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SOME ALGEBRAIC FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS WITH APPLICATIONS

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Abstract

In this paper, some algebraic fixed point theorems for multi-valued discontinuous operators on ordered spaces are proved. These theorems improve the earlier fixed point theorems of Dhage (1988, 1991) Dhage and Regan (2002) and Heikkilä and Hu (1993) under weaker conditions. The main fixed point theorems are applied to the first order discontinuous differential inclusions for proving the existence of the solutions under certain monotonicity condition of multi-functions.

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1. Introduction

It is known that many of dynamical systems of the universe are nonlinear in nature and involve jumps or discontinuities in their behavior and so, such processes are governed by differential equations involving jumps or discontinuities in the state as well as in phase variable. There are mainly two approaches for dealing with such phenomena modeled on discontinuous differential equations. Firstly, we study such differential problems via algebraic fixed point theorems such as those of Tarski [30], Amann [2] and Heikkilä and Lakshmikantham [25]. Secondly, we study the differential inclusions corresponding to the given discontinuous differential equations by applying various topological fixed point theorems of multi-valued operators for proving the existence results for a given discontinuous differential equation (see Aubin and Cellina [4] and the reference therein). A most up-to-date and comprehensive discussion on the topic appears in the monograph of Aubin and Cellina [4], Deimling [7] and Hu and Papageorgiou [26]. We note that in this second approach one needs a certain kind of continuity of the multi-valued functions involved in differential inclusions.

Attempts have been made to remove the continuity conditions of the multi-functions of differential inclusions. Therefore, the fixed point theory for discontinuous multi-valued operators is being developed for the purpose and the author in [9] has proved a first fixed point theorem in this area of research. Later on a few interesting fixed point theorems dealing with discontinuous multi-valued operators have appeared in the literature. See, for example, Dhage and Regan [18], Agarwal *et al.* [1] and Dhage [13] etc. The monotonicity condition used in the above references is very strong and therefore it is desirable to replace it with a mild one following Heikkilä and Hu [24]. In this article, we present some algebraic fixed point theorems for multi-valued operators on ordered spaces and discuss some of their applications to operator inclusions involving two multi-valued operators as well as to first order discontinuous differential inclusions for proving the existence theorems under generalized monotonicity conditions.

2. Preliminaries

As our approach is more applied than merely theoretical, we keep refrain from an abstract treatment of the topic, however, the results presented here can be the abstract setting in a natural way.

In what follows, let X denote an ordered metric space with a metric d and an order relation $\leq$. Then X becomes an ordered topological space, where the topology on X is induced by the metric d on it. A sequence $\{x_n\}$ of points of X is called **monotone increasing** if

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots .$$

Similarly, a sequence $\{x_n\}$ of points of X is called **monotone decreasing** if

$$x_1 \geq x_2 \geq \ldots \geq x_n \geq \ldots .$$

Finally, a sequence $\{x_n\}$ is called **monotone** if it is either monotone increasing or monotone decreasing on X. Again the sets $[a)$ and $(b]$ are defined by

$$[a) = \{x \in X \mid a \leq x\} \quad \text{and} \quad (b] = \{x \in X \mid x \leq b\}.$$

Obviously, $[a)$ and $(b]$ are closed in X. An order interval $[a, b]$ in an ordered metric space X is a set defined by $[a, b] = [a) \bigcap (b]$ (see Heikkila and Lakshmikantham [25]).

The following result is crucial for the rest of the paper.

Lemma 2.1 (Heikkilä and Hu [24])**.** *If a monotone increasing* (*resp. monotone decreasing*) *sequence* $\{x_n\}$ *of points in* X *has a cluster point, then it is the* $\sup\limits_n x_n$ (*resp.* $\inf\limits_n x_n$).

Let $\mathcal{P}(X)$ and $\mathcal{P}_p(X)$ denote respectively the class of all subsets and the class of all non-empty subsets of X with the property p. Thus $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{bd}(X)$ and $\mathcal{P}_{cp}(X)$ denote respectively the classed of all closed, bounded and compact subsets of X. A mapping $Q : X \to \mathcal{P}_p(X)$ is called a multi-valued mapping or a multi-valued operator on X and a point $u \in X$ is called a **fixed point** of Q if $u \in Qu$.

We consider the following notations in the sequel.

Let

$$L = \{x \in X \mid y \leq x \ \text{ for some } \ y \in Qx\},$$

$$L^- = \{x \in X \mid y < x \ \text{ for some } \ y \in Qx\},$$

$$M = \{x \in X \mid x \leq y \ \text{ for some } \ y \in Qx\}$$

and

$$M^+ = \{x \in X \mid x < y \ \text{ for some } \ y \in Qx\}.$$

If $M \neq \emptyset$, then for each $x \in M$ we have a choice function $f : M \to X$ such that $x \leq f(x) = y \in Qx$. Similarly, if $L \neq \emptyset$, then there is choice function $g : L \to X$ satisfying $x \geq g(x) = y \in Qx$ for $x \in L$.

Let $a \in M$. Then a well ordered (w.o.) chain C of generalized Q-iterations of a in X is defined as follows.

Let

$$a = \min C \text{ and } \quad a < x \in C \quad \text{if and only if} \quad x = \sup f\big(C^{<x}\big), \tag{2.1}$$

where $C^{<x} = \{y \in C \mid y < x\}$.

It is clear that a well ordered chain C of Q-iterations of a contains elements of the following form. The element a is the least element of C. The next possible elements of C are of the form $x_n = \sup f(\{x_0, \dots, x_{n-1}\})$ as long as x_n is defined and $x_{n-1} < x_n$ for $n \in \mathcal{N}$. If $x_\omega = \sup f(\{x_n\}_0^\infty)$ exists and is a strict upper bound of $\{x_n\}_0^\infty$, then x_ω is the next element of C, and so on.

Similarly, an inversely well ordered chain (i.w.o.) C of the function g of the point $b \in L$ is defined as

$$b = \max C \text{ and } \quad b > x \in C \quad \text{if and only if} \quad x = \inf g\big(C^{>x}\big), \tag{2.2}$$

where $C^{>x} = \{y \in C \mid y > x\}$.

It can be shown as in Heikkilä and Lakshmikantham [25] that ordinary Q-iterations are inadequate to describe the well ordered chains of generalized Q-iterations, however transfinite sequences are useful to generate the well ordered chains of generalized Q-iterations of some point $a \in X$.

The following result is useful in the sequel.

Lemma 2.2. *Let X be an ordered metric space and let $f : X \to X$ be a mapping such that $x \leq fx$ for each $x \in X$. Suppose that C is a well ordered chain in X at some point $a \in X$ defined by* (2.1). *Then $f(C)$ is again a well ordered chain in X with $f(C) \subset C$. Further if* $\sup f(C)$ *exists, then* $\sup f(C) = \sup C$.

Proof. The assertions that $f(C)$ is again a well ordered chain in X and $f(C) \subset C$ follow respectively from Theorem 1.1.1 of Heikkilä and Lakshmikantham [25] and Corollary 12 of Heikkilä [23]. We only prove that $\sup f(C) = \sup C$. Since $f(C) \subset C$, one has $\sup f(C) \leq \sup C$. Again by nature of f, we get $\sup C = x^* \leq fx^* \leq \sup f(C)$. Hence $\sup f(C) = \sup C$. ∎

Similarly, we have

Lemma 2.3. *Let X be an ordered metric space and let $g : X \to X$ be a mapping such that $x \geq gx$ for each $x \in X$. Suppose that C is an inversely well ordered chain in X at some point $b \in X$ defined by* (2.2). *Then $g(C)$ is again an inversely well ordered chain in X with $g(C) \subset C$. Further if* $\inf g(C)$ *exists, then* $\inf g(C) = \inf C$.

The following generalized lemma which is an improvement of Lemma 2.1 of Heikkilä and Hu [24] forms the basis of our multi-valued fixed point theory in ordered spaces. We give the proof by using the arguments similar to those in Heikkilä and Lakshmikantham [25]. It seems that Lemma 2.1 of Heikkilä and Hu [24] is not correct since the proof of the lemma involves the use of transfinite sequences generated by the choice function f on M^+.

Lemma 2.4. *Let $Q : X \to \mathcal{P}_p(X)$. Assume that*

(a) *$M \neq \emptyset$ and $a \in M$, and*

(b) *the well ordered chain C of generalized Q-iterations of a, if it exists, has a supremum x_* in M.*

Then x_* is a fixed point of Q.

Proof. Let $a \in M$ and let C be a well ordered chain of generalized Q-iterations of a in X. If C contains only one point a, then obviously it is a fixed point of Q. Assume that the well ordered chain C of generalized Q-iterations of a exists and $\sup C = x_*$. Now by Lemma 2.2, $\sup C = \sup f(C) = x_*$, where f is a choice function of Q. Suppose that $x_* \neq fx_*$. Since x_* in M, one has $x_* < fx_* \leq \sup f(C) = x^*$. This is a contradiction and hence $x_* = fx_* \in Qx_*$. Thus x_* is a fixed point of Q. This completes the proof. ∎

Similarly, we have

Lemma 2.5. *Let $Q : X \to \mathcal{P}_p(X)$. Assume that*

(a) *$L \neq \emptyset$ and $b \in L$, and*

(b) *the inversely well ordered chain of generalized Q-iterations of b, if it exists, has an infimum $y^* \in M$.*

Then y^* is a fixed point of Q.

The following example shows that Lemma 2.4 fails to hold if either of the conditions (a) or (b) does not hold.

Example 2.1. Let $X = [0, 1] \cup \{2\}$ and define a multi-valued map $Q : X \to \mathcal{P}_{cp}(X)$ by

$$Qx = \begin{cases} \left[\frac{n+1}{n+2}, \frac{n+2}{n+3}\right], & \frac{n}{n+1} \leq x < \frac{n+1}{n+2},\ n \in \mathbb{N} \cup \{0\}, \\ \{2\} & x = 1, \\ \{1\} & x = 2, \end{cases}$$

and

$$f(x) = \begin{cases} \frac{n+1}{n+2}, & \frac{n}{n+1} \leq x < \frac{n+1}{n+2},\ n \in \mathbb{N}, \\ 2, & x \in \{1,2\}. \end{cases}$$

Choosing $a = 0$ in (2.1), we obtain $C = \{0, \frac{1}{2}, \frac{2}{3}, \dots, 1, 2\}$ so that $\sup C = 2$. Now Q has no fixed point in X, because $2 \notin M$.

In the following section we prove the main results of the paper.

3. Multi-valued fixed point theory

3.1. Ordered metric spaces

First of all, we obtain some basic fixed point theorems in ordered metric spaces and then derive some of their interesting consequences in ordered Banach spaces. The following lemma is useful in the sequel.

Lemma 3.1. *Let C be a well ordered chain of generalized Q-iterations at some point a in an ordered metric space X. If every monotone increasing subsequence of C has a cluster point, then C has a monotone increasing subsequence which converges to the supremum of C.*

Proof. The proof appears in Heikkilä and Lakshmikantham [25]. Also see Heikkila and Hu [24]. We omit the details. ■

Similarly, we have

Lemma 3.2. *Let C be an inversely well ordered chain of generalized Q-iterations at some point b in an ordered metric space X. If every monotone decreasing subsequence of C has a cluster point, then C has a monotone decreasing subsequence which converges to the infimum of C.*

Theorem 3.1. *Let X be an ordered metric space and let $Q : X \to \mathcal{P}_{cp}(X)$. Assume that*

- (Q_0) *the set $M \neq \emptyset$,*
- (Q_1) *$x_1 \leq y_1 \in Qx_1$ implies $y_1 \leq y_2$ for some $y_2 \in Qy_1$,*
- (Q_2) *every monotone increasing sequence $\{y_n\}$ defined by $y_n \in Qx_n$, $n = 0, 2, \dots$; has a cluster point, whenever $\{x_n\}$ is a monotone increasing sequence in X.*

Then Q has a fixed point.

Proof. Let $a \in M$ and let C be a well ordered chain of generalized Q-iterations of a in X. By Lemma 2.2 it suffices to show that $f(C)$ and consequently C has a supremum in M. Now every monotone increasing subsequence of the sequence in $f(C)$ is of the form $\{y_n\}$, where $y_n \in Qx_n$ for $n = 0, 1, \ldots$; and $\{x_n\}$ and $\{y_n\}$ are monotone increasing sequences in X. Therefore by hypothesis (Q_2), every subsequence of the sequence $\{y_n\}$ has a cluster point which further in view of Lemma 2.3 implies that there is a monotone increasing sequence $\{y_n\} \in f(C) \subset C$ converging to the supremum y of $f(C)$ which is also the supremum of C in X. We show that $y \in M$. If a is the only term of C, then $y = a \in M$. Now by the definition of $\{y_n\}$, there is an x_n in C such that

$$x_n \leq y_n \in Qx_n \quad \text{and so,} \quad x_n \leq y \ \forall n \in \mathbb{N}.$$

Next by (Q_1), there exists a $z_n \in Qy$ such that $y_n \leq z_n$. Therefore, $[y_n) \cap Qy \neq \emptyset$ for all $n \in \mathbb{N}$. Again $[y_m) \cap Qy \subset [y_n) \cap Qy$ for all $m \geq n \in \mathbb{N}$. Thus $\{[y_n) \cap Qy \mid n \in \mathbb{N}\}$ is a family of closed non-empty subsets of Qy having the finite intersection property. Since Qy is compact, we have

$$\bigcap_{n=1}^{\infty} \Big\{ [y_n) \cap Qy \Big\} \neq \emptyset.$$

Thus there is a $z \in Qy$ such that $y_n \leq z$ for all $n \in \mathbb{N}$. As y is the least upper bound of $\{y_n\}$ one has $y \leq z$. This shows that $y \in M$. Now the desired conclusion follows from applying Lemma 2.4. This completes the proof. ■

As a special case of Theorem 3.1 we obtain the following fixed point theorem for multi-valued mappings on ordered spaces of Heikkilä and Hu [24].

