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A CLASS OF RETRACTS IN L^p WITH SOME APPLICATIONS TO DIFFERENTIAL INCLUSION

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Consider a differential inclusion

$$-\Delta\psi \in \mathcal{F}(t,\psi) \tag{1}$$

with boundary conditions

$$\psi|_{\partial T} = 0 \tag{2}$$

where t runs over a bounded domain $T \subset \mathbf{R}^n$ with a sufficiently smooth boundary $\Gamma = \partial T$. Δ is Laplace operator in T and $\mathcal{F}$ is a Lipschitzean multifunction with a constant $m \in L^p(T)$ i.e.

$$\operatorname{dist}_H\left(\mathcal{F}(t,x), \mathcal{F}(t,y)\right) \le m(t)\,|x-y|\,.$$

By a solution (1), (2) we mean a function $\psi \in H_0^1(T) \cap W^{2,1}(T)$ such that

$$-\Delta\psi(t) \in \mathcal{F}(t,\psi(t))$$

for a.e. $t \in T$. In [3] we examined the case $n = 1$ i.e.

$$(*) \qquad \begin{cases} -x'' \in \mathcal{F}(t,x), & t \in T \\ x|_{\partial T} = 0. \end{cases}$$

We have proved that if m is sufficiently small than the set of solutions of (*) is an absolute retract. The main tools used in [2] were the spectral properties of the operator $L_m = -\Delta - m$ extended to Sobolev space H_0^1 and in particular the stability property of the principal eigenvalue of the operator L_m, $m \in L^1$. Having this property we were able to renorm L^1, in such a way that the solution set of (*) is the set of fixed points of certain multivalued contraction and then apply the BCF theorem [5] on properties of the set of fixed points. Applying these methods to the case of $\mathbf{R}^n$ is possible, however, it calls for a thorough study of spectral properties of the operator

$$L_m\psi = -\Delta\psi - m \cdot \psi, \qquad \psi \in H_0^1.$$

This will be the subject of a consecutive paper. In particular, we need to examine the stability properties of the principal eigenvalue of the operator L_m in relation to $m \in L^p$ with properly chosen p (c.f. [2] for case $n = 3$). We should point out that spectral properties of the operator L_m are well known, in case m is a sufficiently smooth function. The results, known in the literature, concerning the stability of the principal (or other) eigenvalue of the operator L_m seem not to cover our case $m \in L^p$. In this paper we deal with L_m for $t \in T \subset \mathbf{R}^3$, where a bounded domain T and m are restricted to satisfy the condition

$$(H_+) \qquad \sup\left\{\int_T \mathcal{G}_0(t,\tau)m^2(\tau)\mathcal{G}_0(\tau,s)\,d\tau : t,s \in T\right\} < 1/\,|\,T\,|$$

which is obviously fulfiled if $m \in L^4$ or if $|\,T\,|$ is sufficiently small. Here $\mathcal{G}_0(t,s)$ is a Green function of Dirichlet problem for Laplacean in T.

In Section 2, we extend the result from BCF [5] on retraction in L^1 to the case L^p and in Section 3, we apply this for a differential inclusion of type (1) in H_0^1.

1. Notation

Let $T \subset \mathbf{R}^3$ be a bounded domain with C^∞ smooth boundary Γ and $H_0^1(T)$ be a Sobolev space, i.e. a completion in the norm

$$|\psi| = (\|\nabla\psi\|_2 + \|\psi\|_2)^{1/2},$$

of the space $C_0^\infty(T) = \{\psi : T \to \mathbf{R}^k : \operatorname{supp}\psi \subset T\}$ of infinitely many times differentiable functions where $\|\psi\|_p = (\int_T |\,\psi\,|^p)^{1/p}$ is a norm in L^p with an obvious modification for $p = \infty$. Moreover,

$$W^{2,p} = \Big\{\psi \in L^p : \partial_i\partial_j\psi \in L^p;\ i,j = 1,2,3\Big\}.$$

