Discussiones Mathematicae
Differential Inclusions, Control and Optimization 36 (2016) 131–140
doi:10.7151/dmdico.1187

EXISTENCE OF SOLUTIONS FOR A SECOND ORDER PROBLEM ON THE HALF-LINE VIA EKELAND'S VARIATIONAL PRINCIPLE

D. Bouafia, T. Moussaoui

Laboratory of Fixed Point Theory and Applications
Department of Mathematics
E.N.S. Kouba, Algiers, Algeria

e-mail: alidahmane10@yahoo.fr
moussaoui@ens-kouba.dz

and

D. O'Regan

School of Mathematics, Statistics and Applied Mathematics
National University of Ireland
Galway, Ireland

e-mail: donal.oregan@nuigalway.ie

Abstract

In this paper we study the existence of nontrivial solutions for a nonlinear boundary value problem posed on the half-line. Our approach is based on Ekeland's variational principle.

Keywords: Ekeland's variational principle, critical point.

2010 Mathematics Subject Classification: 35A15, 35B38.

1. Introduction

We consider the problem,

$$(1) \qquad \begin{cases} -u''(x) + u(x) = \lambda q(x) f(x, u(x)), & x \in [0, +\infty), \\ u(0) = u(+\infty) = 0, \end{cases}$$

where $f : [0, +\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and λ is a positive parameter.

Because of the importance of second order differential equations in physics, existence and multiplicity of solutions to boundary value problems on the half-line were studied by many authors. These results were obtained using upper and lower solution techniques, fixed point theory and topological degree theory; see for example, [6, 7, 8] and [11]. There are only a few papers on boundary value problems on the half-line using variational methods; see [3] and [4].

We assume the following are satisfied:

(H_0) there exist constants $a, b \in \mathbb{R}^+ \setminus \{0\}$ and $\theta \in (0,1)$ such that

$$|f(x,u)| \le a|u|^\theta + b, \qquad \forall x \in \mathbb{R}^+, \forall u \in \mathbb{R},$$

(H_1) $p : [0,+\infty) \longrightarrow (0,+\infty)$ is continuously differentiable and bounded, $q : [0,+\infty) \longrightarrow \mathbb{R}^+$, with $q \in L^1[0,+\infty) \cap L^\infty[0,+\infty)$, $\frac{q}{p^\theta}, \frac{q}{p^2} \in L^1[0,+\infty)$, $M_0 = \int_0^{+\infty} q(x) \left(\int_x^{+\infty} \frac{ds}{p(s)}\right) dx < +\infty$, and $M = \max(\|p\|_{L^2}, \|p'\|_{L^2}) < +\infty$.

(H_2) $f(x,0) = 0$, $\lim_{u\to 0} \frac{f(x,u)}{u} = +\infty$ and $\lim_{u\to\infty} \frac{f(x,u)}{u} = 0$, uniformly for $x \in [0,+\infty)$.

Let the space $H_0^1(0,+\infty)$ be defined by

$$H_0^1(0,+\infty) = \Big\{u \text{ measurable}: \ u, \ u' \in \ L^2(0,+\infty), \ u(0) = u(+\infty) = 0\Big\}$$

endowed with its natural norm

$$\|u\| = \left(\int_0^{+\infty} u^2(x)dx + \int_0^{+\infty} u'^2(x)dx\right)^{\frac{1}{2}},$$

associated with the scalar product

$$(u,v) = \int_0^{+\infty} u(x)v(x)dx + \int_0^{+\infty} u'(x)v'(x)dx.$$

Note that if $u \in H_0^1(0,+\infty)$, then $u(0) = u(+\infty) = 0$, (see [2], Corollary 8.9). Let

$$C_{l,p}[0,+\infty) = \Big\{u \in C([0,+\infty),\mathbb{R}) : \lim_{x\to+\infty} p(x)u(x) \text{ exists } \Big\}$$

endowed with the norm

$$\|u\|_{\infty,p} = \sup_{x\in[0,+\infty)} p(x)|u(x)|.$$

Consider the space

$$L_q^2(0,+\infty) = \{u : (0,+\infty) \to \mathbb{R} \text{ measurable such that } \sqrt{q}u \in L^2(0,+\infty)\},$$

equipped with the norm

$$\|u\|_{L^2_q} = \left(\int_0^{+\infty} q(x)u^2(x)dx \right)^{\frac{1}{2}}.$$

We need the following lemmas.