Corollary 3.2. *Let X be an ordered metric space and let $Q : X \to \mathcal{P}_{cp}(X)$. Assume that*

- (Q_0) *the set M is nonempty,*
- (Q_1) *$x_1 \leq y_1 \in Qx_1$ and $x_1 \leq x_2$ imply that $y_1 \leq y_2$ for some $y_2 \in Qx_2$,*
- (Q_2) *each monotone increasing sequence in $\bigcup Q(X)$ has a cluster point.*

Then Q has a fixed point.

Similarly, we have

Theorem 3.3. *Let X be an ordered metric space and let $Q : X \to \mathcal{P}_{cp}(X)$. Assume that*

(Q_0) *the set L is nonempty,*

(Q_1) *$x_1 \geq y_1 \in Qx_1$ implies $y_1 \geq y_2$ for some $y_2 \in Qy_1$,*

(Q_2) *every monotone decreasing sequence $\{y_n\}$ defined by $y_n \in Qx_n$, $n = 0, 2, \ldots$; has a cluster point, whenever $\{x_n\}$ is a monotone decreasing sequence in X.*

Then Q has a fixed point.

Proof. The proof is similar to Theorem 3.1 with appropriate modifications. Now the conclusion follows from applying Lemma 2.5. ■

Corollary 3.4. *Let X be an ordered metric space and let $Q : X \to \mathcal{P}_{cp}(X)$. Assume that*

(Q_0) *the set L is nonempty,*

(Q_1) *$x_1 \geq y_1 \in Qx_1$ and $x_1 \geq x_2$ imply that $y_1 \geq y_2$ for some $y_2 \in Qx_2$,*

(Q_2) *each monotone decreasing sequence in $\bigcup Q(X)$ has a cluster point.*

Then Q has a fixed point.

Note that the hypothesis (Q_1) of Corollary 3.2 implies the hypothesis (Q_1) of Theorem 3.1, but the following simple example shows that the converse may not be true.

Example 3.5. Define a multi-valued map $Q : [0, 1] \to \mathcal{P}_{cp}([0, 1])$ by

$$Qx = \begin{cases} \frac{1}{2} & \text{if } x \in \left[0, \frac{1}{4}\right) \\ \left[0, \frac{1}{3}\right] & \text{if } x = \frac{1}{4} \\ \left[\frac{2}{5}, \frac{2}{3}\right] & \text{if } x \in \left(\frac{1}{4}, \frac{1}{2}\right) \\ 1 - \frac{1}{n+2} & \text{if } 1 - \frac{1}{n+1} \leq x < 1 - \frac{1}{n+2},\ n \in \mathbb{N} \\ \{\frac{3}{4}, 1\} & \text{if } x = 1. \end{cases}$$

Now for any $x \in [0, \frac{1}{4})$ one has $x < \frac{1}{4}$, but there is not $y \in Q(\frac{1}{4}) = [0, \frac{1}{3}]$ such that $\frac{1}{2} \leq y$. However, the condition (Q_1) of Theorem 3.1 is satisfied with $x_1 = x$ for all $x \in [0, \frac{1}{4})$ and the multi-valued map Q has a fixed point in $[0, 1]$, viz., 1.

3.2. Ordered Banach spaces

Let $\mathbb{R}$ denote a real line and X a real Banach space. A closed subset K of X is called a **cone** if it satisfies

(i) $K + K \subseteq K$
(ii) $\lambda K \subseteq K$ for all $\lambda \in \mathbb{R}^+$ and
(iii) $\{-K\} \cap K = \{\theta\}$, where θ is a zero element of X.

A cone K in X is said to be **normal** if the norm is semi-monotone on X, that is, if $x, y \in X$, and $x \leq y$ implies $\|x\| \leq N\|y\|$ for some constant $N > 0$. A cone K is **regular** if every monotone and order bounded sequence in X is convergent in norm. Again a cone K is said to be **fully regular** if every monotone and norm-bounded sequence in X is convergent in norm. The details of cones and their properties may be found in Guo and Lakshmikanthm [20] and Heikkilä and Lakshmikantham [25]. We define an order relation $\leq$ with the help of the cone K in X as follows. Let $x, y \in X$. Then

$$x \leq y \Leftrightarrow y - x \in K. \tag{3.1}$$

The Banach space X together with the order relation $\leq$ is called an ordered Banach space and it is denoted by $(X, \leq)$. Let $a, b \in (X, \leq)$ be such that $a \leq b$. Then the order interval $[a, b]$ is a set in X to be defined by

$$[a, b] = \{x \in X \mid a \leq x \leq b\}. \tag{3.2}$$

In what follows, we define different types of order relations in $\mathcal{P}_p(X)$. Let $A, B \in \mathcal{P}_p(X)$. Then

$$A \overset{i}{\leq} B \Leftrightarrow \text{ for each } a \in A \ \exists \text{ a } \ b \in B \text{ such that } a \leq b, \tag{3.3}$$

$$A \overset{d}{\leq} B \Leftrightarrow \text{ for each } b' \in B \ \exists \text{ an } a' \in A \text{ such that } a' \leq b', \tag{3.4}$$

$$A \overset{id}{\leq} B \Leftrightarrow A \overset{i}{\leq} B \quad \text{and} \quad A \overset{d}{\leq} B \tag{3.5}$$

and

$$A \leq B \ \Leftrightarrow \ a \leq b \quad \text{for all } a \in A \text{ and } \ b \in B. \tag{3.6}$$

The above order relation (3.5) defined in $\mathcal{P}_p(X)$ has been used in Dhage [12, 13] in the study of extremal solutions for differential and integral equations and it is an improvement upon the order relation defined in (3.6) by Dhage [9], Dhage and Regan [18] and Agarwal *et al.* [1].

Definition 3.1. A multi-valued mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone increasing if $x \leq y$, then $Qx \overset{i}{\leq} Qy$ for all $x, y \in X$. Similarly, Q is called left monotone increasing if $x \leq y$, then $Qx \overset{d}{\leq} Qy$ for all $x, y \in X$. Finally, the multi-valued mapping Q is simply called monotone increasing if $x \leq y$, then $Qx \overset{id}{\leq} Qy$ for all $x, y \in X$.

Definition 3.2. A multi-valued mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone decreasing if $x \leq y$, then $Qy \overset{i}{\leq} Qx$ for all $x, y \in X$. Similarly, Q is called left monotone decreasing if $x \leq y$, then $Qy \overset{d}{\leq} Qx$ for all $x, y \in X$. Finally, the multi-valued mapping Q is simply called monotone decreasing if $x \leq y$, then $Qy \overset{id}{\leq} Qx$ for all $x, y \in X$.

Note that if Q is a single-valued mapping, then the monotone increasing multi-valued operator is the same as the monotone increasing operator and the monotone decreasing multi-valued operator is the same as the monotone decreasing operator on X.

Definition 3.3. A multi-valued mapping $Q : X \to \mathcal{P}_p(X)$ is called strictly monotone increasing if $x < y$, then $Qx \leq Qy$ for all $x, y \in X$. Similarly, Q is called strictly monotone decreasing if $x < y$, then $Qx \geq Qy$ for all $x, y \in X$. Finally, the multi-valued mapping Q is simply called strictly monotone on X if it is either strictly monotone increasing or strictly monotone increasing X.

Example 3.6. Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^+$ denote the set of nonnegative real numbers. Define two multi-valued mappings $Q_1, Q_2 : \mathbb{R}^+ \to \mathcal{P}_{cp}(\mathbb{R})$ by

$$Q_1(x) = [-x, x] \quad \text{and} \quad Q_2(x) = [0, x+1].$$

It is very easy to verify that the multi-valued map Q_1 is right monotone increasing, but not left monotone increasing on $\mathbb{R}^+$. However, the multi-valued map Q_2 is right monotone increasing as well as left monotone increasing on $\mathbb{R}^+$. Note that neither Q_1 nor Q_2 is strictly monotone increasing or strictly monotone decreasing on $\mathbb{R}$.

Thus it is clear that every strictly monotone increasing multi-valued mapping is monotone increasing on X, but the converse is not true. The following example illustrates this fact more clearly.

Example 3.7. Let $\mathbb{R}^+$ denote the set of reals and define a strictly monotone increasing multi-valued map $Q_3 : \mathbb{R}^+ \to \mathcal{P}_{cp}(\mathbb{R})$ by $Q_3(x) = [x, x+1]$. It is easy to verify that Q_3 is also right monotone increasing as well as left monotone increasing, that is, a monotone increasing multi-valued map on $\mathbb{R}^+$.

Theorem 3.8. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a right monotone increasing multi-valued mapping. If every monotone increasing sequence $\{y_n\} \subset \bigcup Q([a, b])$ defined by $y_n \in Qx_n, n \in \mathbb{N}$ converges, whenever $\{x_n\}$ is a monotone increasing sequence in $[a, b]$, then Q has a fixed point.*

Proof. We only need to show that hypotheses (Q_0) and (Q_1) of Theorem 3.1 are satisfied. Since $Qx \subset [a, b]$ for each $x \in [a, b]$, we have that $a \leq Qa$ and so $M \neq \emptyset$. Next Q is monotone increasing, therefore the hypothesis (Q_1) holds. Now the desired conclusion follows by Theorem 3.1. This completes the proof. ■

Let X be a metric space. A multi-map $Q : X \to \mathcal{P}_{cp}(X)$ is called **totally compact** if $\overline{\bigcup Q(X)}$ is a compact subset of X. Q is called **compact** if $\bigcup Q(S)$ is a relatively compact subset of X for all bounded subsets S of X. Again Q is called **totally bounded** if for any bounded subset S of X, $\bigcup Q(S)$ is a totally bounded subset of X. It is clear that every compact multi-valued map is totally bounded, but the converse may not be true. However, these two notions are equivalent on bounded subsets of a complete metric space X.

Corollary 3.9. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a right monotone increasing multi-valued mapping. Then Q has a fixed point if any one of the following conditions is satisfied.*

(a) *Q is a totally compact multi-valued mapping.*

(b) *The cone K in X is normal and Q is compact.*

(c) *The cone K is regular.*

Proof. Let $\{x_n\}$ be a monotone increasing sequence in $[a,b]$ and let $\{y_n\}$ be a monotone increasing sequence in $\bigcup Q([a,b])$ defined by $y_n \in Tx_n$ for each $n \in \mathbb{N}$. Clearly such a sequence $\{y_n\}$ exists since the multi-map Q is right monotone increasing on $[a,b]$. Suppose that the hypothesis (a) holds. Then $\overline{\bigcup Q([a,b])}$ is compact and the sequence $\{y_n\}$ has a convergent subsequence. Since $\{y_n\}$ is monotone increasing, it converges to a point in $[a,b]$. Again if the hypothesis (b) holds, then the order interval $[a,b]$ is bounded in norm and $\bigcup Q([a,b])$ is a relatively compact set in X. Therefore the sequence $\{y_n\} \subset \bigcup Q([a,b])$ has a convergent subsequence and so the whole sequence converges to a point in $\bigcup Q([a,b])$. Finally, if the hypothesis (c) holds, then by definition of the cone, the sequence $\{y_n\}$ converges to a point in $\bigcup Q([a,b])$. Thus all the conditions of Theorem 3.1 are satisfied under every hypothesis (a) or (b) or (c). Hence an application of it yields that Q has at least a fixed point in $[a,b]$. This completes the proof. ■

Similarly, we have

Theorem 3.10. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a left monotone increasing multi-valued mapping. If every monotone decreasing sequence $\{y_n\} \subset \bigcup Q([a,b])$ defined by $y_n \in Qx_n, n \in \mathbb{N}$ converges, whenever $\{x_n\}$ is a monotone decreasing sequence in $[a,b]$, then Q has a fixed point.*

Corollary 3.11. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a left monotone increasing multi-valued mapping. Then Q has a fixed point if any one of the following conditions is satisfied.*

(a) *Q is a totally compact multi-map.*

(b) *The cone K in X is normal and Q is compact.*

(c) *The cone K is regular.*

The proofs of Theorem 3.10 and Corollary 3.11 are similar to Theorem 3.8 and Corollary 3.9 with appropriate modifications. We omit the details.

