

UPPER AND LOWER SOLUTIONS METHOD FOR PARTIAL
DISCONTINUOUS FRACTIONAL DIFFERENTIAL
INCLUSIONS WITH NOT INSTANTANEOUS IMPULSES

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Abstract

In this paper, we use the upper and lower solutions method combined with a fixed point theorem for multivalued maps in Banach algebras due to Dhage for investigations of the existence of solutions of a class of discontinuous partial differential inclusions with not instantaneous impulses. Also, we study the existence of extremal solutions under Lipschitz, Carathéodory and certain monotonicity conditions.

Keywords: fractional differential inclusion, left-sided mixed Riemann-Liouville integral, Caputo fractional order derivative, upper solution, lower solution, extremal solution, fixed point, Banach algebras, not instantaneous impulses.

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1. INTRODUCTION

The theory of fractional order differential equations and inclusions represents a powerful tool in applied mathematics to study a lot of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering. Recently, numerous research papers and monographs have appeared devoted to fractional differential equations and inclusions, for example see the monographs of Abbas *et al.* [8, 9], Kilbas *et al.* [24], the papers of Abbas *et al.* [1, 7, 10, 11], Darwish *et al.* [14, 15, 16], Diethelm [19], Kilbas and Marzan [23], Vityuk *et al.* [28, 29] and the references therein.

The method of upper and lower solutions has been successfully applied to study the existence of solutions for ordinary and partial differential equations and inclusions. See the monographs by Benchohra *et al.* [12], Heikkila and Lakshmikantham [20], Ladde *et al.* [25], the papers of Abbas and Benchohra [2, 3, 4, 5], Benchohra and Ntouyas [13] and the references therein.

Recently, in [7], Abbas *et al.* used the upper and lower solutions method to investigate the existence of solutions and extremal solutions to the following class of discontinuous fractional partial differential inclusions at fixed moments of impulse of the form

$$(1) \quad \begin{cases} {}^c D_{\theta_k}^r \left(\frac{u(x,y)}{f(x,y,u(x,y))} \right) \in G(x,y,u(x,y)); (x,y) \in J_k; k = 0, \dots, m, \\ u(x_k^+, y) = u(x_k^-, y) + \bar{I}_k(u(x_k^-, y)); y \in [0, b], k = 1, \dots, m, \\ u(x, 0) = \varphi(x); x \in [0, a], \quad u(0, y) = \psi(y); y \in [0, b], \quad \varphi(0) = \psi(0), \end{cases}$$

where $a, b > 0$, $J_0 = [0, x_1] \times [0, b]$, $J_k := (x_k, x_{k+1}] \times [0, b]$; $k = 1, \dots, m$, $\theta_k = (x_k, 0)$; $k = 0, \dots, m$, $0 = x_0 < x_1 < \dots < x_m < x_{m+1} = a$, ${}^c D_{\theta_k}^r$ is the fractional Caputo derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $G : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$, $J = [0, a] \times [0, b]$, $\mathcal{P}(\mathbb{R})$ is the class of all nonempty subsets of \mathbb{R} , $f : J \times \mathbb{R} \rightarrow \mathbb{R}^*$, $\bar{I}_k : \mathbb{R} \rightarrow \mathbb{R}$; $k = 1, \dots, m$, are given continuous functions, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\varphi : [0, a] \rightarrow \mathbb{R}$ and $\psi : [0, b] \rightarrow \mathbb{R}$ are given absolutely continuous functions.

In pharmacotherapy, the above instantaneous impulses can not describe the certain dynamics of evolution processes. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. From the viewpoint of general theories, in [6, 21, 27] the authors studied some new classes of differential equations with not instantaneous impulses.

In this paper, we use the method of upper and lower solutions for the existence of solutions of the following partial discontinuous fractional differential inclusions

with not instantaneous impulses

$$(2) \quad \begin{cases} {}^c D_{\theta_k}^r \left(\frac{u(x,y)}{f(x,y,u(x,y))} \right) \in G(x,y,u(x,y)); \text{ if } (x,y) \in I_k, \ k = 0, \dots, m, \\ u(x,y) = g_k(x,y,u(x_k^-,y)); \text{ if } (x,y) \in J_k, \ k = 1, \dots, m, \\ u(x,0) = \varphi(x); \ x \in [0,a], \\ u(0,y) = \psi(y); \ y \in [0,b], \\ \varphi(0) = \psi(0), \end{cases}$$

where $a, b > 0$, $I_0 = [0, x_1] \times [0, b]$, $I_k := (s_k, x_{k+1}] \times [0, b]$, $J_k := (x_k, s_k] \times [0, b]$, $k = 1, \dots, m$, $\theta_k = (s_k, 0)$; $k = 0, \dots, m$, $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $0 = s_0 < x_1 \leq s_1 \leq x_2 < \dots < s_{m-1} \leq x_m \leq s_m \leq x_{m+1} = a$, $G : I_k \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$; $k = 0, \dots, m$ is a compact valued multi-valued map, $f : I_k \times \mathbb{R} \rightarrow \mathbb{R}^*$, $g_k : J_k \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions, and φ, ψ are as in problem (1). Our approach in this paper is based on a combination of a fixed-point theorem for multivalued maps in Banach Algebras due to Dhage [17] with the concept of upper and lower solutions. Next, we study the existence of extremal solutions under Lipschitz, Carathéodory and certain monotonicity conditions.

This paper initiates the application of upper and lower solutions for such class of problems.

2. PRELIMINARIES

Let $J := [0, a] \times [0, b]$. Denote by $L^1(J)$ the space of Lebesgue-integrable functions $u : J \rightarrow \mathbb{R}$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b |u(x,y)| dy dx.$$

By $L^\infty(J)$ we denote the Banach space of measurable functions $u : J \rightarrow \mathbb{R}$ which are essentially bounded, equipped with the norm

$$\|u\|_{L^\infty} = \inf \{c > 0 : |u(x,y)| \leq c, \text{ a.e. } (x,y) \in J\}.$$

As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into \mathbb{R} , and $\mathcal{C} := C(J)$ is the Banach space of all continuous functions from J into \mathbb{R} with the supremum (uniform) norm $\|\cdot\|_\infty$.