Lemma 1.1 [10]**.** *$H^1_0(0,+\infty)$ embeds continuously in $C_{l,p}[0,+\infty)$.*

Lemma 1.2 [10]**.** *The embedding $H^1_0(0,+\infty) \hookrightarrow C_{l,p}[0,+\infty)$ is compact.*

Lemma 1.3. *$C_{l,p}[0,+\infty)$ is continuously embedded in $L^2_q(0,+\infty)$.*

Proof. For all $u \in C_{l,p}[0,+\infty)$ we have

$$\|u\|^2_{L^2_q} = \int_0^{+\infty} q(x)u^2(x)dx = \int_0^{+\infty} \frac{q(x)}{p^2(x)} p^2(x)u^2(x)dx \le c\|u\|^2_{\infty,p},$$

where $c = \|\frac{q}{p^2}\|_{L^1}$. Then $\|u\|_{L^2_q} \le \sqrt{c}\|u\|_{\infty,p}$. ∎

Corollary 1.1. *$H^1_0(0,+\infty)$ is compactly embedded in $L^2_q(0,+\infty)$.*

We consider the first eigenvalue λ_1 of the linear problem:

$$\text{(2)} \qquad \begin{cases} -u''(x) + u(x) = \lambda q(x)u(x), & x \ge 0; \\ u(0) = u(+\infty) = 0, \end{cases}$$

namely

$$\lambda_1 = \inf_{u \in H^1_0 \setminus \{0\}} \frac{\|u\|^2}{\|u\|^2_{L^2_q}}.$$

Lemma 1.4. *λ_1 is positive and is achieved for some positive function $\varphi_1 \in H^1_0(0,+\infty) \setminus \{0\}$.*

Proof. We proceed as in [1]. For $u \in H^1_0(0,+\infty)$, let $I_1(u) = \|u\|^2$, $I_2(u) = \|u\|^2_{L^2_q}$, and define the quotient functional $Q : H^1_0(0,+\infty) \setminus \{0\} \to \mathbb{R}$ by

$$Q(u) = \frac{I_1(u)}{I_2(u)}.$$

Then

$$\lambda_1 = \inf_{u \in H^1_0 \setminus \{0\}} Q(u).$$

Let $u \in H_0^1(0,+\infty)$. From Corollary 1.1, we have that $\lambda_1 \geq \frac{1}{\|p\|_{L^\infty} M_0} > 0$. Indeed, for $x > 0$, note

$$\begin{aligned} |u(x)|^2 &= \left| \int_x^{+\infty} u'(s)ds \right|^2 = \left| \int_x^{+\infty} \sqrt{p(s)}u'(s)\frac{1}{\sqrt{p(s)}}ds \right|^2 \\ &\leq \left(\int_x^{+\infty} p(s)u'^2(s)ds \right) \left(\int_x^{+\infty} \frac{ds}{p(s)} \right) \\ &\leq \left(\int_0^{+\infty} p(s)u'^2(s)ds \right) \left(\int_x^{+\infty} \frac{ds}{p(s)} \right), \end{aligned}$$

and so,

$$q(x)u(x)^2 \leq \left(\int_0^{+\infty} p(s)u'^2(s)ds \right) \left(q(x) \int_x^{+\infty} \frac{ds}{p(s)} \right),$$

which yields

$$\|u\|_{L_q^2}^2 \leq \|p\|_{L^\infty} M_0 \|u\|^2,$$

and

$$\lambda_1 = \inf_{u \in H_0^1 \setminus \{0\}} \frac{\|u\|^2}{\|u\|_{L_q^2}^2} \geq \frac{1}{\|p\|_{L^\infty} M_0} > 0.$$

