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ON THE SOLVABILITY OF DIRICHLET PROBLEM FOR THE WEIGHTED p-LAPLACIAN

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Abstract

In this paper we are concerned with the existence and uniqueness of the weak solution for the weighted p-Laplacian. The purpose of this paper is to discuss in some depth the problem of solvability of Dirichlet problem, therefore all proofs are contained in some detail. The main result of the work is the existence and uniqueness of the weak solution for the Dirichlet problem provided that the weights are bounded. Furthermore, under this assumption the solution belongs to the Sobolev space $W_0^{1,p}(\Omega)$.

Keywords: weighted p-Laplacian, weak solutions, solvability, semi-inner product spaces.

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1. Introduction

Boundary value problems for the p-Laplace operator subject to zero Dirichlet boundary conditions on a bounded domain have been studied extensively during the past two decades and many interesting results have been obtained. In this respect we record the Dirichlet problem that lead us to considering generalizations of the weighted Dirichlet problem.

Let Ω be a bounded domain in $\mathbb{R}^N$. Consider the following Dirichlet problem for the Hilbert space case:

$$D(\Omega): \begin{cases} -\Delta v + cv = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Let $c \in L_\infty(\Omega)$ and $f \in L^2(\Omega)$. Then a function $v \in H_0^1(\Omega)$ is a weak solution of Dirichlet problem provided that for all $\phi \in H_0^1(\Omega)$

$$(\nabla\phi, \nabla v)_{L^2(\Omega,\mathbb{R}^N)} + \int_\Omega cv\phi dx = (\phi, f)_{L^2(\Omega)}.$$

A well known theorem states as follows:

Theorem 1.1. *Suppose that $0 \leq c \in L_\infty(\Omega)$. For $f \in L^2(\Omega)$, the Dirichlet problem* (1.1) *has a unique weak solution $v \in H_0^1(\Omega)$.*

In this paper we develop the issues contained in [10]. Thus we are concerned with the existence and uniqueness of the weak solution to the following boundary value problem:

$$D(\Omega) : \begin{cases} -\Delta_{a,p}v + a_0|v|^{p-2}v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

in which $\Delta_{a,p}$, with $1 < p < \infty$, denotes the p-Laplacian weighted by a diagonal matrix $a = (a_1, \ldots, a_N)$, that can be (formally) given by

$$(\Delta_{a,p}v)(x) = \sum_{i=1}^N \frac{\partial}{\partial x_i}\left(a_i(x)\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\right),$$

and a_0 is an arbitrary function. It must be emphasized that $\Delta_{a,p}$ is just a symbol, and may not be a differential operator at all, since the coefficients a_i $(i = 1, \ldots, N)$ are not assumed to be differentiable.

We treat the generalized Dirichlet problem under general conditions on the weight function a, namely, we suppose that the components a_i $(i = 0, 1, \ldots, N)$ of a are measurable functions on Ω such that

$$a_i(x) > 0 \quad \text{for } x \in \Omega \text{ a.e.,} \quad a_i \in L^1_{loc}(\Omega)$$
$$\text{and } \ 1/a_i \in L_\infty(\Omega) \ (i = 1, \ldots, N).$$

Moreover, we assume that the weight

$$a_0 \geq 0 \quad \text{and} \quad a_0 \in L_\infty(\Omega).$$

We established the existence and uniqueness of weak solutions for a non-linear boundary value problem involving the weighted p-Laplacian. Our approach is based on variational principles and representation properties of the associated spaces. In order to carry over Hilbert space type arguments to the theory of nonlinear elliptic equations in Banach spaces, we used on Sobolev space a type of inner product, called a semi-inner product. Moreover, we specify precisely under what conditions on the weights the integral in Theorem 2.2 in [10] makes sense, and when it is just a symbol.

2. Auxiliary results

In this section the reader will be reminded of some important properties of semi-inner product spaces, and some auxiliary results will be quoted or derived.

To apply Hilbert space type methods to the theory of Banach spaces, Lumer [8] constructed a semi-inner product (s.i.p.) on a complex vector space X as a complex function $[\cdot,\cdot]$ on $X \times X$ with the following properties:

$$[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z], \ x,y,z \in X, \ \alpha,\beta \in \mathbb{C}, \tag{2.1}$$

$$[x, \lambda y] = \overline{\lambda}[x,y], \ x,y \in X, \ \lambda \in \mathbb{C}. \tag{2.2}$$

$$[x,x] > 0 \quad \text{for } x \neq 0, \tag{2.3}$$

$$|[x,y]|^2 \le [x,x][y,y], \ x,y \in X. \tag{2.4}$$

$(X,[\cdot,\cdot])$ is called a complex space with semi-inner product.