In all what follows consider the Banach space

$$\mathcal{PC} := \left\{ u : J \rightarrow \mathbb{R} : u \in C \left(I_0 \cup \bigcup_{k=1}^m (x_k, x_{k+1}) \times [0, b] \right); \right. \\ \left. (x_k^-, y) = u(x_k, y), \ y \in [0, b] \right\},$$

with the norm

$$\|u\|_{\mathcal{PC}} = \sup_{(x,y) \in J} |u(x,y)|.$$

Define a multiplication “ \cdot ” by

$$(u \cdot v)(x,y) = u(x,y)v(x,y) \text{ for each } (x,y) \in J.$$

Then \mathcal{PC} is a Banach algebra with the above norm and multiplication.

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Denote $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_{bd}(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $\mathcal{P}_{cp,cv}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

Definition 2.1. A multivalued map $T : X \rightarrow \mathcal{P}(X)$ is convex (closed) valued if $T(x)$ is convex (closed) for all $x \in X$, T is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of X , and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$. T is lower semi-continuous (l.s.c.) if the set $\{t \in X : T(t) \cap B \neq \emptyset\}$ is open for any open set B in X . T is said to be completely continuous if $T(B)$ is relatively compact for every $B \in \mathcal{P}_{bd}(X)$. T has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator T will be denoted by $Fix(T)$. The graph of T will be denoted by $Graph(T) := \{(u,v) \in X \times X : v \in T(u)\}$.

Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty) \cup \{\infty\}$ given by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$, $d(a, B) = \inf_{b \in B} d(a, b)$. Then $(\mathcal{P}_{bd,cl}(X), H_d)$ is a Hausdorff metric space.

Definition 2.2. For each $u \in \mathcal{C}$, define the set of selections of the multivalued $F \circ u : J \times \mathcal{C} \rightarrow \mathcal{P}(\mathcal{C})$ by

$$S_{F \circ u} = \{v \in L^1(J) : v(x,y) \in F(x,y, u(x,y)); (x,y) \in J\}.$$

Definition 2.3. A multivalued map $G : J \rightarrow \mathcal{P}_{cl}(\mathbb{R})$, is said to be measurable if for every $v \in \mathbb{R}$ the function $(x,y) \rightarrow d(v, G(x,y)) = \inf\{|v - z| : z \in G(x,y)\}$ is measurable.

Definition 2.4. A multivalued map $G : J \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

- (i) $(x,y) \mapsto G(x,y, u)$ is measurable for each $u \in \mathbb{R}$;

(ii) $u \mapsto G(x, y, u)$ is upper semicontinuous for almost all $(x, y) \in J$.

G is said to be L^1 -Carathéodory if (i), (ii) are satisfied and the following condition holds.

(iii) For each $c > 0$, there exists a positive function $\sigma_c \in L^1(J)$ such that

$$\begin{aligned} \|G(x, y, u)\|_{\mathcal{P}} &= \sup \{|g| : g \in G(x, y, u)\} \\ &\leq \sigma_c(x, y) \text{ for all } |u| \leq c \text{ and for a.e. } (x, y) \in J. \end{aligned}$$

Lemma 2.5 [22]. *Let G be a completely continuous multivalued map with non-empty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $u_n \rightarrow u, w_n \rightarrow w, w_n \in G(u_n)$ imply $w \in G(u)$).*

Lemma 2.6 [26]. *Let X be a Banach space. Let $G : J \times X \rightarrow \mathcal{P}(X)$ be an L^1 -Carathéodory multivalued mapping with $S_{G \circ u} \neq \emptyset$, and let \mathcal{L} be a linear continuous mapping from $L^1(J, X)$ into $C(J, X)$, then the operator*

$$\mathcal{L} \circ S_{G \circ (\cdot)} : C(J, X) \rightarrow \mathcal{P}_{cp,cv}(C(J, X)),$$

$$u \mapsto \mathcal{L}(S_{G \circ u}),$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Let us recall now some basic definitions and facts on the theory of Banach algebras. Let X be a Banach algebra.

Definition 2.7. An operator $T : X \rightarrow X$ is called *compact* if $T(S)$ is a relatively compact subset of X for any $S \subset X$. Similarly $T : X \rightarrow X$ is called *totally bounded* if T maps a bounded subset of X into the relatively compact subset of X . Finally $T : X \rightarrow X$ is called *completely continuous* operator if it is continuous and totally bounded operator on X .

It is clear that every compact operator is totally bounded, but the converse may not be true.

A non-empty closed set K in a Banach algebra X is called a *cone* if

- (i) $K + K \subseteq K$,
- (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and
- (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X .

The cone K is called to be *positive* if

- (iv) $K \circ K \subseteq K$, where “ \circ ” is a multiplication composition in X .

We introduce an order relation \leq , in X as follows. Let $u, v \in X$. Then $u \leq v$ if and only if $v - u \in K$. A cone K is called to be *normal* if the norm $\|\cdot\|$ is monotone increasing on K . It is known that if the cone K is normal in X , then every order-bounded set in X is norm-bounded.

Now, we introduce notations and definitions concerning to partial fractional calculus theory.

Definition 2.8 [8, 28]. Let $\theta = (0, 0)$, $r_1, r_2 \in (0, \infty)$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t) dt ds,$$

is called the left-sided mixed Riemann-Liouville integral of order r , where $\Gamma(\cdot)$ is the (Euler's) gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt$; $\xi > 0$.

In particular,

$$(I_\theta^\theta f)(x, y) = f(x, y), \quad (I_\theta^\sigma f)(x, y) = \int_0^x \int_0^y f(s, t) dt ds; \text{ for almost all } (x, y) \in J,$$

where $\sigma = (1, 1)$.