Let (u_n) be a minimizing sequence. Since $(|u_n|)$ is a minimizing sequence for Q, we may suppose that $u_n(x) \geq 0$, for $x \in [0,+\infty)$. Moreover the functional Q satisfies $Q(\alpha u) = Q(u)$, for every $\alpha \in \mathbb{R}$. By setting $\widetilde{u_n} = \frac{u_n}{\|u\|_{L_q^2}}$, for every n, we can assume that $\|u_n\|_{L_q^2} = 1$. Note $\lim_{n \to +\infty} Q(u_n) = \inf_{u \in H_0^1 \setminus \{0\}} Q(u) = \lambda_1$, so the sequence $(Q(u_n))$ is bounded. From this and since $Q(u_n) = \|u_n\|^2$, we deduce that (u_n) is bounded in H_0^1. ¿From Lemma 1.2 and the reflexivity and separability of H_0^1, there exists a subsequence (u_{n_k}) of (u_n) such that, as $k \to +\infty$,

$$\begin{cases} u_{n_k} \rightharpoonup \overline{u}, & \text{in } H_0^1; \\ u_{n_k} \to \overline{u}, & \text{in } C_{l,p}, \end{cases}$$

so $u_{n_k}(x) \to \overline{u}(x)$, for all $x \in [0,+\infty)$. ¿From Lemma 1.3, (u_{n_k}) converges in norm to $\overline{u}$ in L_q^2. Thus $\|\overline{u}\|_{L_q^2} = 1$ and $\overline{u}(x) \geq 0$, for $x \in [0,+\infty)$. Finally, the weak lower semi-continuity of the norm guarantees that

$$Q(\overline{u}) = I_1(\overline{u}) \leq \liminf_k I_1(u_{n_k}) = \liminf_k Q(u_{n_k}) = \lambda_1,$$

so $\overline{u} \in H_0^1 \setminus \{0\}$ and $Q(\overline{u}) = \lambda_1$. ∎

To prove our main result, we need the following variational principle.

Theorem 1.1 ([9]). (Weak Ekeland variational principle) *Let (E, d) be a complete metric space and let $J : E \to \mathbb{R}$ a functional that is lower semi-continuous, bounded from below. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in E$ with*

$$J(u_\varepsilon) \leq \inf_E J + \varepsilon,$$

and whenever $w \in E$ with $w \neq u_\varepsilon$, then

$$J(u_\varepsilon) < J(w) + \varepsilon d(u_\varepsilon, w).$$

2. MAIN RESULT

We denote by F the primitive of f with respect to its second variable, i.e., $F(x, u) = \int_0^u f(x, s)ds$. The functional corresponding to (1) is

$$J(u) = \frac{1}{2}\int_0^{+\infty} \Big(u'^2(x)+u^2(x)\Big)dx - \lambda\int_0^{+\infty} q(x)F(x, u(x))dx, \quad u \in H_0^1(0, +\infty).$$

Proposition 2.1. *Suppose that condition (H_0) holds. Then the functional J is continuously differentiable. The Fréchet derivative of J has the form*

$$\langle J'(u), \varphi\rangle = \int_0^{+\infty} \Big(u'(x)\varphi'(x) + u(x)\varphi(x)\Big)dx - \lambda\int_0^{+\infty} q(x)f(x, u(x))\varphi(x)dx.$$