The importance of a semi-inner product space (s.i.p.s.) is that every normed vector space can be represented as a semi-inner product space so that the theory of operators on a Banach space can be penetrated by Hilbert space type arguments.

Theorem 2.1 [5, 8]**.** *A semi-inner product space $(X,[\cdot,\cdot])$ is a normed linear space with the norm*

$$\|x\| = [x,x]^{1/2}, \ x \in X.$$

Every normed linear space can be made into a semi-inner product space (in general, in infinitely many different ways).

As an example, consider the real Banach space $L^p(\Omega)$ where $1 < p < \infty$. It can readily be expressed as a s.i.p. space with s.i.p. defined by

$$[x,y]_p = \begin{cases} \|y\|^{2-p}_{L^p(\Omega)} \displaystyle\int_\Omega x|y|^{p-2}y\,d\mu, & y \neq 0, \\ 0, & y = 0, \end{cases}$$

where $\|y\|_{L^p(\Omega)} = \left(\int_\Omega |y|^p d\mu\right)^{1/p}$.

In a normed vector space X we set

$$S = \{x \in X : \|x\| = 1\}.$$

We introduce additional properties of semi-inner product that will help us to move some arguments of Hilbert space. Note that a semi-inner product is continuous with the respect to the first component. A very convenient property of s.i.p. is continuity with respect to the second variable. First discussion concerning the continuity due to the second variable can be found in paper [5]. Giles made the following definition.

A s.i.p. is called a continuous s.i.p. when the following additional condition is satisfied:

For every $x, y \in S$,

$$\mathrm{Re}[y, x + \lambda y] \to \mathrm{Re}[y, x] \text{ for all real } \lambda \to 0. \tag{2.5}$$

A s.i.p. space X has the representation property when to every continuous functional $f \in X^*$ there exists a unique element $y \in X$ such that

$$f(x) = [x, y] \quad \text{for all} \quad x \in X.$$

In Hilbert spaces the representation theorem for continuous linear functionals shows the natural relationship between vectors and continuous linear functionals using the inner product. The following theorem, which was proved by Giles, is a modification of the representation theorem of Riesz-Fréchet for continuous linear functional.

Theorem 2.2 [5]**.** *Let X be a uniformly convex Banach space with a continuous semi-inner product. Then for each $f \in X^*$ there exists a unique vector $y \in X$ such that*

$$f(x) = [x, y] \quad \textit{for all} \quad x \in X$$

and $\|f\| = \|y\|$.

Now we construct a space, which will be used for solving boundary value problem by examining the properties of a certain s.i.p. space.

Let M be a vector space and let Y be a uniformly convex Banach space with a semi-inner product $[\cdot, \cdot]_Y$. Consequently, Y is a reflexive space. Furthermore, let semi-inner product $[\cdot, \cdot]_Y$ satisfy the semi-Lipschitz condition, i.e., there exists a constant $L > 0$ such that

$$|[x, y]_Y - [x, z]_Y| \le L\|x\|_Y \|y - z\|_Y. \tag{2.6}$$

for x, y, $z \in Y$, where $\|y\|_Y = \|z\|_Y = 1$.

Note that the $L^p(\Omega)$ space $(1 < p < \infty)$ satisfies the semi-Lipschitz condition, i.e.,

$$|[x, y]_p - [x, z]_p| \le 2(p - 1)\|x\|_{L^p(\Omega)} \|y - z\|_{L^p(\Omega)},$$

where $\|y\|_{L^p(\Omega)} = \|z\|_{L^p(\Omega)} = 1$ (cf. [5]).

Let $T : M \to Y$ be an injective linear operator. Consider the functional

$$x \mapsto \|x\|_a := \|Tx\|_Y \quad \text{for} \quad x \in M,$$

which is the norm on M.

We define a semi-inner product on the space M given by the formula:

$$[x, y]_a = [Tx, Ty]_Y \quad \text{for} \quad x, y \in M.$$

Note that the semi-inner product $[\cdot, \cdot]_a$ also satisfies semi-Lipschitz condition with the constant $L > 0$. Indeed, let $x, y, z \in M$ and $\|y\|_a = \|z\|_a = 1$. Then

$$\begin{aligned} |[x, y]_a - [x, z]_a| &= |[Tx, Ty]_Y - [Tx, Tz]_Y| \\ &\leq L\|Tx\|_Y \|Ty - Tz\|_Y = L\|x\|_a \|y - z\|_a. \end{aligned}$$

We denote the complement of the space $(M, \|\cdot\|_a)$ by X_1. We show that the norm in the space X_1 is derived from the semi-inner product $[\cdot, \cdot]_a$.