Then H_0^1 can be continuously embedded in L^6 and compactly embedded in L^2 *e.g.* [10]. The latter means in particular that there exists a constant S such that

$$\Big(\int_T |\psi(t)|^6\,dt\Big)^{1/6} \le\ S\ \|\psi\| \tag{3}$$

for $\psi \in H_0^1$. Moreover, for $m \in L^{3/2}$, the space H_0^1 can be continuously embedded in $L^2(m) = \{u : u^2 m \in L^1\}$, because from (3) and the Hölder inequality we have

$$\int_T m\psi^2 \le \Big(\int_T m^{3/2}\Big)^{2/3}\Big(\int_T |\psi^2|^3\Big)^{1/3} \le\ \|m\|_{3/2}S^2|\psi|^2. \tag{4}$$

Consider a quadratic form

$$D_m[\phi] = \int_T\Big(|\nabla\psi|^2 - m\psi^2\Big)\,dt \tag{5}$$

and let

$$D_m[\phi, v] = \int_T\Big(\nabla\psi\nabla v - m\psi v\Big)\,dt \tag{6}$$

be a corresponding bilinear form. It generates the operator L_m by the formula $\langle L_m\psi, v\rangle = D_m[\psi, v]$ for all $\psi, v \in H_0^1$. The previous remark means, in particular, that the domain of L_m contains H_0^1.

Let us consider the boundary value problem to the inclusion (1) with boundary conditions (2).

We shall assume that the multifunction $\mathcal{F}(t,x)$ satisfies the following hypotheses:

(H1) the sets $\mathcal{F}(t,x)$ are compact subsets of $\mathbf{R}^k$ for any $t \in T$ and $x \in \mathbf{R}^k$, and the multifunctions $t \mapsto \mathcal{F}(t,x)$ are measurable for any $x \in \mathbf{R}^k$;

(H2) there exists $m \in L^{3/2}$ such that for any $x, y \in \mathbf{R}^k$ we have

$$\operatorname{dist}_H(\mathcal{F}(t,x), \mathcal{F}(t,y)) \leq m(t)\,|x-y|,$$

where $\operatorname{dist}(K,L)$ stands for the Hausdorff distance between sets K and $L \subset \mathbf{R}^k$;

(H3)

$$\sup\{|x| : x \in \mathcal{F}(t,0)\} \leq a(t) \quad \text{a.e. and} \quad a \in L^\infty.$$

By a solution to the problem (1), (2) we mean any function $\psi \in W^{2,1} \cap W_0^{1,2}$ such that

$$(L_0\psi)(t) \in \mathcal{F}(t, \psi(t)) \quad \text{a.e. in } T. \tag{7}$$

In the present paper we deal with properties of the solution set $\mathcal{R}$ to the problem (1), (2). We prove that $\mathcal{R}$ is a retract of the whole space $W^{2,1} \cap W_0^{1,2}$.

A by-product is the existence of solutions to the problem (1), (2), since any retract $\mathcal{R} \neq \emptyset$. Our work was motivated by a result by De Blasi and Pianigiani [4], where the authors assumed that the Lipschitz constant $m(t) = \text{const} < 1$, $t \in [0;1]$. In the situation considered in our paper, the above hypothesis has been weakened substantially.

The simplest Schrödinger operator is the operator $L_0\psi = -\Delta\psi$ (for $m = 0$). The equation

$$L_0\psi = u \tag{8}$$

with the boundary conditions (2)

$$\psi|_{\partial T} = 0 \tag{9}$$

has a solution $\psi = Au \in W^{2,1} \cap W_0^{1,2}$ for any $u \in L^1$. This solution is expressed by the formula

$$Au(t) = \int_T \mathcal{G}_0(t,s)u(s)\,ds \tag{10}$$

where $\mathcal{G}_0(t,s)$ is the corresponding Green function in $L^p(T \times T)$ for any $p < \frac{12}{5}$.

The operator $A : L^{6/5} \longrightarrow W^{2,6/5}$ is linear, and bounded, and positive, *i.e.* for any function $u \leq 0$ we have $Au \leq 0$. In particular, it means that for any $u \in L^{6/5}$ the following estimate

$$|Au(t)| \leq \sqrt{k}A(|u|)(t) \quad \text{a.e. in } T \tag{11}$$

holds.