A slightly stronger form of Theorem 3.8 useful in applications to differential inclusions is the following result.

Theorem 3.12. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a right monotone increasing mapping. If every sequence $\{y_n\} \subset \bigcup Q([a,b])$ defined by $y_n \in Qx_n, n \in \mathbb{N}$ has a cluster point, whenever $\{x_n\}$ is a monotone increasing sequence in $[a,b]$, then Q has a fixed point.*

Proof. Suppose that $\{y_n\}$ is a monotone increasing sequence in $[a,b]$ defined by $y_n \in Qx_n, n \in \mathbb{N}$. This is possible in view of the fact that $\{x_n\}$ is monotone increasing and that Q is a monotone increasing multi-valued mapping on $[a,b]$. Since $\{y_n\}$ has a cluster point, the desired conclusion follows from applying Theorem 3.1. ∎

Again a fixed point theorem for monotone increasing operators in the applicable form to differential integral inclusions is

Theorem 3.13. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a,b])$ defined by $y_n \in Qx_n, n \in \mathbb{N}$ has a cluster point, whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then Q has a fixed point.*

Remark 3.1. We note that some fixed point theorems for right monotone increasing multi-valued operators have been proved in Dhage [12] under upper semi-continuity of the multi-valued mappings in ordered Banach spaces. In this case the multi-valued operators are not required to have compact values, but it is enough to have closed values on the domains of definition.

4. Multi-valued hybrid fixed point theory

In this section, we shall prove some fixed point theorems for operator inclusions involving the sum and the product of two multi-valued operators in a Banach space under mixed compactness and monotonicity conditions. It seems that some of the results of this section are also new even to the fixed point theory of single-valued mappings in abstract spaces. The results of this section also have a wide range of applications to perturbed differential equations and inclusions.

4.1. Topo-algebraic hybrid fixed point theorems

In this section, we combine a topological fixed point theorem with an algebraic fixed point theorem to derive a hybrid fixed point theorem called the topo-algebraic hybrid fixed point theorems for multi-valued operators in Banach spaces. Before going to the main results we give some preliminaries needed in the sequel.

Let X and Y be two metric spaces and let $T : X \to \mathcal{P}_c(Y)$. Then T is called **upper semi-continuous** (u.s.c.) if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty and closed subset of Y, and for each open set $N \subset X$ containing $T(x_0)$, there exists an open neighborhood M of x_0 such that $\bigcup T(M) = T(M) \subset N$. The multi-valued mapping $T : X \to \mathcal{P}_{cp}(Y)$ is called **completely continuous** if it is upper semi-continuous and compact on X. If T is non-empty and compact, then T is u.s.c. if and only if T has a closed graph, i.e., given sequences $\{x_n\}_{n=1}^{\infty} \to x_0$, $\{y_n\}_{n=1}^{\infty} \to y_0$, and $y_n \in T(x_n)$ for every $n = 1, 2, \ldots$ imply $y_0 \in T(x_0)$. Note that the complete continuity of T from a metric space X into the complete metric space Y is equivalent to upper semi-continuity together with the totally boundedness of T on X.

Theorem 4.1. *Let $[a, b]$ be a norm-bounded order interval in a subset Y of an ordered Banach space X and let $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ be a multi-valued mapping satisfying*

(a) *$y \mapsto T(x, y)$ is right monotone increasing and completely continuous and has convex values for each $x \in [a, b]$,*

(b) *$x \mapsto T(x, y)$ is right monotone increasing for each $y \in [a, b]$ and*

(c) *every monotone increasing sequence $\{y_n\} \subset \bigcup T([a, b] \times [a, b])$ defined by $y_n \in T(x_n, y), n \in \mathbb{N}$ converges for each $y \in [a, b]$, whenever $\{x_n\}$ is a monotone increasing sequence in $[a, b]$.*

Then the operator inclusion $x \in T(x, x)$ has a solution in $[a, b]$.

Proof. Define the multi-valued operator $Q : [a, b] \to \mathcal{P}_p([a, b])$ by

$$Qx = \big\{ y \in [a, b] \mid y \in T(x, y) \big\}. \tag{4.1}$$

Let $x \in [a, b]$ be fixed and define the operator $T_x(y) : [a, b] \to \mathcal{P}_{cp}([a, b])$ by $T_x(y) = T(x, y)$. Then T_x is a completely continuous multi-valued operator which maps a closed convex and bounded subset $[a, b]$ of the Banach space X into itself. Therefore an application of the multi-valued analogue of a fixed point theorem of Schauder [21] yields that T_x has a fixed point in $[a, b]$ and consequently the set Qx is non-empty for each $x \in [a, b]$. Moreover, Qx is compact for each $x \in [a, b]$. Further, hypothesis (c) implies that Q satisfies all the conditions of Theorem 3.12 on $[a, b]$ and hence an application of it yields that Q has a fixed point. This further implies that the operator inclusion $x \in T(x,x)$ has a solution in $[a,b]$. This completes the proof. ∎

As a consequence of Theorem 4.1 we obtain

Corollary 4.2. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $T : [a,b] \times [a,b] \to \mathcal{P}_{cp,cv}([a,b])$ be a multi-valued mapping satisfying*

(a) *$y \mapsto T(x,y)$ is right monotone increasing and completely continuous for each $x \in [a,b]$ and*
(b) *$x \mapsto T(x,y)$ is right monotone increasing for each $y \in [a,b]$.*

Then the operator inclusion $x \in T(x,x)$ has a solution if any one of the following conditions is satisfied.

(a) *$[a,b]$ is norm-bounded and T is compact multi-valued mapping.*
(b) *The cone K in X is normal and $x \mapsto T(x,y)$ is compact for each $y \in [a,b]$.*
(c) *The cone K is regular.*

The origin of the fixed point theorems involving the sum of two operators in Banach spaces lies in the works of a Russian mathematician, Krasnoselskii [28]. In this case one operator happens to be a contraction and another one happens to be a completely continuous on the domain of their definition. Since every contraction is continuous, both the operators in such theorems are continuous. Below we relax the continuity of one of the mappings in such hybrid fixed point theorems, instead we assume monotonicity and prove a fixed point theorem on ordered Banach spaces.

Theorem 4.3. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X. Let $A, B : [a,b] \to \mathcal{P}_{cp,cv}(X)$ be two multi-valued operators satisfying*

(a) *A is compact and right monotone increasing,*
(b) *B is right monotone increasing and completely continuous and*
(c) *$Ax + By \subset [a,b]$ for all $x, y \in [a,b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a solution in $[a,b]$.

Proof. Define an operator T on $[a,b]\times[a,b]$ by $T(x,y)=Ax+By$. From hypothesis (c) it follows that T defines a multi-valued mapping $T:[a,b]\times[a,b]\to\mathcal{P}_{cp,cv}([a,b])$. Now the desired conclusion follows by Corollary 4.2. ■

Remark 4.1. Note that hypothesis (c) holds if A and B are both right monotone increasing multi-valued operators and there exist elements a and b in $[a,b]$ such that $a \overset{d}{\leq} Aa+Ba$ and $Ab+Bb \overset{i}{\leq} b$.

When A and B are single-valued operators, Theorem 4.3 reduces to

Corollary 4.4. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X. Let $A,B:[a,b]\to X$ be two single-valued operators satisfying*

(a) *A is monotone increasing and compact,*

(b) *B is monotone increasing and completely continuous and*

(c) *$Ax+By\in[a,b]$ for all $x,y\in[a,b]$.*

Further, if the cone K in X is normal, then the operator inclusion $Ax+Bx=x$ has a solution in $[a,b]$.

Remark 4.2. Note that hypothesis (c) holds if A and B are both monotone increasing single-valued operators and there exist elements a and B in $[a,b]$ such that $a\leq Aa+Ba$ and $Ab+Bb\leq b$. In this case the operator equation $Ax+Bx=x$ has the least and the greatest solution in $[a,b]$.

The hybrid fixed point theory involving the product of two operators in a Banach algebra is initiated by the author in [8] and shall be developed further in various directions in due course of time (see Dhage [11, 14] and the references therein). The main feature of these fixed point theorems is again that both the operators are continuous on their domain of definition. Below we remove the continuity of one of the operators and prove a fixed point theorem involving the product of two operators in a Banach algebra. We need the following preliminaries in the sequel.

A cone K in a Banach algebra X is called positive if

(iv) $K\circ K\subseteq K$, where “ $\circ$ ” is a multiplicative composition in X.

Lemma 4.1 (Dhage [14]). *If $u_1,u_2,v_1,v_2\in K$ are such that $u_1\leq v_1$ and $u_2\leq v_2$, then $u_1u_2\leq v_1v_2$, provided K in a positive cone in X.*

Theorem 4.5. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach algebra X with a cone K. Let $A, B : [a, b] \to \mathcal{P}_{cp}(K)$ be two multi-valued operators satisfying*

(a) *A is compact and right monotone increasing,*

(b) *B is right monotone increasing and completely continuous, and*

(c) *$Ax \cdot By \in \mathcal{P}_{cv}([a, b])$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is positive and normal, then the operator inclusion $x \in Ax \cdot Bx$ has a solution in $[a, b]$.

Proof. Define an operator T on $[a, b] \times [a, b]$ by $T(x, y) = Ax \cdot By$. From hypothesis (c) it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp,cv}([a, b])$. Now the desired conclusion follows by applying Corollary 4.2. ■

Remark 4.3. Note that hypothesis (c) holds if (i) A and B are both right monotone increasing multi-valued operators (ii) the cone K in X is positive and (iii) there exist elements a and B in $[a, b]$ such that $a \overset{d}{\le} Aa \cdot Ba$ and $Ab \cdot Bb \overset{i}{\le} b$.

When A and B are single-valued operators, Theorem 4.5 reduces to

Corollary 4.6. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach algebra X. Let $A, B : [a, b] \to X$ be two single-valued operators satisfying*

(a) *A is compact and monotone increasing,*

(b) *B is monotone increasing and completely continuous and*

(c) *$Ax \cdot By \in [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is positive and normal, then the operator inclusion $Ax \cdot Bx = x$ has a solution in $[a, b]$.

Remark 4.4. Note that hypothesis (c) holds if (i) A and B are both positive and monotone increasing multi-valued operators (ii) the cone K in X is positive and (iii) there exist elements a and b in X such that $a \le Aa \cdot Ba$ and $Ab \cdot Bb \le b$. In this case the operator equation $Ax \cdot Bx = x$ has the least and the greatest solution in $[a, b]$.

The study of operator equations involving three operators in a Banach algebras is initiated by the author and for the recent work the reader is referred to Dhage [15]. Below we prove a result concerning the existence of a solution of operator inclusions involving three multi-valued operators in Banach algebras and two of which are discontinuous on the domain of their definitions.

Theorem 4.7. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach algebra X with a cone K. Let $A, B : [a, b] \to \mathcal{P}_{cp}(K)$ and $C : [a, b] \to \mathcal{P}_{cp}(X)$ be three multi-valued operators satisfying*

(a) *A and C are compact and right monotone increasing,*

(b) *B is right monotone increasing and completely continuous, and*

(c) *$Ax \cdot By + Cx \in \mathcal{P}_{cv}([a, b])$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is positive and normal, then the operator inclusion $x \in Ax \cdot Bx + Cx$ has a solution in $[a, b]$.

Proof. Define an operator T on $[a, b] \times [a, b]$ by $T(x, y) = Ax \cdot By + Cx$. From hypothesis (c) it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp,cv}([a, b])$. Now the desired conclusion follows by Corollary 4.2. ■

Remark 4.5. Note that hypothesis (c) holds if (i) A, B and C are right monotone increasing multi-valued operators (ii) the cone K in X is positive and (iii) there exist elements a and b in X such that $a \overset{d}{\leq} Aa \cdot Ba + Ca$ and $Ab \cdot Bb + Cb \overset{i}{\leq} b$.

When A, B and C are single-valued operators, Theorem 4.5 reduces to

Corollary 4.8. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach algebra X. Let $A, B : [a, b] \to K$ and $C : [a, b] \to X$ be three single-valued operators satisfying*

(a) *A and C are compact and monotone increasing,*

(b) *B is monotone increasing and completely continuous and*

(c) *$Ax \cdot By + Cx \in [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is positive and normal, then the operator inclusion $Ax \cdot Bx + Cx = x$ has a solution in $[a, b]$.

Remark 4.6. Note that hypothesis (c) holds if (i) A and B are both monotone increasing multi-valued operators (ii) the cone K in X is positive and (iii) there exist elements a and b in X such that $a \leq Aa \cdot Ba + Ca$ and $Ab \cdot Bb + Cb \leq b$. In this case the operator equation $Ax \cdot Bx + Cx = x$ has the least and the greatest solution in $[a, b]$.