Theorem 2.3. *Let the mapping $[\cdot, \cdot]_a : X_1 \times X_1 \to \mathbb{R}$ be given by the formula*

$$[x, y]_a = \lim_{n\to\infty} [x_n, y_n]_a, \tag{2.7}$$

where $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty} \subset M$ such that $x_n \to x$, $y_n \to y$, as $n \to \infty$, in the norm $\|\cdot\|_a$. Then the mapping $[\cdot, \cdot]_a$ is a semi-inner product on the space X_1.

Proof. We show that the mapping, given by formula (2.7), is well defined. Let x, $y \in X_1$. Then there exist $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty} \subset M$ such that $x_n \to x$ and $y_n \to y$, as $n \to \infty$, in the norm $\|\cdot\|_a$. Using the inequality (2.6), it follows

$$\begin{aligned} &|[x_n, y_n]_a - [x_m, y_m]_a| = \left| \left[x_n \|y_n\|_a, \frac{y_n}{\|y_n\|_a} \right]_a - \left[x_m \|y_m\|_a, \frac{y_m}{\|y_m\|_a} \right]_a \right| \\ &\leq \left| \left[x_n \|y_n\|_a, \frac{y_n}{\|y_n\|_a} \right]_a - \left[x_n \|y_n\|_a, \frac{y_m}{\|y_m\|_a} \right]_a \right| \\ &+ \left| \left[x_n \|y_n\|_a, \frac{y_m}{\|y_m\|_a} \right]_a - \left[x_m \|y_m\|_a, \frac{y_m}{\|y_m\|_a} \right]_a \right| \\ &\leq L\|x_n\|_a \|y_n\|_a \left\| \frac{y_n}{\|y_n\|_a} - \frac{y_m}{\|y_m\|_a} \right\|_a + \|x_n\|y_n\|_a - x_m\|y_m\|_a\|_a \end{aligned}$$

Thus, it is a Cauchy sequence in $\mathbb{R}$. Consequently, the limit $\lim_{n\to\infty}[x_n, y_n]_a$ exists.

Subsequently, we show that the limit does not depend on the choice of representative. Let x, $y \in X_1$ and let $(x_n^{'})_{n=1}^{\infty}$, $(y_n^{'})_{n=1}^{\infty} \subset M$ such that $x_n^{'} \to x$ and

$y_n' \to y$, as $n \to \infty$, in norm $\|\cdot\|_a$. Then

$$\left|[x_n, y_n]_a - [x_n', y_n']_a\right| = \left|\left[x_n\|y_n\|_a, \frac{y_n}{\|y_n\|_a}\right]_a - \left[x_n'\|y_n'\|_a, \frac{y_n'}{\|y_n'\|_a}\right]_a\right|$$
$$\leq \left|\left[x_n\|y_n\|_a, \frac{y_n}{\|y_n\|_a}\right]_a - \left[x_n\|y_n\|_a, \frac{y_n'}{\|y_n'\|_a}\right]_a\right|$$
$$+ \left|\left[x_n\|y_n\|_a, \frac{y_n'}{\|y_n'\|_a}\right]_a - \left[x_n'\|y_n'\|_a, \frac{y_n'}{\|y_n'\|_a}\right]_a\right|$$
$$\leq L\|x_n\|_a\|y_n\|_a\left\|\frac{y_n}{\|y_n\|_a} - \frac{y_n'}{\|y_n'\|_a}\right\|_a + \left\|x_n\|y_n\|_a - x_n'\|y_n'\|_a\right\|_a \to 0.$$

The defined mapping $[\cdot,\cdot]_a$ fulfills the conditions of a semi-inner product. Moreover, semi-inner product $[\cdot,\cdot]_a$ satisfies the semi-Lipschitz condition with a constant $L > 0$. Indeed, set $x, y, z \in X_1$ such that $\|y\|_a = \|z\|_a = 1$. Let $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty} \subset M$ such that $y_n \to y$ and $z_n \to z$, as $n \to \infty$, in norm $\|\cdot\|_a$. By the previous reasoning it suffices to suppose that $\|y_n\|_a = \|z_n\|_a = 1$ for $n \in \mathbb{N}$. Then

$$|[x, y]_a - [x, z]_a| = \lim_{n\to\infty} |[x_n, y_n]_a - [x_n, z_n]_a|$$
$$= \lim_{n\to\infty} |[Tx_n, Ty_n]_Y - [Tx_n, Tz_n]_Y| \leq L \lim_{n\to\infty} \|x_n\|_a\|y_n - z_n\|_a$$
$$= L\|x\|_a\|y - z\|_a.$$

The semi-inner product $[\cdot,\cdot]_a$ is consistent with the norm in X_1. Indeed,

$$[x, x]_a = \lim_{n\to\infty}[x_n, x_n]_a = \lim_{n\to\infty} \|x_n\|_a^2 = \|x\|_a^2,$$

which completes the proof. ■

We will need some properties of the space X_1. Therefore, it should be noted that X_1 is a uniformly convex space.