For instance, $I_\theta^r f$ exists for all $r_1, r_2 \in (0, \infty)$, when $f \in L^1(J)$. Note also that when $u \in \mathcal{C}$, then $(I_\theta^r f) \in \mathcal{C}$, moreover

$$(I_\theta^r f)(x, 0) = (I_\theta^r f)(0, y) = 0; \quad x \in [0, a], \quad y \in [0, b].$$

Example 2.9. Let $\lambda, \omega \in (-1, 0) \cup (0, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r x^\lambda y^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} x^{\lambda+r_1} y^{\omega+r_2}; \quad \text{for almost all } (x, y) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0, 1] \times [0, 1]$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.10 [8, 28]. Let $r \in (0, 1] \times (0, 1]$ and $f \in L^1(J)$. The Caputo fractional-order derivative of order r of f is defined by the expression

$${}^c D_\theta^r f(x, y) = (I_\theta^{1-r} D_{xy}^2 f)(x, y) = \frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)} \int_0^x \int_0^y \frac{D_{st}^2 f(s, t)}{(x-s)^{r_1} (y-t)^{r_2}} dt ds.$$

The case $\sigma = (1, 1)$ is included and we have

$$({}^c D_\theta^\sigma f)(x, y) = (D_{xy}^2 f)(x, y); \quad \text{for almost all } (x, y) \in J.$$

Example 2.11. Let $\lambda, \omega \in (-1, 0) \cup (0, \infty)$ and $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, then

$${}^c D_{\theta}^r x^\lambda y^\omega = \frac{\Gamma(1 + \lambda)\Gamma(1 + \omega)}{\Gamma(1 + \lambda - r_1)\Gamma(1 + \omega - r_2)} x^{\lambda - r_1} y^{\omega - r_2}; \text{ for almost all } (x, y) \in J.$$

Let $a_1 \in [0, a]$, $z^+ = (a_1, 0) \in J$, $J_z = (a_1, a] \times [0, b]$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J_z)$, the expression

$$(I_{z^+}^r u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^x \int_0^y (x - \tau)^{r_1 - 1} (y - \xi)^{r_2 - 1} u(\tau, \xi) d\xi d\tau,$$

is called the left-sided mixed Riemann-Liouville integral of order r of u .

Definition 2.12 [8, 28]. For $u \in L^1(J_z)$ where $D_{tx}^2 u$ is Lebesgue integrable on J_z , the Caputo fractional order derivative of order r of u is defined by the expression

$$({}^c D_{z^+}^r u)(x, y) = (I_{z^+}^{1-r} D_{xy}^2 u)(x, y).$$

Set

$$\mu(x, y) = \frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{\psi(y)}{f(0, y, \psi(y))} - \frac{\varphi(0)}{f(0, 0, \varphi(0))}.$$

Lemma 2.13 [1, 8]. Let $g \in S_{G_{ou}}$. Then the Cauchy problem

$$(3) \quad \begin{cases} {}^c D_{\theta_k}^r \left(\frac{u(x, y)}{f(x, y, u(x, y))} \right) = g(x, y); \text{ if } (x, y) \in J, \\ u(x, 0) = \varphi(x); \text{ } x \in [0, a], \\ u(0, y) = \psi(y); \text{ } y \in [0, b], \\ \varphi(0) = \psi(0), \end{cases}$$

has the following unique solution

$$(4) \quad u(x, y) = f(x, y, u(x, y)) \left(\mu(x, y) + (I_{\theta_k}^r g)(x, y) \right).$$

As a consequence of the previous lemma and Lemma 3.4 in [1], we have the following Lemma

Lemma 2.14. A function $u \in \mathcal{PC}$ is a solution of problem (2) if and only if there exists $g \in S_{G_{ou}}$ such that u is a solution of the fractional integral equations

$$\left\{ \begin{array}{l} u(x, y) = f(x, y, u(x, y))[\mu(x, y) \\ + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds]; \quad \text{if } (x, y) \in I_0, \\ u(x, y) = f(x, y, u(x, y)) \left[\frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{g_k(s_k, y, u(x_k^-, y))}{f(s_k, y, u(s_k, y))} - \frac{g_k(s_k, 0, u(x_k^-, 0))}{f(s_k, 0, u(s_k, 0))} \right. \\ \left. + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds \right]; \quad \text{if } (x, y) \in I_k, \quad k = 1, \dots, m, \\ u(x, y) = g_k(x, y, u(x_k^-, y)); \quad \text{if } (x, y) \in J_k, \quad k = 1, \dots, m, \\ u(x, 0) = \varphi(x); \quad x \in [0, a], \quad u(0, y) = \psi(y); \quad y \in [0, b] \quad \text{and } \varphi(0) = \psi(0). \end{array} \right.$$

3. UPPER AND LOWER SOLUTIONS METHOD RESULT

Definition 3.1. A function $w \in \mathcal{PC}$ such that its mixed derivative D_{xy}^2 exists and is integrable on I_k ; $k = 0, \dots, m$, is said to be a solution of the problem (2) if and only if there exists $g \in S_{G \circ w}$ such that

- (i) the function $(x, y) \mapsto \frac{w(x, y)}{f(x, y, w(x, y))}$ is absolutely continuous, and
- (ii) w satisfies ${}^c D_{\theta_k}^r \left(\frac{w(x, y)}{f(x, y, w(x, y))} \right) = g(x, y)$ on I_k and the conditions

$$\left\{ \begin{array}{l} w(x, y) = g_k(x, y, w(x_k^-, y)); \quad \text{if } (x, y) \in J_k, \quad k = 1, \dots, m, \\ w(x, 0) = \varphi(x); \quad x \in [0, a], \quad w(0, y) = \psi(y); \quad y \in [0, b], \quad \varphi(0) = \psi(0), \end{array} \right.$$

are satisfied.

Definition 3.2. A function $v \in \mathcal{PC}$ such that its mixed derivative D_{xy}^2 exists and is integrable on I_k ; $k = 0, \dots, m$, is said to be a lower solution of the problem (2) if and only if there exists $g^1 \in S_{G \circ v}$ such that

- (i) the function $(x, y) \mapsto \frac{v(x, y)}{f(x, y, v(x, y))}$ is absolutely continuous, and
- (ii) v satisfies ${}^c D_{\theta_k}^r \left(\frac{v(x, y)}{f(x, y, v(x, y))} \right) \leq g^1(x, y)$ on I_k and the conditions

$$\left\{ \begin{array}{l} v(x, y) \leq g_k(x, y, v(x_k^-, y)); \quad \text{if } (x, y) \in J_k, \quad k = 1, \dots, m, \\ v(x, 0) \leq \varphi(x); \quad x \in [0, a], \quad v(0, y) \leq \psi(y); \quad y \in [0, b], \quad \varphi(0) \leq \psi(0), \end{array} \right.$$

are satisfied.