2. Retraction in L^p on fixed points of a contractive multifunction

Let T be a compact Hausdorff space with a *σ-field* Σ of Borel measurable sets given by a nonatomic Radon measure "dt". For $1 \leq p < \infty$ by $L^p = L^p(T,X)$ we mean the Banach space of Bochner integrable functions with the usual norm

$$\| u \|_p = \left(\int_T |u(t)|^p \,dt \right)^{\frac{1}{p}} \quad \text{for} \quad 1 \leq p < \infty.$$

We shall assume that L^p is separable.

A set $K \subset L^p$ is said to be decomposable iff for any $u, v \in K$ and $A \in \Sigma$

$$\chi_A u + (1 - \chi_A)v \in K.$$

Denote by $\mathcal{D}$ the family of all nonempty, closed and decomposable subsets of L^p and take $K \in \mathcal{D}$. Recall that from [9] it follows that for any given $u \in L^p$ and $\varepsilon > 0$ there exists $v \in K$ such that

$$|u(t) - v(t)| \leq \operatorname{essinf}\{|u(t) - z(t)| : \ z \in K\} + \varepsilon \quad \text{a.e. in } T. \tag{12}$$

Therefore

$$\begin{aligned}\operatorname{dist}(u,K) &= \inf\{\|u-z\|_p : z \in K\} \\ &\geq \left(\int_T \{\operatorname{essinf}|u(t)-z(t)| : z \in K\}^p dt\right)^{\frac{1}{p}} \\ &\geq \|u-v\|_p - \varepsilon[\mu(T)]^{\frac{1}{p}}.\end{aligned}$$

But $\varepsilon > 0$ is arbitrary, so

$$\operatorname{dist}(u,K) = \left(\int_T \{\operatorname{essinf}|u(t)-z(t)| : z \in K\}^p dt\right)^{\frac{1}{p}}. \tag{13}$$

Let us consider a mapping $\Phi : L^p \to \mathcal{D}$ which is a contraction, i.e. there is a constant $\alpha \in (0,1)$ such that

$$\operatorname{dist}_H(\Phi(u),\Phi(v)) \leq \alpha \parallel u-v \parallel_p, \tag{14}$$

where $\operatorname{dist}_H(A,B)$ stands for the Hausdorff distance of sets A and B. Consider a mapping $\varphi : L^p(T,X) \to L^p(T,R)$ given by

$$\varphi(u) = \operatorname{essinf}\{|u(t)-w(t)| : w \in \Phi(u)\}. \tag{15}$$

From (14) one can easily observe that

$$\|\varphi(u)\|_p = \operatorname{dist}(u,\Phi(u)). \tag{16}$$

Lemma 1. *The mapping* $\varphi : L^p(T,X) \to L^p(T,R)$ *given by* (15) *is Lipschitz with a constant* $2\alpha+1$.

Proof. Take any $u, v \in L^p$ and fix $\varepsilon > 0$. From (12) there exist $w \in \Phi(u)$ and $z \in \Phi(v)$ such that

$$|u(t)-w(t)| \leq \varphi(u)(t) + \varepsilon \quad \text{a.e. in } T$$

and

$$|v(t)-z(t)| \leq \varphi(v)(t) + \varepsilon \quad \text{a.e. in } T.$$

Moreover, there are $a \in \Phi(u)$ and $b \in \Phi(v)$ such that

$$\|a - z\|_p \leq \operatorname{dist}(z, \Phi(u)) + \varepsilon \leq \alpha \|u - v\|_p + \varepsilon \tag{17}$$

and

$$\|b - w\|_p \leq \operatorname{dist}(w, \Phi(v)) + \varepsilon \leq \alpha \|u - v\|_p + \varepsilon. \tag{18}$$

Then

$$\begin{aligned} \varphi(v)(t) - \varphi(u)(t) &\leq |v(t) - b(t)| - |u(t) - w(t)| + \varepsilon \\ &\leq |v(t) - u(t)| + |u(t) - b(t)| - |u(t) - w(t)| + \varepsilon \\ &\leq |v(t) - u(t)| + |w(t) - b(t)| + |a(t) - z(t)| + \varepsilon. \end{aligned} \tag{19}$$

Similarly

$$\varphi(u)(t) - \varphi(v)(t) \leq |v(t) - u(t)| + |w(t) - b(t)| + |a(t) - z(t)| + \varepsilon \tag{20}$$

and therefore

$$\|\varphi(u) - \varphi(v)\|_p \leq \|u - v\|_p + \|w - b\|_p + \|a - z\|_p + \varepsilon\, [\mu(T)]^{\frac{1}{p}}.$$