Proof. First we show J is Gâteaux-differentiable. Indeed, for all $v \in H_0^1(0, +\infty)$, and for any $t > 0$, we have

$$\begin{aligned}
J(u + tv) - J(u) &= \frac{1}{2}\int_0^{+\infty}(|(u + tv)'|^2 + |u + tv|^2)dx - \lambda\int_0^{+\infty} q(x)F(x, u + tv)dx \\
&- \frac{1}{2}\int_0^{+\infty}(|u'|^2 + |u|^2)dx + \lambda\int_0^{+\infty} q(x)F(x, u)dx \\
&= \frac{t^2}{2}\int_0^{+\infty}|v'|^2dx + \frac{t^2}{2}\int_0^{+\infty}|v|^2dx + t\int_0^{+\infty}u'v'dx \\
&+ t\int_0^{+\infty}uvdx - \lambda\int_0^{+\infty}q(x)\Big[F(x, u + tv) - F(x, u)\Big]dx \\
&= \frac{t^2}{2}\int_0^{+\infty}|v'|^2dx + \frac{t^2}{2}\int_0^{+\infty}|v|^2dx + t\int_0^{+\infty}u'v'dx \\
&+ t\int_0^{+\infty}uvdx - t\lambda\int_0^{+\infty}q(x)f(x, u + t\theta v)vdx,
\end{aligned}$$

where $0 < \theta < 1$ (from the mean value theorem). Then

$$\begin{aligned}\frac{J(u+tv)-J(u)}{t} &= \frac{1}{2}\int_0^{+\infty}|v'|^2dx+\frac{1}{2}\int_0^{+\infty}|v|^2dx+\int_0^{+\infty}u'v'dx\\ &+\int_0^{+\infty}uvdx-\lambda\int_0^{+\infty}q(x)f(x,u+t\theta v)vdx.\end{aligned}$$

Let $t \to 0$. Note assumption (H_0) and the Lebesgue dominated convergence theorem guarantees that

$$\langle J'(u),v\rangle=\int_0^{+\infty}\Big(u'v'+uv\Big)dx-\lambda\int_0^{+\infty}q(x)f(x,u)vdx,\quad \forall v\in H_0^1(0,+\infty).$$

Next we show J' is continuous. Indeed, let $(u_n)\subset H_0^1(0,+\infty)$, where $u_n\to u$, when $n\to+\infty$. It follows from (H_0), that

$$\begin{aligned}q(x)|f(x,u_n(x))| &\leq aq(x)|u(x)|^\theta+bq(x)\\ &\leq a\sup_{x\in[0,+\infty)}|(pu)(x)|^\theta\Big|\frac{q(x)}{p^\theta(x)}\Big|+bq(x)\\ &= a\|u\|_{\infty,p}^\theta\Big|\frac{q(x)}{p^\theta(x)}\Big|+bq(x)\in L^1(0,+\infty).\end{aligned}$$

Then from the Lebesgue dominated convergence theorem we obtain

$$\lim_{n\to+\infty}\int_0^{+\infty}q(x)f(x,u_n(x))dx=\int_0^{+\infty}q(x)f(x,u(x))dx,$$

so, we have

$$\begin{aligned}\langle J'(u_n)-J'(u),v\rangle &= \int_0^{+\infty}\Big(u_n'v'+u_nv\Big)dx-\lambda\int_0^{+\infty}q(x)f(x,u_n)vdx\\ &-\int_0^{+\infty}\Big(u'v'+uv\Big)dx+\lambda\int_0^{+\infty}q(x)f(x,u)vdx\\ &=\int_0^{+\infty}\Big[(u_n'-u')v'+(u_n-u)v\Big]dx\\ &-\lambda\int_0^{+\infty}q(x)(f(x,u_n)-f(x,u))vdx.\end{aligned}$$

Passing to the limit in $\langle J'(u_n)-J'(u),v\rangle$ when $n\to+\infty$, using assumption (H_0) and the Lebesgue dominated convergence theorem, we obtain that $J'(u_n)\to J'(u)$, as $n\to+\infty$. ■

Definition 2.1. We say that $u \in H_0^1(0,+\infty)$ is a weak solution of problem (1) if for any $\varphi \in H_0^1(0,+\infty)$ we have

$$\langle J'(u),\varphi\rangle = \int_0^{+\infty}\Big(u'(x)\varphi'(x)+u(x)\varphi(x)\Big)dx-\lambda\int_0^{+\infty} q(x)f(x,u(x))\varphi(x)dx = 0.$$

Remark 1. Since the nonlinear term f is continuous, then a weak solution of problem (1) is a classical solution.