A function $w \in \mathcal{PC}$ such that its mixed derivative D_{xy}^2 exists and is integrable on I_k ; $k = 0, \dots, m$, is said to be an upper solution of the problem (2) if and only if there exists $g^2 \in S_{G \circ w}$ such that

- (i) the function $(x, y) \mapsto \frac{w(x, y)}{f(x, y, w(x, y))}$ is absolutely continuous, and
- (ii) w satisfies ${}^c D_{\theta_k}^r \left(\frac{w(x, y)}{f(x, y, w(x, y))} \right) \geq g^2(x, y)$ on I_k and the conditions

$$\begin{cases} w(x, y) \geq g_k(x, y, w(x_k^-, y)); & \text{if } (x, y) \in J_k, \quad k = 1, \dots, m, \\ w(x, 0) \geq \varphi(x); \quad x \in [0, a], \quad w(0, y) \geq \psi(y); \quad y \in [0, b], \quad \varphi(0) \geq \psi(0), \end{cases}$$

are satisfied.

We use the following fixed point theorem by Dhage [17] for proving the existence of solutions for our problem.

Theorem 3.3. *Let X be a Banach algebra, $A : X \rightarrow X$ be an operator and $B : X \rightarrow \mathcal{P}(X)$ be a multivalued operator. Assume that A and B satisfy*

- (a) A is Lipschitz with a Lipschitz constant α ,
- (b) B is compact and upper semicontinuous, and
- (c) $2M\alpha < 1$, where $M = \|B(X)\| := \sup_{u \in X} \|B(u)\|_{\mathcal{P}}$.

Then either

- (i) the operator inclusion $u \in Au Bu$ has a solution, or
- (ii) the set $\{u \in X : \lambda u \in Au Bu; \lambda > 1\}$ is unbounded.

Consider the following modified problem

$$(5) \quad \begin{cases} {}^c D_{\theta_k}^r \left(\frac{u(x, y)}{f(x, y, u(x, y))} \right) \in G(x, y, (hu)(x, y)); & (x, y) \in I_k; \quad k = 0, \dots, m, \\ u(x, y) = g_k(x, y, (hu)(x_k^-, y)); & \text{if } (x, y) \in J_k, \quad k = 1, \dots, m, \\ u(x, 0) = \varphi(x); \quad x \in [0, a], \\ u(0, y) = \psi(y); \quad y \in [0, b], \\ \varphi(0) = \psi(0), \end{cases}$$

where $h : \mathcal{PC} \rightarrow \mathcal{PC}$ is the truncation operator defined by

$$(hu)(x, y) = \begin{cases} v(x, y), & u(x, y) < v(x, y) \\ u(x, y), & v(x, y) \leq u(x, y) \leq w(x, y) \\ w(x, y), & w(x, y) < u(x, y). \end{cases}$$

A solution to (5) is a fixed point of the operator $N : \mathcal{PC} \rightarrow \mathcal{P}(\mathcal{PC})$ defined by:

$$(Nu)(x, y) = \left\{ z \in \mathcal{PC} : \begin{cases} z(x, y) = f(x, y, u(x, y)) \\ \times \left[\mu(x, y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds \right]; \\ \text{if } (x, y) \in I_0, \\ \\ z(x, y) = f(x, y, u(x, y)) \\ \times \left[\frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{g_k(s_k, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))} - \frac{g_k(s_k, 0, (hu)(x_k^-, 0))}{f(s_k, 0, u(s_k, 0))} \right] \\ + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds; \\ \text{if } (x, y) \in I_k, \quad k = 1, \dots, m, \\ \\ z(x, y) = g_k(x, y, (hu)(x_k^-, y)); \\ \text{if } (x, y) \in J_k, \quad k = 1, \dots, m, \end{cases} \right\}$$

where

$$g \in \tilde{S}_{G^1}^1 = \left\{ g \in S_{G^1}^1 : g(x, y) \geq g^1(x, y) \text{ on } A_1, \right. \\ \left. \text{and } g(x, y) \leq g^2(x, y) \text{ on } A_2 \right\},$$

$$A_1 = \{(x, y) \in I_k; \quad k = 0, \dots, m : u(x, y) < v(x, y) \leq w(x, y)\},$$

$$A_2 = \{(x, y) \in I_k; \quad k = 0, \dots, m : v(x, y) \leq w(x, y) < u(x, y)\},$$

and

$$S_{G^1}^1 = \left\{ g \in L^1(I_k) : g(x, y) \in G(x, y, (hu)(x, y)), \right. \\ \left. \text{for } (x, y) \in I_k; \quad k = 0, \dots, m \right\}.$$

The following hypotheses will be used in the sequel.

(H₁) There exists a strictly positive function $\alpha \in \mathcal{C}$ such that for each $u, \bar{u} \in \mathbb{R}$, we have

$$|f(x, y, u) - f(x, y, \bar{u})| \leq \alpha(x, y)|u - \bar{u}|; \quad (x, y) \in I_k; \quad k = 0, \dots, m,$$

(H₂) The multifunction G is L^1 -Carathéodory, and $G(x, y, u)$ has compact and convex values for each $(x, y, u) \in I_k \times \mathbb{R}; \quad k = 0, \dots, m,$

(H₃) There exist v and $w \in \mathcal{PC}$, lower and upper solutions for the problem (2) such that $v(x, y) \leq w(x, y)$ for each $(x, y) \in J,$

(H₄) For each $(x, y) \in J_k; \quad k = 1, \dots, m$ we have

$$\begin{aligned} \frac{v(x, y)}{|f(s_k, y, v(x, y))|} &\leq \min_{u \in [v, w]} \frac{g_k(x, y, h(u)(x_k^-, y))}{|f(s_k, y, u(x, y))|} \\ &\leq \max_{u \in [v, w]} \frac{g_k(x, y, h(u)(x_k^-, y))}{|f(s_k, y, u(x, y))|} \leq \frac{w(x, y)}{|f(s_k, y, w(x, y))|}. \end{aligned}$$

Remark 3.4. (A) For each $u \in \mathcal{PC}$, the set \tilde{S}_{Gohou} is nonempty. In fact, (H_2) implies that there exists $g^3 \in S_{Gohou}$. So we set

$$g = g^1 \chi_{A_1} + g^2 \chi_{A_2} + g^3 \chi_{A_3},$$

where χ_{A_i} is the characteristic function of A_i ; $i = 1, 2, 3$ and

$$A_3 = \{(x, y) \in J : v(x, y) \leq u(x, y) \leq w(x, y)\}.$$

Then, by decomposability, $g \in \tilde{S}_{Gohou}$.