This together with (17) and (18) means that

$$\|\varphi(u) - \varphi(v)\|_p \leq (2\alpha + 1)\|u - v\|_p + 2\varepsilon + \varepsilon\, [\mu(T)]^{\frac{1}{p}}. \tag{21}$$

But $\varepsilon > 0$ is arbitrary, so the latter shows our claim. ∎

Theorem 1. *Let $\Phi : L^p \to \mathcal{D}$ be a contraction. Then the set*

$$\operatorname{Fix}(\Phi) = \{u : u \in \Phi(u)\}$$

is a retract of L^p.

Proof. Denote by $S = L^p \setminus \mathrm{Fix}(\Phi)$ and observe that S is open. For any given $u \in S$ define

$$\psi = \varphi(u) + \frac{1-\alpha}{2\alpha}\,[\mu(T)]^{-\frac{1}{p}}\|\phi(u)\|_p \tag{22}$$

and

$$K(u) = \mathrm{cl}\left(\{v \in \Phi(u) : |u(t) - v(t)| < \psi(t) \ \text{ a.e. in } \ T\}\right), \tag{23}$$

where cl stands for the closure in L^p.

Employing similar arguments as in Proposition 2 in [6], [5] and Proposition 3 in [9] one can see that $K : S \to \mathcal{D}$ is lower semicontinuous. Therefore from BCF Theorem there exists a continuous mapping $k : S \to L^p$ such that for $u \in S$ we have $k(u) \in \Phi(u)$ and $|k(u)(t) - u(t)| \leq \psi(t)$ a.e. in T. Therefore

$$\begin{aligned} &\|k(u) - u\|_p \leq \|\psi\|_p \leq \|\varphi(u)\|_p + \frac{1-\alpha}{2\alpha}\,\|\varphi(u)\|_p \\ &= \frac{1+\alpha}{2\alpha}\,\mathrm{dist}\,(u, \Phi(u))\,. \end{aligned} \tag{24}$$

Extend k on L^p by setting $k(u) = u$ if $u \in \mathrm{Fix}(\Phi)$.

By construction we have that for all u

$$k(u) \in \Phi(u) \tag{25}$$

and by (24)

$$\|k(u) - u\|_p \leq \frac{1+\alpha}{2\alpha}\,\mathrm{dist}\,(u, \Phi(u))\,. \tag{26}$$

Such k remains continuous on the whole L^p. To see this it is enough to check continuity for $u \in \mathrm{Fix}(\Phi)$, since it clearly holds on open S. Fix $u \in \mathrm{Fix}(\Phi)$ and let $u_n \to u$. Then by (14) and Lemma 1 we have

$$\mathrm{dist}\,(u_n, \Phi(u_n)) = \|\varphi(u_n)\|_p \to \|\varphi(u)\|_p = 0 \tag{27}$$

and therefore by (26)

$$k(u_n) \to u = k(u) \tag{28}$$

what shows continuity of the mapping k.

Set $r_1(u) = k(u)$ and, by induction,

$$r_{n+1}(u) = k(r_n(u)). \tag{29}$$

Clearly, each r_n is continuous and by (25)

$$r_{n+1}(u) \in \Phi(r_n(u)). \tag{30}$$

We shall show that r_n tends locally uniformly to r and that r is a required retraction. Indeed, from (24), (29) and (30) we have

$$\begin{aligned} \|r_{n+1}(u) - r_n(u)\|_p &\le \frac{1+\alpha}{2\alpha} \operatorname{dist}(r_n(u), \Phi(r_n(u)) \\ \le \frac{1+\alpha}{2\alpha} \operatorname{dist}(\Phi(r_{n-1}(u)), \Phi(r_n(u)) &\le \frac{1+\alpha}{2} \|r_n(u) - r_{n-1}(u)\|_p. \end{aligned} \tag{31}$$

Therefore

$$\|r_{n+1}(u) - r_n(u)\|_p \le \left(\frac{1+\alpha}{2}\right)^n \operatorname{dist}(u, \Phi(u))\,. \tag{32}$$