4.2. Geome-algebraic hybrid fixed point theorems

In this section, we combine a geometrical fixed point theorem with an algebraic fixed point theorem in Banach algebras to yield a hybrid fixed point theorem with mixed conditions called the Geome-algebraic hybrid fixed point theorem for multi-valued operators in Banach spaces. Before going to the main results we give some preliminaries needed in the sequel.

Define a function $H : \mathcal{P}_{bd,cl}(X) \times \mathcal{P}_{bd,cl}(X) \to \mathbb{R}^+$ by

$$H(A, B) = \max\{\rho(A, B)\,,\ \rho(B, A)\}.$$

where $\rho(A, B) = \sup_{a\in A} d(a, B)$ and $d(a, B) = \inf\{d(a, b) : b \in B\}$.

The function H is called a Hausdorff metric on X. Note that $\|Y\| = H(Y, \{0\}) = \sup\{\|y\| : y \in Y\}$. The details of the properties of the Hausdorff metric are given in Hu and Papageorgiou [26].

Definition 4.1. Let $T : X \to \mathcal{P}_f(X)$ be a multi-valued operator. Then T is called a multi-valued contraction if there exists a constant $\lambda \in (0, 1)$ such that

$$H(T(x), T(y)) \leq \lambda\|x - y\|$$

for all $x, y \in X$. The constant λ is called a contraction constant of T.

Theorem 4.9 (Covitz and Nadler [6])**.** *Let $T : X \to \mathcal{P}_{cl}(X)$ be a multi-valued contraction. Then the fixed point set F_T of T is non-empty and closed in X.*

Remark 4.7. It the multi-valued map T in Theorem 4.9 is compact-valued, then the fixed point set F_T of T is compact in X.

Theorem 4.10. *Let $[a, b]$ be an order interval in an ordered Banach space X and let $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ be a mapping satisfying*

(a) *$y \mapsto T(x, y)$ is right monotone increasing and multi-valued contraction for each $x \in [a, b]$,*

(b) *$x \mapsto T(x, y)$ is right monotone increasing for each $y \in [a, b]$ and*

(c) *every monotone increasing sequence $\{y_n\} \subset \bigcup T([a, b] \times [a, b])$ defined by $y_n \in T(x_n, y), n \in \mathbb{N}$ converges for each $y \in [a, b]$, whenever $\{x_n\}$ is a monotone increasing sequence in $[a, b]$.*

Then the operator inclusion $x \in T(x, x)$ has a solution in $[a, b]$.

Proof. Define the multi-valued operator $Q : [a, b] \to \mathcal{P}_p([a, b])$ by

$$Qx = \{y \in [a, b] \mid y \in T(x, y)\}. \tag{4.2}$$

Let $x \in [a, b]$ be fixed and define the operator $T_x(y) : [a, b] \to \mathcal{P}_{cp}([a, b])$ by $T_x(y) = T(x, y)$. Then T_x is a multi-valued contraction which maps a closed convex and bounded subset $[a, b]$ into itself. Therefore an application of the fixed point theorem of Covitz and Nadler [6] yields that the set Qx is non-empty for each $x \in [a, b]$. Moreover, Qx is compact for each $x \in [a, b]$. Further, hypothesis (c) implies that Q satisfies all the conditions of Theorem 3.12 on $[a, b]$ and hence an application of it yields that Q has a fixed point. This further implies that the operator inclusion $x \in T(x, x)$ has a solution in $[a, b]$. This completes the proof. ■

As a consequence of Theorem 4.9 we obtain

Corollary 4.11. *Let $[a, b]$ be an order interval in an ordered Banach space X and let $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ be a mapping satisfying*

(a) *$y \mapsto T(x, y)$ is right monotone increasing and multi-valued contraction for each $x \in [a, b]$,*

(b) *$x \mapsto T(x, y)$ is right monotone increasing for each $y \in [a, b]$.*

Then the operator inclusion $x \in T(x, x)$ has a solution if any one of the following conditions is satisfied.

(a) *$x \mapsto T(x, y)$ is compact multi-map for each $y \in [a, b]$.*

(b) *The cone K in X is normal and $x \mapsto T(x, y)$ is totally bounded for each $y \in [a, b]$.*

(c) *The cone K is regular.*

Below we prove a fixed point theorem involving the sum of two operators in ordered Banach spaces with hybrid conditions. We need the following lemma in the sequel.

Lemma 4.2. *For any $A, B \in \mathcal{P}_{cl}(X)$, we have*

$$H(A + C, B + C) \leq H(A, B).$$

Theorem 4.12. *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $A, B : [a, b] \to \mathcal{P}_{cp}(X)$ be two multi-valued operators satisfying*

(a) *A is totally bounded and right monotone increasing,*

(b) *B is right monotone increasing and multi-valued contraction with the contraction constant k, and*

(c) *$Ax + By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a solution in $[a, b]$.

Proof. Let $x \in [a, b]$ be fixed and define a multi-valued operator $T_x : [a, b] \to \mathcal{P}_{cp}([a, b])$ by $T_x(y) = Ax + By$. Then for any $y_1, y_2 \in [a, b]$ we have by Lemma 4.2,

$$\begin{aligned} H(T_x(y_1), T_x(y_2)) &= H(Ax + By_1, Ax + By_2) \\ &\leq H(Ax + By_1, Ax + By_2) \\ &\leq k\|y_1 - y_2\|. \end{aligned}$$

This shows that $y \mapsto Ax + By = T_x(y)$ is a multi-valued contraction on $[a, b]$ with a contraction constant k. From hypothesis (c) it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$. Now the desired conclusion follows by Corollary 4.11. ■

Remark 4.8. Note that hypothesis (c) of Theorem 4.10 holds if A and B satisfy the conditions of Remark 4.3.

Definition 4.2. A single-valued mapping $T : X \to X$ is called Lipschitz if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|$$

for all $x, y \in X$. The constant k is called the Lipschitz constant of T on X. Further, if $k < 1$, then T is called a contraction on X with a contraction constant k.

Remark 4.9. It is known that a Lipschitz map with a Lipschitz constant k is not a multi-valued Lipschitz map with a Lipschitz constant k, but a multi-valued Lipschitz map with the Lipschitz constant $2k$. In particular, a contraction map T with a contraction constant $k < 1/2$ is a multi-valued contraction with the contraction constant k.

Corollary 4.13. *Let $[a,b]$ be an order interval in an ordered Banach space X. Let $A : [a,b] \to \mathcal{P}_{cp}(X)$ be a multi-valued and $B : [a,b] \to X$ be a single-valued operators satisfying*

(a) *A is totally bounded and right monotone increasing,*

(b) *B is monotone increasing single-valued contraction with a contraction constant $k < \frac{1}{2}$, and*

(c) *$Ax + By \subset [a,b]$ for all $x, y \in [a,b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a solution in $[a,b]$.

When A and B are single-valued operators, we obtain some interesting corollaries to Theorem 4.10 applicable to nonlinear differential and integral equations involving discontinuous nonlinearities. Before these results we need the following preliminaries in the sequel.

Corollary 4.14. *Let $[a,b]$ be an order interval in an ordered Banach space X. Let $A, B : [a,b] \to X$ be two single-valued operators satisfying*

(a) *A is totally bounded and monotone increasing,*

(b) *B is monotone increasing and contraction with a contraction constant k, and*

(c) *$Ax + By \in [a,b]$ for all $x, y \in [a,b]$.*

Further, if the cone K in X is normal, then the operator equation $Ax+Bx = x$ has a solution in $[a,b]$.

Proof. Define a mapping $T : [a,b] \times [a,b] \to X$ by $T(x,y) = Ax + By$. By hypothesis (c), T maps $[a,b] \times [a,b]$ into itself. Since the cone K is normal, the order interval $[a,b]$ is a closed convex and bounded subset of X. Let $x \in [a,b]$ be fixed. Then for any $y_1, y_2 \in [a,b]$, one has

$$\|T_x(y_1) - T_x(y_2)\| \leq \|By_1 - By_2\| \leq k\|y_1 - y_2\|$$

where $k < 1$. This shows that the operator T_x is a contraction on $[a,b]$ and hence by the Banach fixed point theorem, T_x has a unique fixed point in $[a,b]$. Thus there is a unique point $z \in [a,b]$ such that $T_x(z) = Ax + Bz = z$. Define a mapping $Q : [a,b] \to X$ by $Qx = z$, where z is a unique solution to the operator equation $Ax + Bz = z$. We show that Q is a nondecreasing mapping on $[a,b]$. Let $x_1, x_2 \in [a,b]$ be such that $x_1 \le x_2$. Then there are unique elements $z_1, z_2 \in [a,b]$ such that

$$Qx_1 = z_1 = Ax_1 + Bz_1 = T_{x_1}(z_1)$$

and

$$Qx_2 = z_2 = Ax_2 + Bz_2 = T_{x_2}(z_2).$$

From the monotonicity of A, it follows that

$$T_{x_1}(y) = Ax_1 + By \le Ax_2 + By = Tx_{x_1}(y)$$

for all $y \in [a,b]$. Hence for any $y \in [a,b]$

$$T^n_{x_1}(y) \le T^n_{x_2}(y)$$

for all $n \in \mathbb{N}$. Since T_{x_1} and T_{x_2} are contractions, by the Banach fixed point theorem,

$$z_1 = \lim_{n\to\infty} T^n_{x_1}(y) \le T^n_{x_2}(y) = z_2.$$

This shows that Q defines a nondecreasing totally bounded operator $Q : [a,b] \to [a,b]$. Now the desired conclusion follows by Theorem 1.1.1 of Heikkilä and Lashmikantham [25]. ■

Remark 4.10. Note that hypothesis (c) of Corollary 4.11 holds if A and B satisfy the conditions of Remark 4.4.

To prove a hybrid fixed point theory involving the product of two operators in an ordered Banach algebra, we need the following definition.

Definition 4.3. A multi-valued mapping $T : X \to \mathcal{P}_p(X)$ is **Lipschitz** if there is a constant $k > 0$ such that

$$H(Tx, Ty) \le k \, \|x - y\|$$

for all $x, y \in X$. The constant k is called the Lipschitz constant of T.

Lemma 4.3 (Dhage [14]). *Let X be a Banach algebra and let $A, B, C \in \mathcal{P}_p(X)$. Then*

$$H(AC, BC) \leq H(0, C)\, H(A, B).$$

Theorem 4.15. *Let $[a, b]$ be an order interval in an ordered Banach algebra X with a cone K. Let $A : [a, b] \to \mathcal{P}_{cp}(K)$ and $B : [a, b] \to \mathcal{P}_{cp}(K)$ be two multi-valued operators satisfying*

(a) *A is totally bounded and right monotone increasing,*
(b) *B is right monotone increasing and multi-valued Lipschitz with a Lipschitz constant k, and*
(c) *$Ax \cdot By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is positive and normal and $kM < 1$, then the operator inclusion $x \in Ax \cdot Bx$ has a solution in $[a, b]$ where $M = \|A([a, b])\| = \sup\{\|Ax\| : x \in [a, b]\}$.

Proof. Let $x \in [a, b]$ be fixed and define a multi-valued operator $T_x : [a, b] \to \mathcal{P}_{cp}([a, b])$ by $T_x(y) = Ax \cdot By$. Then for any $y_1, y_2 \in [a, b]$, by Lemma 4.3 we have

$$\begin{aligned} H(T_x(y_1), T_x(y_2) &= H(Ax \cdot By_1, Ax \cdot By_2) \\ &\leq H(Ax, 0)\, H(By_1, By_2) \\ &\leq \lambda \|y_1 - y_2\| \end{aligned}$$

where $\lambda = kM < 1$. This shows that $y \mapsto Ax \cdot By = T_x(y)$ is a multi-valued contraction on $[a, b]$ with a contraction constant kM. From hypothesis (c) it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$. Now the desired conclusion follows by Corollary 4.11. ■

Remark 4.11. Note that hypothesis (c) of Theorem 4.12 holds if A and B satisfy all the conditions of Remark 4.1.

Corollary 4.16. *Let $[a, b]$ be an order interval in an ordered Banach algebra X with a cone K. Let $A : [a, b] \to \mathcal{P}_{cp}(K)$ and $B : [a, b] \to K$ be two operators satisfying*

(a) *A is totally bounded and right monotone increasing,*

(b) *B is monotone increasing and single-valued Lipschitz with a Lipschitz constant k, and*

(c) *$Ax \cdot By \subset [a,b]$ for all $x, y \in [a,b]$.*

Further, if the cone K in X is normal and $2kM < 1$, then the operator inclusion $x \in Ax \cdot Bx$ has a solution in $[a,b]$ where $M = \|A([a,b])\| = \sup\{\|Ax\| : x \in [a,b]\}$.

When A and B are single-valued operators, Theorem 4.12 reduces to

Corollary 4.17. *Let $[a,b]$ be an order interval in an ordered Banach algebra X. Let $A, B : [a,b] \to K$ be two single-valued operators satisfying*

(a) *A is totally bounded and monotone increasing,*

(b) *B is monotone increasing and Lipschitz with the Lipschitz constant k, and*

(c) *$Ax \cdot By \in [a,b]$ for all $x, y \in [a,b]$.*

Further, if the cone K in X is positive and normal and $Mk < 1$, then the operator inclusion $Ax \cdot Bx = x$ has a solution in $[a,b]$ where $M = \|A([a,b])\| = \sup\{\|Ax\| : x \in [a,b]\}$.