(B) By the definition of h it is clear that $G(\cdot, \cdot, (hu)(\cdot, \cdot))$ is an L^1 -Carathéodory multi-valued map with compact convex values and there exists $\phi \in L^\infty(J, \mathbb{R}_+)$ such that

$$\|G(x, y, (hu)(x, y))\|_{\mathcal{P}} \leq \phi(x, y); \text{ for each } u \in \mathcal{PC} \text{ and } (x, y) \in I_k; k = 0, \dots, m.$$

(C) By the definition of h and from (H_4) , for each $(x, y) \in J_k$; $k = 1, \dots, m$, we have

$$\frac{v(x, y)}{|f(s_k, y, v(x, y))|} \leq \frac{g_k(x, y, h(u)(x_k^-, y))}{|f(s_k, y, u(x, y))|} \leq \frac{w(x, y)}{|f(s_k, y, w(x, y))|}.$$

Set

$$\beta := \max_{k=1, \dots, m} \max_{(x, y) \in J_k} \left(\left| \frac{v(x, y)}{f(s_k, y, v(x, y))} \right|, \left| \frac{w(x, y)}{f(s_k, y, w(x, y))} \right| \right).$$

Theorem 3.5. Assume that hypotheses (H_1) – (H_4) hold. If

$$(6) \quad M := \|\alpha\|_\infty \left[\|\mu\|_\infty + 2\beta + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right] < \frac{1}{2},$$

then the problem (2) has at least one solution u such that

$$v(x, y) \leq u(x, y) \leq w(x, y); \text{ for all } (x, y) \in J.$$

Proof. From Lemma 2.14 and the fact that $h(u) = u$ for all $v \leq u \leq w$, the problem of finding the solutions of (5) is reduced to finding the solutions of the inclusion $u \in N(u)$. Let $A : \mathcal{PC} \rightarrow \mathcal{PC}$ be the operator defined by

$$(7) \quad \begin{cases} (Au)(x, y) = f(x, y, u(x, y)); & \text{if } (x, y) \in I_k, \quad k = 0, \dots, m, \\ (Au)(x, y) = f(s_k, y, u(s_k, y)); & \text{if } (x, y) \in J_k, \quad k = 1, \dots, m, \end{cases}$$

and $B : \mathcal{PC} \rightarrow \mathcal{P}_{cp,cv}(\mathcal{PC})$ be the multivalued operator defined by

$$(8) \quad (Bu)(x, y) = \left\{ z \in \mathcal{PC} : \begin{cases} z(x, y) = \mu(x, y) \\ + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds; \\ \text{if } (x, y) \in I_0, \\ \\ z(x, y) = \frac{\varphi(x)}{f(x, 0, \varphi(x))} \\ + \frac{g_k(s_k, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))} - \frac{g_k(s_k, 0, (hu)(x_k^-, 0))}{f(s_k, 0, u(s_k, 0))} \\ + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds; \\ \text{if } (x, y) \in I_k, \quad k = 1, \dots, m, \\ \\ z(x, y) = \frac{g_k(x, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))}; \\ \text{if } (x, y) \in J_k, \quad k = 1, \dots, m, \end{cases} \right\}$$

where $g \in \tilde{S}_{G\text{ohou}}^1$. Clearly $(Nu)(x, y) = (Au)(x, y)(Bu)(x, y); (x, y) \in J$. Solving the problem (5) is equivalent to solving the operator inclusion

$$(9) \quad u(x, y) \in (Au)(x, y)(Bu)(x, y); (x, y) \in J.$$

We show that operators A and B satisfy all the assumptions of Theorem 3.3. The proof will be given in several steps and claims.

Step 1. A is a Lipschitz operator.

Let $u_1, u_2 \in \mathcal{PC}$. Then by (H_1) , for each $(x, y) \in J$, we have

$$\begin{aligned} |(Au_1)(x, y) - (Au_2)(x, y)| &= |f(x, y, u_1(x, y)) - f(x, y, u_2(x, y))| \\ &\leq \alpha(x, y)|u_1(x, y) - u_2(x, y)| \leq \|\alpha\|_\infty \|u_1 - u_2\|_{\mathcal{PC}}. \end{aligned}$$

Thus,

$$\|Au_1 - Au_2\|_{\mathcal{PC}} \leq \|\alpha\|_\infty \|u_1 - u_2\|_{\mathcal{PC}}.$$

Hence, A is a Lipschitz with a Lipschitz constant $\|\alpha\|_\infty$.

Step 2. B is compact and upper semicontinuous with convex values on \mathcal{PC} .

The proof of this step will be given in several claims.

Claim 1. B has convex values on \mathcal{PC} .