Theorem 2.1. *Suppose* (H_0), (H_1) *and* (H_2) *hold. Then problem* (1) *possesses at least one solution* u_λ *for every* $\lambda \in (0, \frac{1}{\|p\|_{L^\infty} M_0})$.

Proof. It follows from (H_2) that $\exists \delta_1 > 0$ such that

$$|F(x,u)| \le \frac{1}{2}u^2, \quad \text{for all } |u| > \delta_1;$$

and from (H_0) that $\exists M_1 > 0$ such that

$$|F(x,u)| \le M_1, \text{ for all } u \in [-\delta_1, \delta_1] \text{ and } x \in (0,+\infty).$$

Therefore, we deduce that

$$|F(x,u)| \le M_1 + \frac{1}{2}u^2, \quad \text{for all } u \in \mathbb{R} \text{ and } x \in [0,+\infty). \tag{3}$$

Now (3) together with (H_1) (also note the continuous embedding of $H_0^1(0,+\infty)$ in $L_q^2(0,+\infty)$ (i.e., note $\|u\|^2_{L^2_q} \le \|p\|_{L^\infty} M_0 \|u\|^2$ for $u \in H_0^1(0,+\infty)$, see Lemma 1.4)) yields

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \lambda\int_0^{+\infty} q(x)F(x,u(x))dx \\
&\ge \frac{1}{2}\|u\|^2 - \lambda\int_0^{+\infty} q(x)\Big(M_1 + \frac{1}{2}u^2(x)\Big)dx \\
&= \frac{1}{2}\|u\|^2 - \lambda M_1\int_0^{+\infty} q(x)dx - \frac{\lambda}{2}\|u\|^2_{L^2_q} \\
&\ge \frac{1}{2}(1-\lambda\|p\|_{L^\infty}M_0)\|u\|^2 - \lambda M_1\|q\|_{L^1}.
\end{aligned}$$

Thus there exist $\rho > \left(\frac{2\lambda M_1\|q\|_{L^1}}{1-\lambda\|p\|_{L^\infty}M_0}\right)^{\frac{1}{2}} > 0$ with

$$J(u) > 0 \quad \text{if } \|u\| = \rho, \quad \text{and then } \inf_{u\in\partial \overline{B_\rho(0)}} J(u) > 0,$$

and $J(u) \geq -C_2$ if $\|u\| \leq \rho$, where $C_2 = \lambda M_1 \|q\|_{L^1}$. Then the functional J is bounded from below on $\overline{B_\rho(0)}$. Let $\varphi_1 \in H_0^1(0,+\infty)$ be defined as in Lemma 1.4. Fix λ in $(0, \frac{1}{\|p\|_{L^\infty} M_0})$, and let $M_2 = \frac{\lambda_1}{\lambda}$. From (H_2), there exists $\delta_2 > 0$ such that

$$F(x,u) \geq M_2 |u|^2, \qquad \text{for all } \; -\delta_2 < u < \delta_2. \tag{4}$$