Lemma 2.1. *Let Y be a uniformly convex space. Then the space X_1 is a uniformly convex space.*

Proof. Let $\varepsilon \in (0, 2]$, $x, y \in M$ such that

$$\|x\|_a \leq 1, \|y\|_a \leq 1 \quad \text{and} \quad \|x - y\|_a > \varepsilon.$$

Hence, we obtain that

$$\|Tx\|_Y \leq 1, \ \|Ty\|_Y \leq 1 \quad \text{and} \quad \|Tx - Ty\|_Y > \varepsilon.$$

By uniform convexity of the space Y, there exists $\delta(\varepsilon) > 0$ such that

$$\frac{\|Tx + Ty\|_Y}{2} \leq 1 - \delta(\varepsilon).$$

Consequently,

$$\frac{\|x + y\|_a}{2} \leq 1 - \delta(\varepsilon).$$

Under the assumption of density of the set M and continuity of the norm it follows that for $\varepsilon \in (0, 2]$ and arbitrary $x, y \in X_1$ such that

$$\|x\|_a \leq 1, \|y\|_a \leq 1, \qquad \|x - y\|_a > \varepsilon$$

there exists $\delta(\varepsilon) > 0$ such that

$$\frac{\|x + y\|_a}{2} \leq 1 - \delta(\varepsilon).$$

Thus, the space X_1 is a uniformly convex space, which completes the proof. ∎

Furthermore, we show that there is a representation theorem for continuous linear functionals in the space X_1.

Lemma 2.2. *Let Y be a uniformly convex Banach space such that the semi-inner product satisfies the semi-Lipschitz inequality with a constant $L > 0$. Then the space X_1 has the representation property.*

Moreover, for every $y \in X_1$ and for every sequence $(y_n)_{n=1}^{\infty} \subset X_1$ converging to y it follows

$$\lim_{n\to\infty} [x, y_n]_a = [x, y]_a \quad \textit{for all} \quad x \in X_1.$$

Proof. We prove first that $[\cdot, \cdot]_a$ is a continuous semi-inner product. Let x, $y \in X_1$ such that $\|x\|_a = \|y\|_a = 1$. From (2.6) we have

$$\left| \left[x, \frac{y + \lambda x}{\|y + \lambda x\|_a} \right]_a - [x, y]_a \right| \leq L\|x\|_a \left\| \frac{y + \lambda x}{\|y + \lambda x\|_a} - y \right\|_a \to 0$$

for all real $\lambda \to 0$. As a consequence of continuity of the norm we obtain that

$$|[x, y + \lambda x]_a - [x, y]_a| \to 0$$

for all real $\lambda \to 0$. By virtue of Theorem 2.2, it follows that the space X_1 has the representation property.

To prove the second statement, let a sequence $(y_n)_{n=1}^{\infty} \subset X_1$ converge to y. Then

$$
\begin{aligned}
&|[x, y_n]_a - [x, y]_a| = \left| \left[x\|y_n\|_a, \frac{y_n}{\|y_n\|_a} \right]_a - \left[x\|y\|_a, \frac{y}{\|y\|_a} \right]_a \right| \\
&\le \left| \left[x\|y_n\|_a, \frac{y_n}{\|y_n\|_a} \right]_a - \left[x\|y\|_a, \frac{y_n}{\|y_n\|_a} \right]_a \right| + \left| \left[x\|y\|_a, \frac{y_n}{\|y_n\|_a} \right]_a - \left[x\|y\|_a, \frac{y}{\|y\|_a} \right]_a \right| \\
&\le \|x\|y_n\|_a - x\|y\|_a\|_a + L\, \|x\|y\|_a\|_a \left\| \frac{y_n}{\|y_n\|_a} - \frac{y}{\|y\|_a} \right\|_a .
\end{aligned}
$$

Clearly, the right-hand side expression tends to zero, which completes the proof. ■

Consider the following problem.

For a given functional $f \in X_1^*$ we seek $y \in X_1$ such that

$$[x, y]_a = \langle x, f \rangle \quad \text{for all} \quad x \in X_1. \tag{2.8}$$

Then y is called a weak solution of the problem (2.8).