Let $z_1, z_2 \in B(u)$. Then there exist $g_{01}, g_{02} \in \tilde{S}_{G \circ h \circ u}^1$ such that for each $(x, y) \in I_0$ we have

$$z_l(x, y) = \mu(x, y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_{0l}(s, t) dt ds; \quad l \in \{1, 2\},$$

and for each $(x, y) \in I_k; k = 1, \dots, m$, we have

$$\begin{aligned} z_l(x, y) &= \frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{g_k(s_k, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))} - \frac{g_k(s_k, 0, (hu)(x_k^-, 0))}{f(s_k, 0, u(s_k, 0))} \\ &\quad + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_{0l}(s, t) dt ds; \quad l \in \{1, 2\}, \end{aligned}$$

and for each $(x, y) \in J_k; k = 1, \dots, m$, we have

$$z_l(x, y) = \frac{g_k(x, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))}; \quad l \in \{1, 2\}.$$

Let $0 \leq \lambda \leq 1$. Then, for each $(x, y) \in I_0$, we have

$$\begin{aligned} [\lambda z_1 + (1 - \lambda)z_2](x, y) &= \mu(x, y) \\ &\quad + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [\lambda g_{01} + (1 - \lambda)g_{02}](s, t) dt ds, \end{aligned}$$

and, for each $(x, y) \in I_k; k = 1, \dots, m$, we have

$$\begin{aligned} [\lambda z_1 + (1 - \lambda)z_2](x, y) &= \frac{\varphi(x)}{f(x, 0, \varphi(x))} \\ &\quad + \frac{g_k(s_k, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))} - \frac{g_k(s_k, 0, (hu)(x_k^-, 0))}{f(s_k, 0, u(s_k, 0))} \\ &\quad + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [\lambda g_{01} + (1 - \lambda)g_{02}](s, t) dt ds. \end{aligned}$$

Since $\tilde{S}_{G \circ h \circ u}^1$ is convex (because G has convex values), we have that

$$[\lambda z_1 + (1 - \lambda)z_2](x, y) \in B(u); \quad (x, y) \in I_k; \quad k = 0, \dots, m.$$

Also, for each $(x, y) \in J_k$; $k = 1, \dots, m$, we have

$$[\lambda z_1 + (1 - \lambda)z_2](x, y) = \frac{g_k(x, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))} \in B(u).$$

Hence

$$[\lambda z_1 + (1 - \lambda)z_2](x, y) \in B(u); (x, y) \in J.$$

Claim 2. B maps bounded sets into bounded sets of \mathcal{PC} .

Let $z \in B(u)$ for some $u \in S$, where S is a bounded set of \mathcal{PC} . Then there exists $g \in \tilde{S}_{G \circ h \circ u}^1$ such that for each $(x, y) \in I_0$

$$z(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s, t) dt ds,$$

and for each $(x, y) \in I_k$; $k = 1, \dots, m$, we have

$$\begin{aligned} z(x, y) &= \frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{g_k(s_k, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))} - \frac{g_k(s_k, 0, (hu)(x_k^-, 0))}{f(s_k, 0, u(s_k, 0))} \\ &\quad + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds, \end{aligned}$$

and for each $(x, y) \in J_k$; $k = 1, \dots, m$, we have

$$z(x, y) = \frac{g_k(x, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))}.$$

Set

$$L := \sup_{x \in [0, b]} \left| \frac{\varphi(x)}{f(x, 0, \varphi(x))} \right|.$$

From (H_2) and (H_3) , for each $(x, y) \in I_0$, we get

$$\|z\|_{\mathcal{PC}} \leq \|\mu\|_{\infty} + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \leq \|\mu\|_{\infty} + 2\beta + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)},$$

and for each $(x, y) \in I_k$; $k = 1, \dots, m$, we get

$$\|z\|_{\mathcal{PC}} \leq L + 2\beta + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \leq \|\mu\|_{\infty} + 2\beta + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)},$$

and for each $(x, y) \in J_k$; $k = 1, \dots, m$, easily we get

$$\|z\|_{\mathcal{PC}} \leq \beta \leq \|\mu\|_{\infty} + 2\beta + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}.$$

Hence, for each $(x, y) \in J$, we get

$$\|z\|_{\mathcal{PC}} \leq \|\mu\|_{\infty} + 2\beta + \frac{a^{r_1}b^{r_2}\|\phi\|_{L^{\infty}}}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{M}{\|\alpha\|_{\infty}} := \ell.$$

Claim 3. B maps bounded sets into equicontinuous sets of \mathcal{PC} .

Let $z \in B(u)$ for some $u \in S$, where S is a bounded set of \mathcal{PC} , and let $(\tau_1, y_1), (\tau_2, y_2) \in J$, with $\tau_1 < \tau_2$ and $y_1 < y_2$. Then there exists $g \in \tilde{S}_{G^1}^1$ such that for each $(x, y) \in I_0$, we have

$$\begin{aligned} & |z(\tau_2, y_2) - z(\tau_1, y_1)| \leq |\mu(\tau_1, y_1) - \mu(\tau_2, y_2)| \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_0^{y_1} [(\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} - (\tau_1 - s)^{r_1-1}(y_1 - t)^{r_2-1}] \\ & \quad \times |g(s, t)| dt ds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_{y_1}^{y_2} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} |g(s, t)| dt ds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_{y_1}^{y_2} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} |g(s, t)| dt ds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_0^{y_1} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} |g(s, t)| dt ds \\ & \leq |\mu(\tau_1, y_1) - \mu(\tau_2, y_2)| + \frac{\|\phi\|_{L^{\infty}}}{1 + \Gamma(r_1)\Gamma(1+r_2)} [2y_2^{r_2}(\tau_2 - \tau_1)^{r_1} + 2\tau_2^{r_1}(y_2 - y_1)^{r_2} \\ & \quad + \tau_1^{r_1}y_1^{r_2} - \tau_2^{r_1}y_2^{r_2} - 2(\tau_2 - \tau_1)^{r_1}(y_2 - y_1)^{r_2}] \longrightarrow 0, \text{ as } \tau_1 \rightarrow \tau_2 \text{ and } y_1 \rightarrow y_2. \end{aligned}$$

Again, for each $(x, y) \in I_k; k = 1, \dots, m$, we have

$$\begin{aligned} & |z(\tau_2, y_2) - z(\tau_1, y_1)| \leq \left| \frac{\varphi(\tau_2)}{f(\tau_2, 0, \varphi(\tau_2))} - \frac{\varphi(\tau_1)}{f(\tau_1, 0, \varphi(\tau_1))} \right| \\ & + \left| \frac{g_k(s_k, y_2, (hu)(x_k^-, y_2))}{f(s_k, y_2, u(s_k, y_2))} - \frac{g_k(s_k, y_1, (hu)(x_k^-, y_1))}{f(s_k, y_1, u(s_k, y_1))} \right| \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_0^{y_1} [(\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} - (\tau_1 - s)^{r_1-1}(y_1 - t)^{r_2-1}] \\ & \quad \times |g(s, t)| dt ds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_{y_1}^{y_2} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} |g(s, t)| dt ds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\tau_1} \int_{y_1}^{y_2} (\tau_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} |g(s, t)| dt ds \end{aligned}$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\tau_1}^{\tau_2} \int_0^{y_1} (\tau_2 - s)^{r_1-1} (y_2 - t)^{r_2-1} |g(s, t)| dt ds.$$