Proof. Define a mapping $T : [a,b] \times [a,b] \to X$ by $T(x,y) = Ax \cdot By$. By hypothesis (c), T maps $[a,b] \times [a,b]$ into itself. Since the cone K is normal, the order interval $[a,b]$ is a closed convex and bounded subset of X. Let $x \in [a,b]$ be fixed. Then for any $y_1, y_2 \in [a,b]$, one has

$$\begin{aligned}\|T_x(y_1) - T_x(y_2)\| &\leq \|Ax \cdot By_1 - Ax \cdot By_2\| \\ &\leq \|Ax\|\|y_1 - y_2\| \\ &\leq Mk\|y_1 - y_2\|\end{aligned}$$

where $Mk < 1$. This shows that the operator T_x is a contraction on $[a,b]$ and hence by the Banach fixed point theorem, T_x has a unique fixed point in $[a,b]$. Thus there is a unique point $z \in [a,b]$ such that $T_x(z) = Ax \cdot Bz = z$. Define a mapping $Q : [a,b] \to X$ by $Qx = z$, where z is a unique solution to the operator equation $Ax \cdot Bz = z$. We show that Q is a nondecreasing mapping on $[a,b]$. Let $x_1, x_2 \in [a,b]$ be such that $x_1 \leq x_2$. Then there are unique elements $z_1, z_2 \in [a,b]$ such that

$$Qx_1 = z_1 = Ax_1 \cdot Bz_1 = T_{x_1}(z_1)$$

and

$$Qx_2 = z_2 = Ax_2 \cdot Bz_2 = T_{x_2}(z_2).$$

From the monotonicity of A, it follows that

$$T_{x_1}(y) = Ax_1 \cdot By \leq Ax_2 \cdot By = Tx_{x_1}(y)$$

for all $y \in [a, b]$. Hence for any $y \in [a, b]$

$$T^n_{x_1}(y) \leq T^n_{x_2}(y)$$

for all $n \in \mathbb{N}$. Since T_{x_1} and T_{x_1} are contractions, by the Banach fixed point theorem,

$$z_1 = \lim_{n\to\infty} T^n_{x_1}(y) \leq T^n_{x_2}(y) = z_2.$$

This shows that Q defines a nondecreasing totally bounded operator $Q : [a, b] \to [a, b]$ (see also Dhage [16] and the references therein). Now the desired conclusion follows by Theorem 1.1.1 of Heikkilä and Lashmikantham [25]. ■

Remark 4.12. Note that hypothesis (c) of Corollary 4.14 holds if A and B satisfy all the conditions of Remark 4.2.

Theorem 4.18. *Let $[a, b]$ be an order interval in an ordered Banach algebra X with a cone K. Let $A, B : [a, b] \to \mathcal{P}_{cp}(K)$ and $C : [a, b] \to \mathcal{P}_{cp}(X)$ be three multi-valued operators satisfying*

(a) *A is totally bounded and right monotone increasing,*
(b) *B and C are right monotone increasing and multi-valued Lipschitz with Lipschitz constants α and β respectively, and*
(c) *$Ax \cdot By + Cy \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is positive and normal and $\alpha M + \beta < 1$, then the operator inclusion $x \in Ax \cdot Bx + Cx$ has a solution in $[a, b]$ where $M = \|A([a, b])\| = \sup\{\|Ax\| : x \in [a, b]\}$.

Proof. Let $x \in [a, b]$ be fixed and define a multi-valued operator $T_x : [a, b] \to \mathcal{P}_{cp}([a, b])$ by $T_x(y) = Ax \cdot By + Cy$. Then for any $y_1, y_2 \in [a, b]$, by Lemma 4.3 we have

$$\begin{aligned} H(T_x(y_1), T_x(y_2)) &= H(Ax \cdot By_1 + Cy_1, Ax \cdot By_2 + Cy_2) \\ &\le H(Ax, 0)\, H(By_1, By_2) + H(Cy_1, Cy_2) \\ &\le \lambda \|y_1 - y_2\| \end{aligned}$$

where $\lambda = \alpha M + \beta < 1$. This shows that $y \mapsto Ax \cdot By + Cy = T_x(y)$ is a multi-valued contraction on $[a, b]$ with a contraction constant $\alpha M + \beta$. From hypothesis (c) it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$. Now the desired conclusion follows by Corollary 4.11. ∎

Remark 4.13. Note that hypothesis (c) of Theorem 4.18 holds if A, B and C satisfy all the conditions of Remark 4.5.

Corollary 4.19. *Let $[a, b]$ be an order interval in an ordered Banach algebra X with a cone K. Let $A : [a, b] \to K$, $B : [a, b] \to \mathcal{P}_{cp}(K)$ and $C : [a, b] \to X$ be three operators satisfying*

(a) *B is totally bounded and right monotone increasing,*
(b) *A and C are monotone increasing single-valued Lipschitz with Lipschitz constants α and β respectively, and*
(c) *$Ax \cdot By + Cy \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is positive and normal and $2(\alpha M + \beta) < 1$, then the operator inclusion $x \in Ax \cdot Bx + Cx$ has a solution in $[a, b]$ where $M = \|A([a, b])\| = \sup\{\|Ax\| : x \in [a, b]\}$.

When A, B and C are single-valued operators, Theorem 4.12 reduces to

Corollary 4.20. *Let $[a, b]$ be an order interval in an ordered Banach algebra X. Let $A, B : [a, b] \to K$ and $C : [a, b] \to X$ be three single-valued operators satisfying*

(a) *B is totally bounded and monotone increasing,*
(b) *A and C are monotone increasing and Lipschitz with Lipschitz constants α and β respectively, and*
(c) *$Ax \cdot By + Cy \in [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is positive and normal and $\alpha M + \beta < 1$, then the operator inclusion $Ax \cdot Bx + Cx = x$ has a solution in $[a, b]$, where $M = \|A([a, b])\| = \sup\{\|Ax\| : x \in [a, b]\}$.

Proof. The proof is similar to Corollaries 4.14 and 4.17 and so we omit the details. ■

Remark 4.14. Note that hypothesis (c) of Corollary 4.14 holds if A, B and C satisfy all the conditions of Remark 4.6.

5. Discontinuous differential inclusions

The method of upper and lower solutions has been successfully applied to the problems of nonlinear differential equations and inclusions. We refer to Heikkilä and Laksmikantham [25], Halidias and Papageorgiou [22] and Benchohra [5]. In this section, we apply the results of previous sections to the first order initial value problem of ordinary discontinuous differential inclusions for proving the existence of solution between the given upper and lower solutions under monotonicity conditions.

5.1. Initial value problems

Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the differential inclusion (in short DI)

$$\begin{cases} x'(t) \in F(t, x(t)), \text{ a.e. } t \in J \\ x(0) = x_0 \end{cases} \tag{5.1}$$

where $F : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$.

By a solution of DI (5.1) we mean a function $x \in AC(J, \mathbb{R})$ such that

$$x'(t) = v(t),\ t \in J, \quad x(0) = x_0,$$

for some $v \in L^1(J, \mathbb{R})$ satisfying $v(t) \in F(t, x(t))$, a.e. for $t \in J$, where $AC(J, \mathbb{R})$ is a space of all absolutely continuous real-valued functions on J.

The DI (5.1) has been discussed in the literature very extensively for different aspects of the solution under different continuity conditions. See Aubin and Cellina [4], Deimling [7] and the references therein. Recently the

DI (5.1) with discontinuous F has been discussed in Dhage [8, 9], Dhage and O'Regan [18] and Agarwal *et al.* [1] for the existence of extremal solutions via the lattice theoretic approach to differential and integral inclusions. In this section, we shall prove the existence theorems for DI (5.1) via the functional theoretic approach embodied in Theorem 3.1 under the weaker order relation than in Dhage and O'Regan [18] and Agarwal [1] on the lines of Hu and Heikkilä [24].

Define a norm $\|\cdot\|$ and an order relation " $\leq$ " in $AC(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{5.2}$$

and

$$x \leq y \iff x(t) \leq y(t) \text{ for all } t \in J. \tag{5.3}$$

Here the cone K in $AC(J,\mathbb{R})$ is defined by

$$K = \{x \in AC(J,\mathbb{R}) \mid x(t) \geq 0\},$$

which is obviously normal. See Amann [2] and Guo and Lakshmikantham [20] and Heikkila and Lakshmikantham [25].

We need the following definition in the sequel.

Definition 5.1. A function $a \in AC(J,\mathbb{R})$ is called a lower solution of the DI (5.1) if $a'(t) \leq v(t)$, a.e. $t \in J$, and $a(0) \leq x_0$, for all $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t, a(t))$ a.e. $t \in J$. Similarly, a function $b \in AC(J,\mathbb{R})$ is called an upper solution of the DI (5.1) if $b'(t) \geq v(t)$, for all $t \in J$, and $a(0) \leq x_0$, for all $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t, b(t))$ a.e. $t \in J$.

We use the following notations in the sequel. Denote

$$|F(t,x)| = \{|u| \mid u \in F(t,x)\},$$

and

$$\|F(t,x)\| = \sup\{|u| \mid u \in F(t,x)\}.$$

Let $\beta : J\times\mathbb{R} \to \mathbb{R}$ be a multifunction. Then the set of all Lebesgue integrable selectors S^1_β of β is defined by

$$S^1_\beta(x) = \{v \in L^1(J,\mathbb{R}) \mid v(t) \in \beta(t, x(t)) \text{ a.e. } t \in J\}$$

for $x \in AC(J,\mathbb{R})$. The problem of non-emptiness of the set $S^1_\beta(x)$ has been of great interest for a long time. Some crucial results concerning $S^1_\beta(x) \neq \emptyset$ have been given in Lasota and Opial [29], Covitz and Nadler [6] and Wagner [31] (see also the monographs of Aubin and Cellina [4], Deimling [7] and the references therein).

We consider the following set of assumptions:

(A_1) There exists a Lebesgue integrable function $m \in L^1(J,\mathbb{R})$ such that

$$|F(t,x)| \leq m(t) \text{ a.e. } t \in J$$

for all $x \in \mathbb{R}$.

(A_2) $F(t,x)$ is a closed and bounded subset of $\mathbb{R}$ for each $(t,x) \in J \times \mathbb{R}$.

(A_3) There exists a Lebesgue integrable function $h \in L^1(J,\mathbb{R})$ such that the function $x \mapsto F(t,x) + h(t)x$ is right monotone increasing for a.e. $t \in J$.

(A_4) $S^1_{F+h}(x) \neq \emptyset$ and the map $x \mapsto S^1_{F+h}(x)$ is right monotone increasing in $x \in AC(J,\mathbb{R})$.

(A_5) There exist a lower solution a and an upper solution b of the DI (5.1) on J such that $a \leq b$.

Remark 5.1. Assume that hypothesis (A_1)–(A_3) hold and define a mapping $G : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ by

$$G(t,x) = F(t,x) + h(t)x.$$

Then $G(t,x)$ is compact for each $(t,x) \in J \times \mathbb{R}$ and $S^1_G(x) \neq \emptyset$ for each $x \in AC(J,\mathbb{R})$. Again

$$\begin{aligned} |G(t,x)| &= |F(t,x)| + h(t)x(t) \\ &\leq m(t) + h(t)[\|a\| + \|b\|] \\ &= \gamma(t) \end{aligned}$$

for all $t \in J$ and $x \in [a,b]$. Note that $\gamma(\cdot) = m(\cdot) + h(\cdot)[\|a\| + \|b\|] \in L^1(J,\mathbb{R})$.

Theorem 5.1. *Assume that hypotheses* (A_1)–(A_5) *hold. Then the DI* (5.1) *has a minimal and a maximal solution on J.*

Proof. Let $X = BM(J, \mathbb{R})$ and $Y = AC(J, \mathbb{R})$ and consider the DI

$$\left\{ \begin{array}{c} x' + h(t)x(t) \in G(t, x(t)) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}, \end{array} \right. \tag{5.4}$$

which is equivalent to the DI (5.1). Obviously, the lower solution a of the DI (5.1) is the lower solution for the DI (5.4) and the upper solution b of the DI (5.1) is the upper solution for the DI (5.4) with $a \leq b$. Now consider the order interval $[a, b]$ in Y. Define a multi-map T on $[a, b]$ by

$$\begin{aligned} Tx &= \left\{ u \in X \mid u(t) = e^{-H(t)} \Big(x_0 + \int_0^t e^{H(s)} v(s)\, ds \Big),\ v \in S_G^1(x) \right\} \\ &= \mathcal{K} \circ S_G^1(x) \end{aligned} \tag{5.5}$$

where $H(t) = \int_0^t h(s)\, ds$ and the continuous operator $\mathcal{K} : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ is defined by

$$\mathcal{K}v(t) = x_0 e^{-H(t)} + e^{-H(t)} \int_0^t e^{H(s)} v(s)\, ds. \tag{5.6}$$

Obviously, the multi-valued operator T is well defined since $S_G^1(x) \neq \emptyset$ for all $x \in X$ in view of Remark 5.1. We shall show that the multi-valued operator T satisfies all the conditions of Theorem 3.1.