The function φ_1 is continuous on $[0,+\infty)$ (note $\varphi_1 \in H_0^1(0,+\infty)$) and $\varphi_1(0) = \varphi_1(+\infty) = 0$ so $\sup_{x\in[0,+\infty)} \varphi_1(x) \leq c^\star$ for some $c^\star > 0$. Hence for every $0 < t < \frac{\delta_2}{c^\star}$, and (4), we have

$$\begin{aligned} J(t\varphi_1) &= \frac{t^2}{2}\|\varphi_1\|^2 - \lambda \int_0^{+\infty} q(x) F(x, t\varphi_1(x))dx \\ &\leq \frac{t^2}{2}\|\varphi_1\|^2 - \lambda M_2 t^2 \int_0^{+\infty} q(x)\varphi_1^2(x)dx \\ &= \frac{t^2}{2}\|\varphi_1\|^2 - \frac{\lambda M_2 t^2}{\lambda_1}\|\varphi_1\|^2 = \frac{t^2}{2}\|\varphi_1\|^2 - t^2\|\varphi_1\|^2 = -\frac{t^2}{2}\|\varphi_1\|^2 < 0. \end{aligned}$$

Thus, when $t \to 0$, we have $J(t\varphi_1) < 0$. Then we deduce that

$$\inf_{u \in \overline{B_\rho(0)}} J(u) < 0 < \inf_{u \in \partial \overline{B_\rho(0)}} J(u). \tag{5}$$

By applying Ekeland's variational principle (Theorem 1.1) in the complete metric space $\overline{B_\rho(0)}$, there is a sequence $(u_n) \subset \overline{B_\rho(0)}$ such that

$$J(u_n) \leq \inf_{u \in \overline{B_\rho(0)}} J(u) + \frac{1}{n}, \qquad J(u_n) \leq J(w) + \frac{1}{n}\|w - u_n\|, \quad \forall w \in \overline{B_\rho(0)}.$$

From (5), $u_n \notin \partial \overline{B_\rho(0)}$. Thus, $\forall n \in \mathbb{N}, u_n \in B_\rho(0)$ and if we put, $w = u_n + th$, for all $t > 0$, $\; h \in H_0^1(0,+\infty)$, and $n \in \mathbb{N}$, then $w = u_n + th$ belongs to the open ball $B_\rho(0)$ when $t \to 0$, and then $J(u_n) \leq J(u_n + th) + \frac{1}{n} t \|h\|$, so

$$\frac{J(u_n) - J(u_n + th)}{t} \leq \frac{1}{n}\|h\|,$$

and we have

$$-\langle J'(u_n), h\rangle \leq \frac{1}{n}\|h\|, \;\text{ for all } n \in \mathbb{N}^*.$$

If we put $w = u_n - th$, then we obtain $\langle J'(u_n), h\rangle \leq \frac{1}{n}\|h\|, \;\; \forall n \in \mathbb{N}^*$. Thus

$$\sup_{\|h\|\leq 1} |\langle J'(u_n), h\rangle| \leq \frac{1}{n}, \;\text{ for all } n \in \mathbb{N}^*.$$

Therefore, we have

$$\|J'(u_n)\| \to 0, \text{and } \ J(u_n) \to c_\lambda \quad \text{as } n \to +\infty,$$

where c_λ stands for the infimum of $J(u)$ on $\overline{B_\rho(0)}$. Since (u_n) is bounded and $\overline{B_\rho(0)}$ is a closed convex set, there exists a subsequence still denoted by (u_n), and there exists $u_\lambda \in \overline{B_\rho(0)} \subset H_0^1(0,+\infty)$ such that

$$\begin{cases} u_n \to u_\lambda \text{ weakly in } H_0^1(0,+\infty); \\ u_n(x) \to u_\lambda(x) \text{ for } x \text{ in } (0,+\infty); \\ u_n \to u_\lambda \text{ strongly in } C_{l,p}[0,+\infty). \end{cases}$$