It follows from homogeneity property of the semi-inner product that if y is a weak solution of the problem (2.8) for f, then λy is a weak solution of the problem (2.8) for λf. Indeed,

$$\langle x, \lambda f \rangle = \lambda \langle x, f \rangle = \lambda [x, y]_a = [x, \lambda y]_a \quad \text{for all} \quad x \in X_1. \tag{2.9}$$

The theorem to be proved is the following.

Theorem 2.4. *Let M be a vector space and let Y be a uniformly convex Banach space such that the semi-inner product satisfies the semi-Lipschitz inequality with a constant $L > 0$. For every $f \in X_1^*$, the variational problem* (2.8) *has a unique weak solution $y \in X_1$, i.e.,*

$$[x, y]_a = \langle x, f \rangle$$

for all $x \in M$ (or, equivalently, for any $x \in X_1$).

Moreover, the set of all weak solutions, where f runs through X_1^*, is the entire space X_1.

Proof. By Lemma 2.2 for any functional $f \in X_1^*$ there exists a unique element $y \in X_1$ satisfying the identity (2.8). ■

Theorem 2.4 will form the basis for our subsequent results.

3. Dirichlet problem

Let Ω be a bounded domain in $\mathbb{R}^N$. Consider the following boundary problem (the generalized Dirichlet problem for second order)

$$D(\Omega): \begin{cases} -\Delta_{a,p}v + a_0|v|^{p-2}v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

in which $\Delta_{a,p}$, with $1 < p < \infty$, denotes the p-Laplacian weighted by a diagonal matrix $a = (a_1, \dots, a_N)$ and a_0 is an arbitrary function.

Let $a_0 \equiv 0$. It is a particular case of the problem (3.1). Then we obtain the following boundary problem

$$D(\Omega): \begin{cases} -\Delta_{a,p}v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $\Delta_{a,p}$, $1 < p < \infty$, denotes as previously weighted p-Laplacian with weights given by the diagonal matrix $a = (a_1, \dots, a_N)$.

We treat the generalized Dirichlet problem under general conditions on the weight function a, namely, we suppose that the components a_i $(i = 0, 1, \dots, N)$ of a are measurable functions on Ω such that

$$\begin{aligned} a_i(x) > 0 \quad \text{for } x \in \Omega \text{ a.e.}, \quad a_i \in L^1_{loc}(\Omega) \\ \text{and } \ 1/a_i \in L_\infty(\Omega) \ (i = 1, \dots, N). \end{aligned} \tag{3.3}$$

Moreover, we assume that the weight

$$a_0 \geq 0 \quad \text{and} \quad a_0 \in L_\infty(\Omega). \tag{3.4}$$

We shall establish the existence and uniqueness of weak solutions for a non-linear boundary value problem involving the weighted p-Laplacian. We start by writing the integral identity, which will contribute to define the notion of a weak solution for this particular issue.

In addition, we assume that Ω is a bounded domain in $\mathbb{R}^N$ with C^1 boundary $\partial\Omega$. Let $f \in W_0^{-1,p}(\Omega)$. Assuming that the functions a_i $(i = 0, 1, \dots, N)$ are sufficiently smooth and using the Green–Gauss–Ostrogradzki formula for u, $v \in C_0^\infty(\Omega)$ we obtain

$$\begin{aligned} \langle u, f \rangle &= \int_\Omega a_0(x) u|v|^{p-2} v dx - \int_\Omega u \Delta_{a,p} v dx \\ &= \int_\Omega a_0(x) u|v|^{p-2} v dx + \sum_{i=1}^N \int_\Omega a_i(x) \frac{\partial u}{\partial x_i} \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i}\, dx \\ &+ \sum_{i=1}^N \int_{\partial\Omega} u\, a_i(x) \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial n} ds, \end{aligned} \tag{3.5}$$

where $\frac{\partial v}{\partial n}$ denotes the outer normal derivative of v with respect to $\partial\Omega$. We consider the Dirichlet problem with zero condition on the boundary. Taking into account this condition, we demand that the trace of the function v on the boundary of the domain be zero. Then we obtain the following integral identity

$$\langle u, f\rangle = \int_\Omega a_0(x)u|v|^{p-2}v dx + \sum_{i=1}^N \int_\Omega a_i(x)\frac{\partial u}{\partial x_i}\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\, dx.$$

Note that the particular integrals are well defined for all functions which have first order weak derivatives, such that the integrands are integrable functions.