Step I. First, we show that T is a right monotone increasing on $[a, b]$. Let $x, y \in [a, b]$ be such that $x \leq y$ and let $u_1 \in Tx$ be arbitrary. Then there exists an element $v_1 \in S_G^1(x)$, that is, $v_1(t) \in G(t, x(t))$ a.e. $t \in J$ such that

$$u_1(t) = x_0 e^{-H(t)} + e^{-H(t)} \int_0^t e^{H(s)} v_1(s)\, ds. \tag{5.7}$$

Since (A_4) holds, there is an element $v_2(t) \in G(t, y(t))$ a.e. $t \in J$, such that $v_1(t) \leq v_2(t)$ for all $t \in J$. As a result we have an element $u_2 \in Ty$ such that

$$u_2(t) = x_0 e^{-H(t)} + e^{-H(t)} \int_0^t e^{H(s)} v_2(s)\, ds. \tag{5.8}$$

Now for any $t \in J$ we have

$$\int_0^t e^{H(s)} v_1(s)\, ds \leq \int_0^t e^{H(s)} v_2(s)\, ds.$$

As a result we have from (5.7)–(5.8),

$$\begin{aligned} u_1(t) &= x_0 e^{-H(t)} + e^{-H(t)} \int_0^t e^{H(s)} v_1(s)\,ds \\ &\le x_0 e^{-H(t)} + e^{-H(t)} \int_0^t e^{H(s)} v_2(s)\,ds \\ &= u_2(t) \end{aligned}$$

for all $t \in J$. Hence $u_1 \le u_2$. Therefore $Tx \overset{i}{\le} Ty$, that is, T is right monotone increasing on X and in particular on $[a,b]$.

Step II. Next we claim that T has compact-values and maps $[a,b]$ into itself. First, we show that T has compact values on $[a,b]$. Observe that the operator B is equivalent to the composition $\mathcal{L} \circ S^1_G$ of two operators on $L^1(J,\mathbb{R})$, where $\mathcal{L} : L^1(J,\mathbb{R}) \to X$ is the continuous operator defined (5.6). To show that B has compact values, it suffices to prove that the composition operator $\mathcal{L} \circ S^1_G$ has compact values on $[a,b]$. Let $x \in [a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_G(x)$. Then, by the definition of S^1_G, $v_n(t) \in G(t,x(t))$ a.e. for $t \in J$. Since $G(t,x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in G(t,x(t))$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let $t,\ \tau \in J$, then

$$\begin{aligned} |\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| &\le \left| x_0 e^{-H(t)} - x_0 e^{-H(\tau)} \right| \\ &\quad + \left| e^{-H(t)} \int_0^t e^{H(s)} v(s)\,ds - e^{-H(\tau)} \int_0^\tau e^{H(s)} v(s)\,ds \right| \\ &\le |x_0| \left| e^{-H(t)} - e^{-H(\tau)} \right| + \left| e^{-H(t)} \right| \left| \int_\tau^t \left| e^{H(s)} v_n(s) \right| ds \right| \\ &\quad + \left| e^{-H(t)} - e^{-H(\tau)} \right| \left| \int_0^\tau \left| e^{H(s)} v_n(s) \right| ds \right|. \end{aligned} \tag{5.9}$$

The function H is continuous on the compact set J, so it is uniformly continuous there. In addition, $v_n \in L^1(J,\mathbb{R})$, so the right hand side of (5.9) tends to 0 as $t \to \tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S^1_G)(x)$ as $j \to \infty$, and so $(\mathcal{L} \circ S^1_G)(x)$ is compact. Therefore, T is a compact-valued multi-valued operator on $[a, b]$.

Again let $u \in Tb$ be arbitrary. Then there is a $v \in S^1_G(b)$ such that

$$u(t) = e^{-H(t)} \left[x_0 + \int_0^t e^{H(t)} v(s)\, ds \right], \ t \in J.$$

Since b is an upper solution of DI (5.1), we have

$$\begin{aligned}
u(t) &= e^{-H(t)} \left[x_0 + \int_0^t e^{H(s)} v(s)\, ds \right] \\
&\leq e^{-H(t)} \left[x_0 + \int_0^t e^{H(s)} \left[b'(s) + h(s)b(s) \right] ds \right] \\
&\leq e^{-H(t)} \left[x_0 + \int_0^t e^{H(s)} b'(s)\, ds + \int_0^t e^{H(s)} h(s)b(s)\, ds \right] \\
&\leq e^{-H(t)} \left[x_0 + \left(e^{H(s)} b(s) \right)_0^t - \int_0^t e^{H(s)} h(s)b(s)\, ds \right] \\
&\qquad + e^{-H(t)} \int_0^t e^{H(s)} h(s)b(s)\, ds \\
&\leq e^{-H(t)} \left(x_0 + e^{H(t)} b(t) - x_0 \right) \\
&= b(t)
\end{aligned}$$

for all $t \in J$. Hence $u \leq b$ and consequently $Tb \overset{i}{\leq} b$. Similarly, it is proved that $a \overset{d}{\leq} Ta$. Since T is monotone increasing, we have that for any $x,\ a \leq x \leq b$,

$$a \overset{d}{\leq} Ta \overset{i}{\leq} Tx \overset{i}{\leq} Tb \overset{i}{\leq} b.$$

Hence T defines a multi-map $T : [a, b] \to \mathcal{P}_{cp}([a, b])$ and the claim follows.

Step III. Let $\{x_n\}$ be monotone increasing sequence in $[a, b]$ and let $\{y_n\}$ be a sequence in $\bigcup T([a, b])$ defined by $y_n \in Tx_n$, $n \in \mathbb{N}$. We shall show that $\{y_n\}$ is a uniformly bounded and equi-continuous set in $[a, b]$.

Since $y_n \in Tx_n$, there exists a $v_n \in S^1_G(x_n)$ such that

$$y_n(t) = e^{-H(t)}\left(x_0 + \int_0^t e^{H(s)} v_n(s)\,ds\right)$$

for all $t \in J$. Therefore, by Remark 5.1,

$$\begin{aligned}
|y_n(t)| &= \left|e^{-H(t)}\right| \left|\left(x_0 + \int_0^t e^{H(s)} v_n(s)\,ds\right)\right| \\
&\le |x_0| + \left|\int_0^t e^{H(s)} v_n(s)\,ds\right| \\
&\le |x_0| + \int_0^t \left|e^{H(s)} v_n(s)\right| ds \\
&\le |x_0| + \int_0^t e^{\|h\|_{L^1}} |v_n(s)|\,ds \\
&\le |x_0| + e^{\|h\|_{L^1}} \int_0^t |\gamma(s)|\,ds \\
&\le |x_0| + e^{\|h\|_{L^1}} \|\gamma\|_{L^1}
\end{aligned}$$

for all $t \in J$ and so, $\{y_n\}$ is uniformly bounded.

Again let $t, \tau \in J$. Then

$$\begin{aligned}
\left|y_n(t) - y_n(\tau)\right| &\le |x_0| \left|e^{-H(t)} - e^{-H(\tau)}\right| \\
&\quad + \left|e^{-H(t)} \int_0^t e^{H(s)} v_n(s)\,ds - e^{-H(\tau)} \int_0^\tau e^{H(s)} v_n(s)\,ds\right| \\
&\le |x_0| \left|e^{-H(t)} - e^{-H(\tau)}\right| \\
&\quad + \left|e^{-H(t)} \int_0^t e^{H(s)} v_n(s)\,ds - e^{-H(t)} \int_0^\tau e^{H(s)} v_n(s)\,ds\right| \\
&\quad + \left|e^{-H(t)} \int_0^\tau e^{H(s)} v_n(s)\,ds - e^{-H(\tau)} \int_0^\tau e^{H(s)} v_n(s)\,ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq |x_0| \left| e^{-H(t)} - e^{-H(\tau)} \right| + \left| e^{-H(t)} \right| \left| \int_\tau^t \left| e^{H(s)} v_n(s) \right| ds \right| \\
&\quad + \left| e^{-H(t)} - e^{-H(\tau)} \right| \left| \int_0^\tau \left| e^{H(s)} v_n(s) \right| ds \right| \\
&\leq \left(|x_0| + e^{\|h\|_{L^1}} \|\gamma\|_{L^1} \right) \left| e^{-H(t)} - e^{-H(\tau)} \right| + |p(t) - p(\tau)|
\end{aligned}
$$

where $p(t) = e^{\|h\|_{L^1}} \int_0^t \gamma(s)\, ds$.

Since H and p are continuous functions on a compact interval, they are uniformly continuous on J. Hence from the above inequality it follows that

$$
\left| y_n(t) - y_n(\tau) \right| \longrightarrow 0 \quad \text{as} \quad t \to \tau.
$$

This shows that $\{y_n\}$ is an equi-continuous sequence of functions in $[a,b]$. Now $\{y_n\}$ is a uniformly bounded and equi-continuous, so it has a convergent subsequence by Arzelá-Ascoli theorem. Now we apply Theorem 3.12 to yield that the operator inclusion $x \in Tx$ has a solution which corresponds to the solution of the DI (5.1) on J. This completes the proof. ■

5.2. Perturbed initial value problem

Let $\mathbb{R}$ denote the real line and let $J = [0,1]$ be a closed and bounded interval in $\mathbb{R}$. Consider the initial value problem of the first order perturbed differential inclusion (in short PDI)

$$
\begin{cases} x'(t) \in F(t, x(t)) + G(t, x(t)) \;\; a.e. \;\; t \in J, \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{5.10}
$$

where $F, G : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$.

By a solution of PDI (5.10) we mean a function $x \in AC(J, \mathbb{R})$ whose first derivative x' exists and is a member of $L^1(J, \mathbb{R})$ in $F(t,x)$, i.e., there exists a $v \in L^1(J, \mathbb{R})$ such that $v(t) \in F(t, x(t)) + G(t, x(t))$ a.e. $t \in J$, and $x'(t) = v(t)$, $t \in J$ and $x(0) = x_0 \in \mathbb{R}$, where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

Definition 5.2. A measurable multi-valued function $F : J \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h \in L^1(J, \mathbb{R})$ such that $|v| \leq h(t)$ a.e. $t \in J$ for all $v \in F(t)$.

Remark 5.2. It is known that if $F : J \to \mathcal{P}_{cl}(\mathbb{R})$ is an integrably bounded multi-function, then the set S^1_F of all Lebesgue integrable selections of F is closed and non-empty. See Hu and Papageorgiou [26].

Definition 5.3. A multi-valued map map $F : J \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for every $y \in X$, the function $t \to d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.

Definition 5.4. A multi-valued map $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t \mapsto G(t, x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $x \mapsto G(t, x)$ is upper semi-continuous for almost all $t \in J$, and

(iii) for each real number $k > 0$, there exists a function $h_k \in L^1(J, \mathbb{R})$ such that

$$\|G(t, x)\| = \sup\{|u| : u \in G(t, x)\} \leq h_k(t), \quad a.e. \;\; t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq k$.

Then we have the following lemmas due to Lasota and Opial [29].

Lemma 5.1. *If* $\dim(X) < \infty$ *and* $F : J \times X \to \mathcal{P}_{cp}(X)$ *is* L^1*-Carathéodory, then* $S^1_F(x) \neq \emptyset$ *for each* $x \in X$.

Lemma 5.2. *Let* E *be a Banach space,* F *an* L^1*-Carathéodory multi-valued map with* $S^1_F \neq \emptyset$ *and let* $\mathcal{L} : L^1(J, \mathbb{R}) \to C(J, E)$ *be a linear continuous mapping. Then the operator*

$$\begin{gathered} \mathcal{L} \circ S^1_F : C(J, E) \to \mathcal{P}_{cp,cv}(C(J, E)) \\ u \mapsto (\mathcal{L} \circ S^1_F)(x) := \mathcal{L}(S^1_F(x)) \end{gathered}$$

is a closed graph operator in $C(J, E) \times C(J, E)$.

Remark 5.3. It is known that a compact multi-valued map $T : X \to \mathcal{P}_{cp}(X)$ is upper semi-continuous if and only if it is a closed graph operator.

Definition 5.5. A function $\alpha \in AC(J, \mathbb{R})$ is called a lower solution of the PDI (5.10) if for all $v_1 \in L^1(J, \mathbb{R})$ with $v_1(t) \in F(t, \alpha(t))$ and $v_2 \in L^1(J, \mathbb{R})$ with $v_2(t) \in G(t, \alpha(t))$ a.e. $t \in J$ we have that $\alpha'(t) \leq v_1(t) + v_2(t)$ a.e.