Consequently, passing to the limit in $\langle J'(u_n), \varphi\rangle$, as $n \to +\infty$, we have using the Lebesgue dominated convergence theorem that

$$\int_0^{+\infty} \Big(u_\lambda'(x)\varphi'(x) + u_\lambda(x)\varphi(x)\Big)dx - \lambda \int_0^{+\infty} q(x) f(x, u_\lambda(x))\varphi(x)dx = 0,$$

for all $\varphi \in H_0^1(0,+\infty)$. That is, $\langle J'(u_\lambda), \varphi\rangle = 0$ for all $\varphi \in H_0^1(0,+\infty)$. Thus u_λ is a critical point of the functional J, which is a classical solution of our problem. ■

3. Example

Let $f(x,u) = u^{\frac{1}{5}}$, $q(x) = e^{-kx}$, $p(x) = e^{-\frac{1}{3}kx}$, where $k > 0$ is a constant. Then we get

$$\forall x \in \mathbb{R}^+, \forall u \in \mathbb{R} : |f(x,u)| = \left|u^{\frac{1}{5}}\right| \leq |u|^\theta + 1, \quad \text{where} \quad \theta \in \left(\frac{1}{2}, 1\right).$$

Also

$$\left(\frac{q}{p^{\frac{1}{2}}}\right)(x) = e^{-\frac{5}{6}kx}, \ \left(\frac{q}{p^2}\right)(x) = e^{-\frac{1}{3}kx} \in L^1, \text{ and } q \in L^1[0,+\infty) \cap L^\infty[0,+\infty).$$

Note conditions (H_0), (H_1) and (H_2) hold. Theorem 2.1 can now be applied.

References

[1] M. Badiale and E. Serra, Semilinear Elliptic Equations for Beginners (Universitext, Springer, London, 2011) x+199 pp.

[2] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations (Springer, 2010). doi:10.1007/978-0-387-70914-7

[3] H. Chen, Z. He and J. Li, *Multiplicity of solutions for impulsive differential equation on the half-line via variational methods*, Bound. Value Probl. **14** (2016). doi:10.1186/s13661-016-0524-8

[4] B. Dai and D. Zhang, *The Existence and multiplicity of solutions for second-order impulsive differential equations on the half-line*, Results. Math. **63** (2013) 135–149. doi:10.1007/s00025-011-0178-x

[5] S. Djebali and T. Moussaoui, *A class of second order BVPs on infinite intervals*, Electron. J. Qual. Theory Differ. Equ. (4) (2006) 1–19. doi:10.14232/ejqtde.2006.1.4

[6] S. Djebali, O. Saifi and S. Zahar, *Singular boundary value problems with variable coefficients on the positive half-line*, Electron. J. Differential Equations **2013** (73) (2013) 1–18.

[7] S. Djebali, O. Saifi and S. Zahar, *Upper and lower solutions for BVPs on the half-line with variable coefficient and derivative depending nonlinearity*, Electron. J. Qual. Theory Differ. Equ. (14) (2011) 1–18. doi:10.14232/ejqtde.2011.1.14

[8] S. Djebali and S. Zahar, *Bounded solutios for a derivative dependent boundary value problem on the half-line*, Dynam. Systems Appl. **19** (2010) 545–556.

[9] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974) 324–353. doi:10.1016/0022-247X(74)90025-0

[10] O. Frites, T. Moussaoui and D. O'Regan, *Existence of solutions via variational methods for a problem with nonlinear boundary conditions on the half-line*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **22** (2015) 395–407.

[11] H. Lian and W. Ge, *Solvability for second-order three-point boundary value problems on a half-line*, Appl. Math. Lett. **19** (2006) 1000–1006. doi:10.1016/j.aml.2005.10.018

[12] D. O'Regan, B. Yan and R.P. Agarwal, *Nonlinear boundary value problems on semi-infinite intervals using weighted spaces: An upper and lower solution approach*, Positivity **11** (2007) 171–189. doi:10.1007/s11117-006-0050-5

[13] N.S. Papageorgiou and S.K. Yiallourou, Handbook of Applied Analysis (Springer, New-York, 2009).

Received 5 April 2016