Thus, for each $(x, y) \in I_k$; $k = 1, \dots, m$, we get

$$\begin{aligned} |z(\tau_2, y_2) - z(\tau_1, y_1)| &\leq \left| \frac{\varphi(\tau_2)}{f(\tau_2, 0, \varphi(\tau_2))} - \frac{\varphi(\tau_1)}{f(\tau_1, 0, \varphi(\tau_1))} \right| \\ &+ \left| \frac{g_k(s_k, y_2, (hu)(x_k^-, y_2))}{f(s_k, y_2, u(s_k, y_2))} - \frac{g_k(s_k, y_1, (hu)(x_k^-, y_1))}{f(s_k, y_1, u(s_k, y_1))} \right| \\ &+ \frac{\|h\|_{L^\infty}}{1 + \Gamma(r_1)\Gamma(1+r_2)} [2y_2^{r_2}(\tau_2 - \tau_1)^{r_1} + 2\tau_2^{r_1}(y_2 - y_1)^{r_2} \\ &+ \tau_1^{r_1}y_1^{r_2} - \tau_2^{r_1}y_2^{r_2} - 2(\tau_2 - \tau_1)^{r_1}(y_2 - y_1)^{r_2}] \longrightarrow 0, \text{ as } \tau_1 \rightarrow \tau_2 \text{ and } y_1 \rightarrow y_2. \end{aligned}$$

Also, for each $(x, y) \in J_k$; $k = 1, \dots, m$, we get

$$\begin{aligned} |z(\tau_2, y_2) - z(\tau_1, y_1)| &\leq \left| \frac{g_k(\tau_2, y_2, (hu)(x_k^-, y_2))}{f(s_k, y_2, u(s_k, y_2))} - \frac{g_k(\tau_1, y_1, (hu)(x_k^-, y_1))}{f(s_k, y_1, u(s_k, y_1))} \right| \\ &\longrightarrow 0, \text{ as } \tau_1 \rightarrow \tau_2 \text{ and } y_1 \rightarrow y_2. \end{aligned}$$

As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that B is compact. Moreover, from Claim 2, we get

$$\ell = \|B(\mathcal{PC})\| \leq \frac{M}{\|\alpha\|_\infty}.$$

Then, by assumption (6), we get

$$2\ell\|\alpha\|_\infty \leq 2M < 1.$$

Step 3. B has a closed graph.

Let $u_n \rightarrow u_*$, $z_n \in B(u_n)$ and $z_n \rightarrow z_*$. We need to show that $z_* \in B(u_*)$. $z_n \in B(u_n)$ means that there exists $\tilde{g}_n \in \tilde{S}_{Gohou}^1$ such that

$$\begin{cases} z_n(x, y) = \mu(x, y) \\ + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \tilde{g}_n(s, t) dt ds; \text{ if } (x, y) \in I_0, \\ z_n(x, y) = \frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{g_k(s_k, y, (hu_n)(x_k^-, y))}{f(s_k, y, u_n(s_k, y))} - \frac{g_k(s_k, 0, (hu_n)(x_k^-, 0))}{f(s_k, 0, u_n(s_k, 0))} \\ + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \tilde{g}_n(s, t) dt ds; \text{ if } (x, y) \in I_k, k = 1, \dots, m, \\ z_n(x, y) = \frac{g_k(x, y, (hu_n)(x_k^-, y))}{f(s_k, 0, u_n(s_k, 0))}; \text{ if } (x, y) \in J_k, k = 1, \dots, m. \end{cases}$$

We must show that there exists $g_* \in \tilde{S}_{Gohou}^1$ such that,

$$\begin{cases} z_*(x, y) = \mu(x, y) \\ + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_*(s, t) dt ds; \text{ if } (x, y) \in I_0, \\ z_*(x, y) = \frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{g_k(s_k, y, (hu_*)(x_k^-, y))}{f(s_k, y, u_*(s_k, y))} - \frac{g_k(s_k, 0, (hu_*)(x_k^-, 0))}{f(s_k, 0, u_*(s_k, 0))} \\ + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_*(s, t) dt ds; \text{ if } (x, y) \in I_k, \ k = 1, \dots, m, \\ z_*(x, y) = \frac{g_k(x, y, (hu_*)(x_k^-, y))}{f(s_k, 0, u_*(s_k, 0))}; \text{ if } (x, y) \in J_k, \ k = 1, \dots, m. \end{cases}$$

Clearly, for each $(x, y) \in I_0$, we have

$$\|(z_n - \mu) - (z_* - \mu)\|_{\mathcal{PC}} = \|z_n - z_*\|_{\mathcal{PC}} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and for each $(x, y) \in I_k; k = 1, \dots, m$, we have

$$\begin{aligned} & \left| \left(z_n(x, y) - \frac{\varphi(x)}{f(x, 0, \varphi(x))} \right) - \left(z_*(x, y) - \frac{\varphi(x)}{f(x, 0, \varphi(x))} \right) \right| \\ & \leq \|z_n - z_*\|_{\mathcal{PC}} \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus we have $(z_n - \mu) \rightarrow (z_* - \mu)$, and $\left(z_n(x, y) - \frac{\varphi(x)}{f(x, 0, \varphi(x))} \right) \rightarrow \left(z_*(x, y) - \frac{\varphi(x)}{f(x, 0, \varphi(x))} \right)$ as $n \rightarrow \infty$. Now, consider the continuous linear operator $\mathcal{L} : L^1(J) \rightarrow \mathcal{PC}; g \mapsto (\mathcal{L}g)(x, y)$ such that

$$\begin{cases} (\mathcal{L}g)(x, y) = \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds; \text{ if } (x, y) \in I_0, \\ (\mathcal{L}g)(x, y) = \frac{g_k(s_k, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))} - \frac{g_k(s_k, 0, (hu)(x_k^-, 0))}{f(s_k, 0, u(s_k, 0))} \\ + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds; \text{ if } (x, y) \in I_k, \ k = 1, \dots, m, \\ (\mathcal{L}g)(x, y) = \frac{g_k(x, y, (hu)(x_k^-, y))}{f(s_k, 0, u_*(s_k, 0))}; \text{ if } (x, y) \in J_k, \ k = 1, \dots, m, \end{cases}$$