Moreover, if $f \in W_0^{-1,p}(\Omega)$, then there exist functions $f_0, f_1, \ldots, f_N \in L^q(\Omega)$ such that

$$\langle u, f\rangle = \int_\Omega f_0 u dx + \sum_{i=1}^N \int_\Omega f_i \frac{\partial u}{\partial x_i} dx, \quad u \in W_0^{1,p}(\Omega).$$

¿From the above considerations, we obtain that $f_0 = a_0|v|^{p-2}v$, $f_i = a_i\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}$ $(i = 1, \ldots, N)$.

We now prove a lemma which is interesting in its own right.

Lemma 3.1. *Under the conditions* (3.3), (3.4) *there holds the following inequality*
(3.6)

$$\int_\Omega |u|^p\, dx + \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx \le c\left(\int_\Omega a_0(x)\,|u|^p\, dx + \sum_{i=1}^N \int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^p dx\right)$$

for all $u \in C_0^\infty(\Omega)$.

Proof. Under the assumption $a_i \in L^1_{loc}(\Omega)$ $(i = 0, 1, \ldots, N)$ the integral

$$\int_\Omega a_0(x)\,|u|^p\, dx + \sum_{i=1}^N \int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^p dx$$

exists and is finite for all $u \in C_0^\infty(\Omega)$. Using Friedrichs inequality it follows

$$\begin{aligned}\int_\Omega |u|^p\, dx + \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx &\le m\sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx \\ &= m\sum_{i=1}^N \int_\Omega \frac{1}{a_i(x)}a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^p dx \\ &\le c\left(\int_\Omega a_0(x)\,|u|^p\, dx + \sum_{i=1}^N \int_\Omega a_i(x)\left|\frac{\partial u}{\partial x_i}\right|^p dx\right),\end{aligned}$$

in which a constant $c > 0$ does not depend on u. ■

Taking into account this fact, we can formulate.

Corollary 3.1. *Under the assumptions* (3.3), (3.4) *the functional* $\|\cdot\|_a : C_0^\infty(\Omega) \to \mathbb{R}$ *given by*

$$\|u\|_a = \left(\int_\Omega a_0(x)\, |u|^p \, dx + \sum_{i=1}^N \int_\Omega a_i(x) \left| \frac{\partial u}{\partial x_i} \right|^p dx \right)^{\frac{1}{p}}$$

is a norm on $C_0^\infty(\Omega)$*. This is due to Lemma* 3.1.

We will denote the completion of $C_0^\infty(\Omega)$ with respect to the metric of this norm $\|\cdot\|_a$ by $W_a^{1,p}(\Omega)$.

Next, we consider the mapping $T : C_0^\infty(\Omega) \to L^p(\Omega, \mathbb{R}^{N+1})$ given by

$$Tu = \left(a_0^{1/p} u, a_1^{1/p} \frac{\partial u}{\partial x_1}, \dots, a_N^{1/p} \frac{\partial u}{\partial x_N} \right).$$

By virtue of Lemma 3.1, T is an injective linear operator. Moreover, $\|u\|_a = \|Tu\|_{L^p(\Omega, \mathbb{R}^{N+1})}$ for all $u \in C_0^\infty(\Omega)$.

Due to Theorem 2.3 the mapping $[\cdot,\cdot]_a : C_0^\infty(\Omega) \times C_0^\infty(\Omega) \to \mathbb{R}$ given by

$$[u,v]_a = \|v\|_a^{2-p} \left(\int_\Omega a_0(x) u\, |v|^{p-2}\, v dx + \sum_{i=1}^N \int_\Omega a_i(x) \frac{\partial u}{\partial x_i} \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} dx \right)$$

is a semi-inner product on the space $C_0^\infty(\Omega)$. Moreover, the semi-inner product $[\cdot,\cdot]_a : W_a^{1,p}(\Omega) \times W_a^{1,p}(\Omega) \to \mathbb{R}$ given by

$$[u,v]_a = \lim_{n\to\infty} [u_n, v_n]_a, \tag{3.7}$$

for $(u_n)_{n=1}^\infty$, $(v_n)_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that $u_n \to u$ and $v_n \to v$, as $n \to \infty$, in norm $\|\cdot\|_a$, is consistent with the norm in $W_a^{1,p}(\Omega)$.

Norm of any element u of the space $W_a^{1,p}(\Omega)$ can be approximated by a sequence of functions of class $C_0^\infty(\Omega)$, i.e., there exists a sequence $(u_n)_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that

$$\|u\|_a = \lim_{n\to\infty} \left(\int_\Omega a_0(x) |u_n|^p dx + \sum_{i=1}^N \int_\Omega a_i(x) \left| \frac{\partial u_n}{\partial x_i} \right|^p dx \right)^{\frac{1}{p}}.$$

Next, we denote by $W_a^{-1,p}(\Omega)$ the dual space of $W_a^{1,p}(\Omega)$.