$t \in J$ and $\alpha(0) \leq x_0$. Similarly, a function $\beta \in AC(J, \mathbb{R})$ is called an upper solution of the PDI (5.10) if for all $v_1 \in L^1(J, \mathbb{R})$ with $v_1(t) \in F(t, \beta(t))$ and $v_2 \in L^1(J, \mathbb{R})$ with $v_2(t) \in G(t, \beta(t))$ a.e. $t \in J$ we have that $\beta'(t) \geq v_1(t) + v_2(t)$ a.e. $t \in J$ and $\beta(0) \geq x_0$.

We now introduce the following hypotheses in the sequel.

(H_1) The multi-function $t \mapsto F(t, x)$ is integrally bounded for each $x \in \mathbb{R}$.

(H_2) $G : J \times \mathbb{R} \to \mathcal{P}_{cp,cv}(\mathbb{R})$ is a Carathéodory multi-function.

(H_3) The multi-function F is integrably bounded and there exists a function $\ell \in L^1(J, \mathbb{R})$ such that

$$H(G(t, x), G(t, y)) \leq \ell(t)|x - y| \quad \text{a.e.} \quad t \in J,$$

for all $, y \in \mathbb{R}$.

(H_4) The multi-valued maps $x \mapsto S^1_F(x)$ and $x \mapsto S^1_G(x)$ are right monotone increasing in $x \in \mathbb{R}$ for almost everywhere $t \in J$.

(H_5) The PDI (5.10) has a lower solution a and an upper solution b with $a \leq b$.

Theorem 5.2. *Assume that hypotheses* (H_1)–(H_2) *and* (H_4)–(H_5) *hold. Then the PDI* (5.10) *has a solution in* $[a, b]$.

Proof. Define an order interval $[a, b]$ in $AC(J, \mathbb{R})$ which does exist in view of hypothesis (H_5). Now the PDI (5.10) is equivalent to the integral inclusion

$$x(t) \in x_0 + \int_0^t F(s, x(s))\, ds + \int_0^t G(t, x(t))\, ds, \ t \in J. \tag{5.11}$$

Define two multi-valued operators $A, B : [a, b] \to AC(J, \mathbb{R})$ by

$$Ax(t) = \int_0^t F(s, x(s))\, ds, \ t \in J, \tag{5.12}$$

and

$$Bx(t) = x_0 + \int_0^t G(s, x(s))\, ds, \ t \in J. \tag{5.13}$$

Clearly, the multi-valued operators A and B are well defined on $[a, b]$ in view of hypotheses (H_1)–(H_2). We shall show that A and B satisfy all the conditions of Theorem 4.3 on $[a, b]$.

Step I. We show that A and B are compact and convex-valued multi-valued operators on $[a,b]$. First, we prove A and B has convex values on $[a,b]$. Let $u_1, u_2 \in Ax$ for some $x \in [a,b]$. Then there exist v_1 and v_2 in $S_F^1(x)$ such that

$$u_1(t) = \int_0^t v_1(s)\,ds \quad \text{and} \quad u_2(t) = \int_0^t v_2(s)\,ds$$

for $t \in J$. Then for any $\lambda \in [0,1]$, one has

$$\begin{aligned}
\lambda u_1(t) + (1-\lambda)u_2(t) = \lambda u_1(t) &= \int_0^t v_1(s)\,ds + (1-\lambda)u_2(t) = \int_0^t v_2(s)\,ds \\
&= \int_0^t \lambda v_1(s)\,ds + \int_0^t (1-\lambda)v_2(s)\,ds \\
&= \int_0^t \big[\lambda v_1(s) + (1-\lambda)v_2(s)\big]\,ds \\
&= \int_0^t v_3(s)\,ds
\end{aligned}$$

where $v_3(t) = \lambda v_1(s) + (1-\lambda)v_2(s) \in F(t,x(t))$ for all $t \in J$, because $F(t,x)$ is convex for each $(t,x) \in J \times \mathbb{R}$. Therefore the multi-valued operator A is convex-valued on $[a,b]$. Similarly, it can be shown that B is also convex-valued on $[a,b]$.

Next we show that A has compact values on $[a,b]$. Now the operator A is equivalent to the composition $\mathcal{L} \circ S_F^1$ of two operators on $L^1(J,\mathbb{R})$, where $\mathcal{L} : L^1(J,\mathbb{R}) \to X$ is a continuous operator defined by

$$\mathcal{L}v(t) = \int_0^t v(s)\,ds. \tag{5.14}$$

To show that A has compact values, it suffices to prove that the composition operator $\mathcal{L} \circ S_F^1$ has compact values on $[a,b]$. Let $x \in [a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_F^1(x)$. Then, by the definition of S_F^1, $v_n(t) \in F(t,x(t))$ a.e. for $t \in J$. Since $F(t,x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity denoted also by $v_n(t)$) that converges in measure to some $v(t)$, where $v(t) \in F(t,x(t))$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on J as $n \to \infty$.

In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let t, $\tau \in J$, then

$$
\begin{aligned}
|\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| &\le \left| \int_0^t v(s)\, ds - \int_0^\tau v(s)\, ds \right| \\
&\le \left| \int_\tau^t |v_n(s)|\, ds \right|. \qquad (5.15)
\end{aligned}
$$

The function $v_n \in L^1(J, \mathbb{R})$, so the right hand side of (5.15) tends to 0 as $t \to \tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Arzelá-Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S^1_G)(x)$ as $j \to \infty$, and so $(\mathcal{L} \circ S^1_G)(x)$ is compact. Therefore, A is a compact-valued multi-valued operator on $[a, b]$.

Similarly, define a continuous operator $\mathcal{K} : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$
\mathcal{K}v(t) = x_0 + \int_0^t v(s)\, ds. \qquad (5.16)
$$

Then the multi-valued operator B is equivalent to the composition $\mathcal{K} \circ S^1_G$ of two operators so that

$$
\begin{aligned}
Bx(t) &= x_0 + \int_0^t G(s, x(s))\, ds \\
&= \left(\mathcal{K} \circ S^1_G\right) x(t)
\end{aligned}
$$

for all $t \in J$. Now applying the arguments similar to the case of the multi-valued operator A, it can be proved that B is also a compact-valued multi-valued operator on $[a, b]$.

Step II. Next we show that A and B are right monotone increasing on $[a, b]$. Let $x, y \in [a, b]$ be such that $x \le y$ and let $u_1 \in Ax$. Then there is a $v_1 \in S^1_F(x)$ such that $u_1(t) = \int_0^t v_1(s)\, ds$. Since ($\mathrm{H}_4$) holds, there is a $v_2 \in S^1_F(y)$ such that $v_1 \le v_2$. Therefore,

$$
u_1(t) = \int_0^t v_1(s)\, ds \le \int_0^t v_2(s)\, ds = u_2
$$

for all $t \in J$, where $u_2 \in Ay$. This shows that A is right monotone increasing on $[a, b]$. Similarly, it is shown that B is also right monotone increasing on $[a,b]$. This in view of hypothesis (H_5) and Remark 4.3 further implies that A and B define the multi-valued operators $A, B : [a,b] \to \mathcal{P}_{cp,cv}(X)$ satisfying

$$Ax + By \subset \mathcal{P}_{cp,cv}([a,b])$$

for all $x, y \in [a,b]$.

Step III. Finally, we show that A and B are respectively compact and completely continuous multi-valued operators on $[a,b]$.

Let $y \in A(S)$ be arbitrary. Then following the arguments as in Step I it is proved that

$$\|y\| \leq \|h\|_{L^1} \quad \text{and} \quad |y(t) - y(\tau)| \to 0 \quad \text{as} \quad t \to \tau,$$

for all $t, \tau \in J$. This shows that $A(S)$ is a relatively compact set in $ACJ, \mathbb{R})$. As a result A is a compact multi-valued operator on $[a,b]$.

From the definition of B it follows that

$$Bx(t) = x_0 + \int_0^t G(s, x(s))\, ds = x_0 + \left(\mathcal{K} \circ S_G^1\right)(x)(t)$$

where $\mathcal{K}$ is a continuous linear operator on $L^1(J, \mathbb{R})$ into $C(J, \mathbb{R})$ defined by

$$\mathcal{K}v(t) = \int_0^t v(s)\, ds.$$

It is clear from Lemma 5.2 that $\mathcal{K} \circ S_G^1$ is a closed graph operator. Let $\{x_n\}$ be a sequence in $L^1(J, \mathbb{R})$ such that $x_n \to x_*$ as $n \to \infty$. Consider a sequence $\{y_n\}$ in $C(J, \mathbb{R})$ defined by $y_n \in x_0 + \mathcal{K} \circ S_G^1(x_n)$ for each $n \in \mathbb{N}$ sch that $y_n \to y_*$. But then $\{y_n - x_0\} \in \mathcal{K} \circ S_G^1(x_n)$ and $(y_n - x_0) \to (y_* - x_0)$. Since $\mathcal{K} \circ S_G^1$ is a closed graph operator, one has $y_* - x_0 \in \mathcal{K} \circ S_G^1(x_*)$ and consequently $y_* \in x_0 + \mathcal{K} \circ S_G^1(x_*)$. As a result B is a closed graph operator and which is further upper semi-continuous in view of Remark 5.1.

Next we show that B is compact on $[a,b]$. Let S be a subset of $[a,b]$. Since the cone K is normal in $AC(J, \mathbb{R})$, S is bounded in norm, and so there is a constant $k = \|a\| + \|b\|$ such that $\|x\| \leq k$ for all $x \in S$. To conclude, it is enough to show that $\bigcup B(S)$ is a uniformly bounded and equi-continuous set in $AC(J, \mathbb{R})$. Let $y \in \bigcup B(S)$ be arbitrary. Then there is a $v \in S_G^1(x)$

such that

$$y(t) = x_0 + \int_0^t v(s)\,ds, \quad t \in J$$

for some $x \in S$. Now by (H_2),

$$\begin{aligned}
|y(t)| &= |x_0| + \left| \int_0^t v(s)\,ds \right| \\
&\le |x_0| + \int_0^t |v(s)|\,ds \\
&\le |x_0| + \int_0^t \|G(s,x)\|\,ds \\
&\le |x_0| + \int_0^t h_k(s)\,ds \\
&\le |x_0| + \|h_k\|_{L^1}.
\end{aligned}$$

This shows that the set $\bigcup B(S)$ is uniformly bounded in $AC(J,\mathbb{R})$. Similarly, let $t, \tau \in J$. Then we have

$$\begin{aligned}
|y(t) - y(\tau)| &= \left| \int_0^t v(s)\,ds - \int_0^\tau v(s)\,ds \right| \\
&\le \left| \int_\tau^t |v(s)|\,ds \right| \\
&\le \left| \int_\tau^t \|G(s,x)\|\,ds \right| \\
&\le \left| \int_\tau^t h_k(s)\,ds \right| \\
&\le |p(t) - p(\tau)|.
\end{aligned}$$

where $p(t) = \int_0^t h_k(s)\,ds$. Since the function p is continuous on compact interval J, it is uniformly continuous, and therefore we have

$$|y(t) - y(\tau)| \to 0 \quad \text{as} \quad t \to \tau,$$

for all $y \in \bigcup B(S)$. Hence $\bigcup B(S)$ is an equi-continuous set in $AC(J,\mathbb{R})$.

Thus $\bigcup B(S)$ is a compact subset of $AC(J, \mathbb{R})$ in view of Arzelá-Ascoli theorem. Therefore B is a completely continuous multi-valued operator on $[a, b]$.

Thus A and B satisfy all the conditions of Theorem 4.3 and hence an application of it yields that the operator inclusion $x \in Ax + Bx$ has a solution in $[a, b]$. Consequently, the PDI (4.2) has a solution in $[a, b]$. This completes the proof. ■

Theorem 5.3. *Assume that hypotheses* (H_1) *and* (H_3)–(H_5) *hold. Then the PDI* (5.10) *has a solution in* $[a, b]$.

Proof. Define an order interval $[a, b]$ in $AC(J, \mathbb{R})$ and define two multi-valued operators A and B on $[a, b]$ by (5.12) and (5.13) respectively. Then proceeding as in the proof of Theorem 5.1 it is proved that A and B are right monotone increasing and compact-valued multi-valued operators satisfying

$$Ax + By \in \mathcal{P}_{cp}([a, b]).$$

It is clear that the operator A is totally bounded on $[a, b]$. To conclude we simply show that B is Lipschitz with a Lipschitz constant k satisfying $\|\ell\|_{L^1} < 1$. Let $x, y \in [a, b]$ be arbitrary and let $u_1 \in B(x)$. Then $u_1 \in [a, b]$ and

$$u_1(t) = x_0 + \int_0^t v_1(s)\, ds$$

for some $v_1 \in S_F^1(x)$. Since

$$H(G(t, x(t)), G(t, y(t)) \leq \ell(t)|x(t) - y(t)|,$$

one obtains that there exists a $w \in G(t, y(t))$ such that

$$|v_1(t) - w| \leq \ell(t)|x(t) - y(t)|.$$

Thus the multi-valued operator U defined by $U(t) = S_G^1(y)(t) \cap K(t)$, where

$$K(t) = \{w : |v_1(t) - w| \leq \ell(t)|x(t) - y(t)|\}$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U, which exists by Kuratowskii-Ryll-Nardzewski's selection theorem.