Since $\operatorname{dist}(u, \Phi(u))$ is locally bounded, r_n converges locally uniformly. This implies that $r(u) = \lim r_n(u)$ is continuous. Moreover, for $u \in \Phi(u)$ we have $r(u) = u$, since $r_n(u) = u$. Passing to the limit in (30) we obtain $r(u) \in \Phi(r(u))$, so r is a retraction. ■

3. Application of the retraction result to differential inclusion

Let us consider the problem of the existence of solution ψ to the differential inclusion

$$-\Delta\psi \in \mathcal{F}(t,\psi) \tag{33}$$

in the class of functions $\psi \in W^{2,1} \cap W_0^{1,2}$ and therefore ψ satisfies the boundary conditions

$$\psi|_{\partial T} = 0. \tag{34}$$

Let us impose the conditions (H1), (H2) and (H3) on the right hand side $\mathcal{F}(t,x)$ and let us assume that the operator L_m where m is a "Lipschitz constant" of the multifunction $\mathcal{F}(t,\cdot)$ satisfies (H_+). The solution set $\mathcal{R}$ is the set of all ψ such that (33) is fulfiled almost everywhere in T with (34) on the boundary of T. The main result in this paper is the following:

Theorem 2. *Let us assume that for the multifunction* $\mathcal{F}(t,x)$ (H1), (H2), (H3) *and* (H_+) *hold. Then the set of solutions to the problem* (33) *with* (34) *is a retract of the space* $W^{2,1} \cap W_0^{1,2}$.

Proof. Denote by

$$\mathcal{K}(u) = \Big\{v \in L^2 : v(t) \in \mathcal{F}(t, A(u)(t)) \text{ a.e. in } \; T\Big\} \tag{35}$$

and

$$\alpha = |\,T\,| \sup_{t,s\in T} \int_T \mathcal{G}_0(t,\tau) m^2(\tau) \mathcal{G}_0(\tau,s)\, d\tau. \tag{36}$$

We shall prove that $\mathcal{K} : L^2 \longrightarrow \operatorname{dec}(L^2)$ is a contraction.

First, let us observe that the sets $\mathcal{K}(u) \neq \emptyset$. Indeed, let v be a measurable selection of the multifunction $t \mapsto \mathcal{F}(t, A(u)(t))$. The existence of v follows from the Kuratowski and Ryll-Nardzewski Theorem. The hypothesis (H2) implies

$$\operatorname{dist}(v(t), \mathcal{F}(t,0)) \leq m(t)\; |\,A(u)(t)\,|$$

for a.e. $t \in T$. Then from (H3) follows an estimate

$$|\,v(t)\,| \leq a(t) + m(t)\; |\,A(|\,u\,|)(t)\,|$$

and further $v \in L^2$.

Secondly, for the contractivity of the map $u \mapsto \mathcal{K}(u)$, let us fix u_1, u_2 and $v_1 \in \mathcal{K}(u_1)$. Let $v_2(\tau) \in \mathcal{F}(\tau, A(u_2)(\tau))$ be a measurable selection such that $|v_1(\tau) - v_2(\tau)| \leq m(\tau)|A(u_1)(\tau) - A(u_2)(\tau)|$ a.e. in T. Hence together with (H2) we have

$$\begin{aligned} \int_T |v_1(\tau) - v_2(\tau)|^2 d\tau &\leq \int_T m^2(\tau)|A(u_1 - u_2)(\tau)|^2 d\tau \\ &\leq \alpha |T|^{-1} \int_T |u_1(\tau) - u_2(\tau)|^2 d\tau. \end{aligned} \tag{37}$$

Now, we have that this is nothing but the contractivity of $\mathcal{K}$. Now, Theorem 1 else implies that the set $\mathrm{Fix}(\mathcal{K})$ of fixed points of multifunction $\mathcal{K}(u)$ is a retract of the space L^2. Let $\phi : L^2 \longrightarrow \mathrm{Fix}(\mathcal{K})$ be the retraction. Since the map A is continuous into $W^{2,1} \cap W_0^{1,2}$ then the map $r : W^{2,1} \cap W_0^{1,2} \longrightarrow \mathcal{R}$ given by

$$r(\psi) = A(\phi(-\Delta\psi))$$

is the retraction from the theorem. ∎

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