On the basis of the considerations in Section (3.5) we give now the definition of a weak solution.

A function v is a weak solution of the Dirichlet problem (3.1), where f is a given functional of $W_a^{-1,p}(\Omega)$, provided that

$$[u, v]_a = \langle u, f \rangle \quad \text{for all} \quad u \in W_a^{1,p}(\Omega).$$

We will need some properties of the obtained space $W_a^{1,p}(\Omega)$. In this context, note that it is a uniformly convex space and there holds a representation theorem for linear continuous functionals defined on it (see Lemma 2.2).

Our main result is the following.

Theorem 3.1. *Suppose that the conditions* (3.3) *and* (3.4) *are fulfilled. For* $f \in W_a^{-1,p}(\Omega)$, *the Dirichlet problem* (3.1) *has a unique weak solution* $v \in W_a^{1,p}(\Omega)$, *i.e.,*

$$[u, v]_a = \langle u, f \rangle$$

for all $u \in C_0^\infty(\Omega)$ (*or, equivalently, for any* $u \in W_a^{1,p}(\Omega)$).

Moreover, the set of all weak solutions, where f runs through $f \in W_a^{-1,p}(\Omega)$ *is the entire space* $W_a^{1,p}(\Omega)$.

Proof. See Theorem 2.4. ■

¿From Theorem 3.1 we infer a useful corollary.

Corollary 3.2. *Under the conditions* (3.3), (3.4) *for any* $f \in W_0^{-1,p}(\Omega)$ *there exists a sequence of functions* $(v_n)_{n=1}^\infty \subset C_0^\infty(\Omega)$ *such that*

$$\lim_{n\to\infty} \|v_n\|_a^{2-p} \left(\int_\Omega a_0(x) u |v_n|^{p-2} v_n dx + \sum_{i=1}^N \int_\Omega a_i(x) \frac{\partial u}{\partial x_i} \left| \frac{\partial v_n}{\partial x_i} \right|^{p-2} \frac{\partial v_n}{\partial x_i} \, dx \right) = \langle u, f \rangle$$

for all $u \in C_0^\infty(\Omega)$.

Suppose additionally that the weights $a_i \in L_\infty(\Omega)$ $(i = 0, \dots, N)$. Then, obviously $W_a^{1,p}(\Omega)$ becomes the Sobolev space $W_0^{1,p}(\Omega)$ and we can look for a weak solution in $W_0^{1,p}(\Omega)$. The significance of this fact for our purposes is captured by Theorem 3.2.

Theorem 3.2. *Suppose that the conditions* (3.3) *and* (3.4) *are fulfilled. If the weights* $a_i \in L_\infty(\Omega)$ $(i = 0, \dots, N)$, *then for any* $f \in W_0^{-1,p}(\Omega)$ *there exists a unique weak solution* $v \in W_0^{1,p}(\Omega)$ *such that*

$$\|v\|_a^{2-p} \left(\int_\Omega a_0(x) u |v|^{p-2} v dx + \sum_{i=1}^N \int_\Omega a_i(x) \frac{\partial u}{\partial x_i} \left| \frac{\partial v}{\partial x_i} \right|^{p-2} \frac{\partial v}{\partial x_i} \, dx \right) = \langle u, f \rangle$$

for all $u \in C_0^\infty(\Omega)$.

Proof. Let the weights $a_i \in L_\infty(\Omega)$ $(i = 0, \dots, N)$. Then under the inequality (3.6) the space $W_a^{1,p}(\Omega)$ becomes the Sobolev space $W_0^{1,p}(\Omega)$. Moreover, the operator $T : C_0^\infty(\Omega) \to L^p(\Omega, \mathbb{R}^{N+1})$ is bounded. Indeed, for $u \in C_0^\infty(\Omega)$

$$\begin{aligned}\|Tu\|^p_{L^p(\Omega,\mathbb{R}^{N+1})} &= \int_\Omega a_0(x)\,|u|^p\,dx + \sum_{i=1}^N \int_\Omega a_i(x) \left|\frac{\partial u}{\partial x_i}\right|^p dx \\ &\le c\left(\int_\Omega |u|^p\,dx + \sum_{i=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^p dx\right).\end{aligned}$$

Furthermore, by continuity the operator T can be uniquely extended to a continuous linear operator $\tilde{T}$ on the whole space $W_0^{1,p}(\Omega)$ and

$$\tilde{T}u = \left(a_0^{1/p}u, a_1^{1/p}\frac{\partial u}{\partial x_1}, \dots, a_N^{1/p}\frac{\partial u}{\partial x_N}\right)$$

for all $u \in W_0^{1,p}(\Omega)$.

