,3}([0, 2]) : u(0) = -u(2)\}$, such that

$$0 < u_i(x) < 8, \qquad \forall x \in [0, 2],\ i = 1, 2, 3.$$

The following result is a special case of Theorem 3.4 with $\mu = 0$.

Theorem 3.6. *Assume that* (F_1)–(F_3) *and* (I) *hold. Furthermore, suppose that there exist three positive constants c_1, c_2 and d such that conditions* (3.13) *and* (3.14) *and assumptions* (F_6)–(F_8) *hold. Then, for each $\lambda \in \Lambda_2$, where Λ_2 is given by* (3.11)*, the problem* (3.10) *has at least three solutions u_i, $i = 1, 2, 3$, in X, such that*

$$0 < u_i(x) < c_2, \qquad \forall x \in [0, T],\ i = 1, 2, 3.$$

Finally, we prove Theorem 1.1 from the Introduction.

***Proof of Theorem* 1.1.** Fix $\lambda > \lambda^* := \frac{\frac{d^2}{2}T - \sum_{i=1}^m \int_0^{d(\frac{T}{2}-x_i)} I_i(s)ds}{\int_0^T \min \int_0^{d(\frac{T}{2}-x)} F(s)dsdx}$ for some $d > 0$. Since

$$\liminf_{\xi \to 0} \frac{\sup_{|u| \le \xi} \min \int_0^u F(s)ds}{\xi^2} = 0,$$

there is a sequence $\{c_n\} \subset (0, +\infty)$ such that $\lim\limits_{n \to +\infty} c_n = 0$ and

$$\lim_{n \to +\infty} \frac{\sup_{|u| \le c_n} \min \int_0^u F(s)ds}{c_n^2} = 0.$$

Indeed, one has

$$\lim_{n\to+\infty}\frac{\sup_{|u|\le c_n}\min\int_0^u F(s)ds}{c_n^2}=\lim_{n\to+\infty}\frac{\min\int_0^{\xi_{c_n}} F(s)ds}{\xi_{c_n}^2}\frac{\xi_{c_n}^2}{c_n^2}=0,$$

where $\min\displaystyle\int_0^{\xi_{c_n}} F(s)ds=\sup_{|u|\le c_n}\min\int_0^u F(s)ds$. Hence, there exists $\bar{c}>0$ such that

$$\frac{\sup_{|u|\le\bar{c}}\min\int_0^u F(s)ds}{\frac{1}{2}\left(\frac{2\bar{c}}{T}\right)^2}<\min\left\{\frac{\int_0^T\min\int_0^{d(\frac{T}{2}-x)}F(s)dsdx}{\frac{d^2}{2}T-\sum\limits_{i=1}^m\int_0^{d(\frac{T}{2}-x_i)}I_i(s)ds},\frac{1}{\lambda}\right\}$$

and $\bar{c}<\frac{dT}{2}$. The conclusion follows from Theorem 3.1 with $p=2$ and $M=0$. ■

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