See Hu and Papageorgiou [26]. Then $v_2 \in G(t, y(t))$ and

$$|v_1(t) - v_2(t)| \leq \ell(t)|x(t) - y(t)| \text{ a.e. } \ t \in J.$$

Define

$$u_2(t) = x_0 + \int_0^t v_2(s)\, ds.$$

It follows that $u_2 \in B(y)$ and

$$\begin{aligned} |u_1(t) - u_2(t)| &\leq \left| \int_0^t v_1(s)\, ds - \int_0^t v_2(s)\, ds \right| \\ &\leq \int_0^t |v_1(s) - v_2(s)|\, ds \\ &\leq \int_0^t \ell(s)|x(s) - y(s)|\, ds \\ &\leq \|\ell\|_{L^1} \|x - y\|. \end{aligned}$$

Taking the supremum over t, we obtain

$$\|u_1 - u_2\| \leq \|\ell\|_{L^1} \|x - y\|.$$

From this and the analogous inequality obtained by interchanging the roles of x and y we get that

$$H(B(x), B(y)) \leq \|\ell\|_{L^1} \|x - y\|,$$

for all $x, y \in [a, b]$. This shows that B is a multi-valued contraction, since $\|\ell\|_{L^1} < 1$. Now the desired conclusion follows by applying of Theorem 4.12. This completes the proof. ■

6. Differential inclusions in Banach algebras

Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$ consider the first order differential inclusion (DI)

$$\begin{cases} \left(\dfrac{x(t)}{f(t, x(t))} \right)' \in G(t, x(t)) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R}^+ \end{cases} \tag{6.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is continuous and $G : J \times \mathbb{R} \to \mathcal{P}_f(\mathbb{R})$.

By a solution of the DI (6.1) we mean a function $x \in AC(J,\mathbb{R})$ such that $x' \in L^1(J,\mathbb{R})$ and $\left(\frac{x(t)}{f(t,x(t))}\right)' = v(t)$ for all $t \in J$ and $x(0) = x_0 \in \mathbb{R}^+$ for some $v \in L^1(J,\mathbb{R})$ with $v(t) \in G(t,x(t))$, a.e. $t \in J$.

The DI (6.1) has been studied in Dhage [14] for the existence theorem under the Carathéodory condition of the multi-function G. Again the special case of (6.1) in the form of differential equation (DE)

$$\begin{cases} \left(\dfrac{x(t)}{f(t,x(t))}\right)' = g(t,x(t)) \text{ a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R} \end{cases} \tag{6.2}$$

has been discussed in Dhage and Regan [18] for the existence results. In this section, we prove the existence results for the DI (6.1) under a weaker Carathéodory condition of the multi-function G. Here we do not require any continuity condition of the multi-function G.

We need the following definition in the sequel.

Definition 6.1. A function $a \in AC(J,\mathbb{R})$ is called a lower solution if $\left(\frac{a(t)}{f(t,a(t))}\right)' \leq v(t)$ for all $t \in J$ and $a(0) \leq x_0 \in \mathbb{R}^+$ for all $v \in L^1(J,\mathbb{R})$ with $v(t) \in G(t,a(t))$, a.e. $t \in J$. Similarly, a function $b \in AC(J,\mathbb{R})$ is called an upper solution of the DI (6.1) if the above inequalities hold with reverse sign.

We consider the following set of assumptions:

- (B_1) f defines the mappings $f : J \times \mathbb{R} \to \mathbb{R}^+ \setminus \{0\}$.
- (B_2) $f(t,x)$ is monotone increasing in x almost everywhere for $t \in J$.
- (B_3) G defines the multi-valued mapping $G : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R}^+)$.
- (B_4) The multi-function $G(t,x)$ is integrably bounded.
- (B_3) The multi-valued map $x \mapsto S^1_G(x)$ is right monotone increasing in $x \in \mathbb{R}$ almost everywhere for $t \in J$.
- (B_5) The DI (6.1) has a lower solution a and an upper solution b on J with $a \leq b$.

Theorem 6.1. *Assume that hypotheses* (B_1)–(B_5) *hold. Then the DI* (6.1) *has at least a solution in* $[a,b]$.

Proof. Let $X = AC(J, \mathbb{R})$ and define a norm $\|\cdot\|$ and an order relation $\leq$ in X by (5.2) and (5.3) respectively. Then X is an ordered Banach algebra with respect to the multiplication " $\cdot$ " defined by $(x.y)(t) = x(t)y(t)$ for $t \in J$. Consider the order interval $[a, b]$ in X which does exists in view of hypothesis (B_5). Define two operators $A : [a, b] \to X$ and $B : [a, b] \to \mathcal{P}_p(X)$ by

$$Ax(t) = \{f(t, x(t))\}, \quad t \in J \tag{6.3}$$

and

$$Bx(t) = \Big\{u(t) : u(t) = \frac{x_0}{f(0, x_0)} + \int_o^t v(s)\,ds, \;\; v \in S^1_G(x)\Big\}, \quad t \in J. \tag{6.4}$$

We shall show that the mappings A and B satisfy all the conditions of Theorem 4.5 on $[a, b]$.

Step I. Next we show that $AxBy$ is a convex subset of $[a, b]$ for each $x, y \in S$. Let $x, y \in S$ be arbitrary. Then there are $u, v \in S_G(x)$ such that

$$w = [f(t, x(t))]\left(\frac{x_0}{f(0, x_0)} + \int_0^t u(s)\,ds\right)$$

and

$$z = [f(t, x(t))]\left(\frac{x_0}{f(0, x_0)} + \int_0^t v(s)\,ds\right).$$

Now for any $\lambda \in [0, 1]$,

$$\begin{aligned}
\lambda w + (1-\lambda) z \; &= \; \lambda[f(t, x(t))]\left(\frac{x_0}{f(0, x_0)} + \int_0^t v(s)\,ds\right) \\
&\qquad + (1-\lambda)[f(t, x(t))]\left(\frac{x_0}{f(0, x_0)} + \int_0^t v(s)\,ds\right) \\
&= [f(t, x(t))]\left(\lambda\frac{x_0}{f(0, x_0)} + \int_0^t \lambda v(s)\,ds\right) \\
&\qquad + [f(t, x(t))]\left((1-\lambda)\frac{x_0}{f(0, x_0)} + \int_0^t (1-\lambda) v(s)\,ds\right) \\
&= [f(t, x(t))]\left(\frac{x_0}{f(0, x_0)} + \int_0^t [\lambda u(s) + (1-\lambda) v(s)]\,ds\right).
\end{aligned}$$

Since $G(t,y(t))$ is convex, $\lambda z + (1-\lambda)w \in G(t,x(t))$ for all $t \in J$ and so $\lambda z + (1-\lambda)w \in S^1_G(x)$. As a result $\lambda z + (1-\lambda)w \in AxBy$. Hence $AxBy$ is a convex subset of X.

Step II. Next we show that A and B are monotone increasing and $AxBy \subset [a,b]$ for all $x,y \in [a,b]$. It follows from hypothesis (B_1) that A and B define the mappings $A : [a,b] \to K$ and $B : [a,b] \to \mathcal{P}_{cp}(K)$. Let $x,y \in [a,b]$ be such that $x \le y$. Then by (B_3),

$$Ax(t) = f(t,x(t)) \le f(t,x(t)) = Ay(t)$$

for all $t \in J$. Hence $Ax \le Ay$. Similarly, let $u_1 \in Bx$. Then there is a $v_1 \in S^1_G(x)$ such that

$$u_1(t) = \frac{x_0}{f(0,x_0)} + \int_0^t v_1(s)\,ds, \quad t \in J.$$

Since S^1_G is right monotone increasing in $\mathbb{R}$, we have $S^1_G(x) \le S^1_G(y)$, and so there is a $v_2 \in S^1_G(y)$ such that $v_1 \le v_2$ on J. Therefore we have

$$\begin{aligned} u_1(t) &= \frac{x_0}{f(0,x_0)} + \int_0^t v_1(s)\,ds \\ &\le \frac{x_0}{f(0,x_0)} + \int_0^t v_1(s)\,ds \\ &= u_2(t) \end{aligned}$$

for all $t \in J$, where $u_2 \in By$. Thus A and B are right monotone increasing on $[a,b]$. By (B_5), $a \overset{d}{\le} AaBa$ and $AbBb \overset{i}{\le} b$. Since the cone K in X is positive, an application of Remark 4.3 yields that $AxBy \in \mathcal{P}_{cp,cv}([a,b])$ for all $x,y \in [a,b]$.

Step III. Next we show that A is completely continuous on $[a,b]$. Now the cone K in X is normal, so the order interval $[a,b]$ is norm-bounded. Hence there exists a constant $r > 0$ such that $\|x\| \le r$ for all $x \in [a,b]$. As f is continuous on compact $J \times [-r,r]$, it attains its maximum, say M.

Therefore for any subset S of $[a,b]$ we have

$$\begin{aligned}\|A(S)\| &= \sup\{\|Ax\| : x \in S\}\\ &= \sup\Big\{\sup_{t\in J}|f(t,x(t))| : x \in S\Big\}\\ &\le \sup\Big\{\sup_{t\in J}|f(t,x)| : x \in [-r,r]\Big\}\\ &\le M.\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next we note that the function $f(t,x)$ is uniformly continuous on $[0,1]\times[-r,r]$. Therefore for any $t,\tau \in [0,1]$ we have

$$|f(t,x) - f(\tau,x)| \to 0 \text{ as } t \to \tau$$

for all $x \in [-r,r]$. Similarly for any $x,y \in [-r,r]$

$$|f(t,x) - f(t,y)| \to 0 \text{ as } x \to y$$

for all $t \in [0,1]$. Hence any $t,\tau \in [0,1]$ and for any $x \in S$ one has

$$\begin{aligned}|Ax(t) - Ax(\tau)| &= |f(t,x(t)) - f(\tau,x(\tau))|\\ &\le |f(t,x(t)) - f(\tau,x(t)| + |f(\tau,x(t)) - f(\tau,x(\tau))|\\ &\to 0 \quad \text{as} \quad t \to \tau.\end{aligned}$$

This shows that $A(S)$ is an equi-continuous set in X. Now an application of the Arzela-Ascoli theorem yields that A is a completely continuous operator on $[a,b]$.

Step IV. Finally, we show that B is a compact multi-valued operator on $[a,b]$. To finish, we shall show that $B(S)$ is uniformly bounded and equi-continuous set in X for any subset S of $[a,b]$. Let $y \in B(S)$ be arbitrary. Then there is a $v \in S^1_G(x)$ such that

$$y(t) = \frac{x_0}{f(0,x_0)} + \int_0^t v(s)\,ds$$

for some $x \in S$. By hypothesis (B_2) one has

$$\begin{aligned} |y(t)| &= \left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t |v(s)|\,ds \\ &\le \left|\frac{x_0}{f(0,x_0)}\right| + \int_0^t |h(s)|\,ds \\ &\le \left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1}. \end{aligned}$$

Taking the supremum over t,

$$\|y\| \le \left|\frac{x_0}{f(0,x_0)}\right| + \|h\|_{L^1},$$

which shows that $B(S)$ is a uniformly bounded set in X. Similarly, let $t, \tau \in J$. Then for any $y \in B(S)$,

$$\begin{aligned} |y(t) - y(\tau)| &= \left|\int_0^t v(s)\,ds - \int_0^\tau v(s)\,ds\right| \\ &\le \left|\int_\tau^t |v(s)|\,ds\right| \\ &\le \left|\int_\tau^t h(s)\,ds\right| \\ &\le |p(t) - p(\tau)| \end{aligned}$$

where $p(t) = \int_0^t h(s)\,ds$. Since the function p is continuous on a compact interval J, it is uniformly continuous, and therefore we have

$$|y(t) - y(\tau)| \to 0 \quad \text{as} \quad t \to \tau,$$

for all $y \in B(S)$. Hence $B(S)$ is an equi-continuous set in X. Thus B is totally bounded in view of the Arzela-Ascoli theorem. Now an application of Corollary 4.16 yields that the operator inclusion $x \in AxBx$ and consequently the DI (6.1) has a solution in $[a, b]$. This completes the proof. ∎

7. Remarks and conclusion

In this paper, we have been able to extend the fixed point principles and the generalized iteration method of single-valued mappings given in Heikkila and Lakshmikantham [25] to monotone multi-valued mappings in ordered spaces. Unlike Heikkilä and Lakshmikantham [25] the results proved in this paper are only of existential nature and do not provide any information about the qualitative behavior of the fixed points. However, in a forthcoming paper we shall deal with discontinuous strictly monotone multi-valued mappings and prove the existence of a greatest and a least fixed point in the given order intervals in ordered Banach spaces. Finally, we also mention that the results presented here have a wide range of applications to a variety of discontinuous differential inclusions for proving the existence of solution.

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