Due to Theorem 3.1, there exists a unique $v \in W_0^{1,p}(\Omega)$ such that

$$[u, v]_a = \langle u, f\rangle$$

for all $u \in W_0^{1,p}(\Omega)$. This is equivalent that there exists a sequence $(v_n)_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that $v_n \to v$ as $n \to \infty$ and

$$\lim_{n\to\infty} \|v_n\|_a^{2-p}\left(\int_\Omega a_0(x)u|v_n|^{p-2}v_n dx + \sum_{i=1}^N \int_\Omega a_i(x)\frac{\partial u}{\partial x_i}\left|\frac{\partial v_n}{\partial x_i}\right|^{p-2}\frac{\partial v_n}{\partial x_i}\,dx\right) = \langle u, f\rangle$$

for all $u \in C_0^\infty(\Omega)$. By continuity of the operator $\tilde{T}$ and continuity of the semi-inner product in $L^p(\Omega, \mathbb{R}^{N+1})$ (cf. Lemma 2.2), it follows

$$\begin{aligned}\langle u, f\rangle &= \lim_{n\to\infty}[Tu, Tv_n]_{L^p(\Omega,\mathbb{R}^{N+1})} = [Tu, \tilde{T}v]_{L^p(\Omega,\mathbb{R}^{N+1})} \\ &= \|v\|_a^{2-p}\left(\int_\Omega a_0(x)u|v|^{p-2}v dx + \sum_{i=1}^N \int_\Omega a_i(x)\frac{\partial u}{\partial x_i}\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\,dx\right)\end{aligned}$$

for $u \in C_0^\infty(\Omega)$. ■

Due to Theorem 3.2 it follows that for every $\tilde{f}$ there exist $v_{\tilde{f}}$ such that

$$(3.8)\quad \left\langle u, \tilde{f}\|v_{\tilde{f}}\|_a^{p-2}\right\rangle = \int_\Omega a_0(x)u|v_{\tilde{f}}|^{p-2}v_{\tilde{f}}dx + \sum_{i=1}^N \int_\Omega a_i(x)\frac{\partial u}{\partial x_i}\left|\frac{\partial v_{\tilde{f}}}{\partial x_i}\right|^{p-2}\frac{\partial v_{\tilde{f}}}{\partial x_i}\,dx.$$

Let $\tilde{f} = f\|v_f\|_a^{\frac{2-p}{p-1}}$. Then by the equality (2.9) the left-hand side of expression (3.8) can be suitably modified, i.e.,

$$\left\langle u, \tilde{f}\|v_{\tilde{f}}\|_a^{p-2} \right\rangle = \left\langle u, f\|v_f\|_a^{\frac{2-p}{p-1}}\|v_{\tilde{f}}\|_a^{p-2} \right\rangle$$
$$= \left\langle u, f\|v_f\|^{\frac{2-p}{p-1}}\|v_f\|^{\frac{(2-p)(p-2)}{p-1}}\|v_f\|_a^{p-2} \right\rangle = \langle u, f \rangle .$$

Therefore,

$$\langle u, f\rangle = \int_\Omega a_0(x)u|v_{\tilde{f}}|^{p-2}v_{\tilde{f}}dx + \sum_{i=1}^N \int_\Omega a_i(x)\frac{\partial u}{\partial x_i}\left|\frac{\partial v_{\tilde{f}}}{\partial x_i}\right|^{p-2}\frac{\partial v_{\tilde{f}}}{\partial x_i}\,dx.$$

We conclude this section with a useful corollary.

Corollary 3.3. *Suppose that the conditions* (3.3) *and* (3.4) *are fulfilled. Suppose that the weights* $a_i \in L_\infty(\Omega)$ $(i = 0, \dots, N)$*. Then to every* $f \in W_0^{-1,p}(\Omega)$ *there exists a unique weak solution* $v \in W_0^{1,p}(\Omega)$ *such that*

$$\int_\Omega a_0(x)u|v|^{p-2}vdx + \sum_{i=1}^N \int_\Omega a_i(x)\frac{\partial u}{\partial x_i}\left|\frac{\partial v}{\partial x_i}\right|^{p-2}\frac{\partial v}{\partial x_i}\,dx = \langle u, f\rangle$$

for all $u \in C_0^\infty(\Omega)$.

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