From Lemma 2.6, it follows that $\mathcal{L} \circ \tilde{S}_{Gohou}^1$ is a closed graph operator. Moreover, we have

$$(z_n(x, y) - \mu(x, y)), \left(z_n(x, y) - \frac{\varphi(x)}{f(x, 0, \varphi(x))} \right) \in \mathcal{L} \left(\tilde{S}_{Gohou}^1 \right).$$

Since $u_n \rightarrow u_*$, we have that $(z_*(x, y) - \mu(x, y)), \left(z_*(x, y) - \frac{\varphi(x)}{f(x, 0, \varphi(x))} \right) \in \mathcal{L}(\tilde{S}_{Gohou_*}^1)$. Therefore, there exists $g_* \in \tilde{S}_{Gohou_*}^1$ such that

$$\begin{cases} z_*(x, y) = \mu(x, y) \\ + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_*(s, t) dt ds; \text{ if } (x, y) \in I_0, \\ z_*(x, y) = \frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{g_k(s_k, y, (hu_*)(x_k^-, y))}{f(s_k, y, u_*(s_k, y))} - \frac{g_k(s_k, 0, (hu_*)(x_k^-, 0))}{f(s_k, 0, u_*(s_k, 0))} \\ + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g_*(s, t) dt ds; \text{ if } (x, y) \in I_k, \ k = 1, \dots, m, \\ z_*(x, y) = \frac{g_k(x, y, (hu_*)(x_k^-, y))}{f(s_k, 0, u_*(s_k, 0))}; \text{ if } (x, y) \in J_k, \ k = 1, \dots, m. \end{cases}$$

Thus the multivalued operator B has closed graph. Consequently, in view of compactness of B , it is (u.s.c.) on \mathcal{PC} .

Step 4. *The conclusion (ii) of Theorem 3.3 is not possible.*

Set

$$f^* = \sup\{|f(x, y, 0)| : (x, y) \in I_k; \ k = 1, \dots, m, \}.$$

Let $u \in \mathcal{PC}$ be any solution to (5), such that for any $\lambda > 1$ we have $\lambda u \in N(u)$. Then, there exists $g \in \tilde{S}_{G \circ h \circ u}^1$, such that

$$\begin{cases} \lambda u(x, y) = f(x, y, u(x, y)) \\ \times \left[\mu(x, y) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds \right]; \text{ if } (x, y) \in I_0, \\ \lambda u(x, y) = f(x, y, u(x, y)) \\ \times \left[\frac{\varphi(x)}{f(x, 0, \varphi(x))} + \frac{g_k(s_k, y, (hu)(x_k^-, y))}{f(s_k, y, u(s_k, y))} - \frac{g_k(s_k, 0, (hu)(x_k^-, 0))}{f(s_k, 0, u(s_k, 0))} \right. \\ \left. + \int_{s_k}^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} g(s, t) dt ds \right]; \text{ if } (x, y) \in I_k, \ k = 1, \dots, m, \\ \lambda u(x, y) = g_k(x, y, (hu)(x_k^-, y)); \text{ if } (x, y) \in J_k, \ k = 1, \dots, m. \end{cases}$$

Therefore,

$$\begin{aligned} |u(x, y)| &\leq |f(x, y, u(x, y))| \left(|\mu(x, y)| + 2\beta + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \\ &\leq [|f(x, y, u(x, y)) - f(x, y, 0)| + |f(x, y, 0)|] \\ &\quad \times \left(|\mu(x, y)| + 2\beta + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \\ &\leq [\|\alpha\|_\infty |u(x, y)| + f^*] \left(\|\mu\|_\infty + 2\beta + \frac{a^{r_1} b^{r_2} \|\phi\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right). \end{aligned}$$

Hence

$$\|u\|_{PC} \leq \frac{f^*L}{\|\alpha\|_\infty(1-L)} := M^*.$$

Thus the conclusion (ii) of Theorem 3.3 does not hold for $\lambda > 1$. Consequently, the problem (5) has a solution on J .

Step 5. *The solution u of (5) satisfies*

$$v(x, y) \leq u(x, y) \leq w(x, y); \text{ for all } (x, y) \in J.$$

Case 1. If $(x, y) \in J_k; k = 1, \dots, m$, Then from (H_4) it is clear that

$$v(x, y) \leq u(x, y) = g_k(x, y, (hu)(x_k^-, y)) \leq w(x, y); k = 1, \dots, m.$$

Case 2. Now, we prove that the solution u of (5) satisfies

$$v(x, y) \leq u(x, y) \leq w(x, y); \text{ for all } (x, y) \in I_k, k = 0, \dots, m.$$

First, we prove that

$$u(x, y) \leq w(x, y) \text{ for all } (x, y) \in I_k, k = 0, \dots, m.$$

Assume that $u - w$ attains a positive maximum on $(s_k^+, x_{k+1}^-] \times [0, b]$ at $(\bar{x}_k, \bar{y}) \in (s_k^+, x_{k+1}^-] \times [0, b]$, for some $k = 0, \dots, m$, that is,

$$(u - w)(\bar{x}_k, \bar{y}) = \max\{u(x, y) - w(x, y) : (x, y) \in (s_k^+, x_{k+1}^-] \times [0, b]\} > 0,$$

for some $k = 0, \dots, m$. There exists $(x_k^*, y^*) \in (s_k^+, x_{k+1}^-] \times [0, b]$ such that

$$\left[\frac{u(x, y^*)}{f(x, y^*, u(x, y^*))} - \frac{w(x, y^*)}{f(x, y^*, w(x, y^*))} \right] + \left[\frac{u(x_k^*, y)}{f(x_k^*, y, u(x_k^*, y))} - \frac{w(x_k^*, y)}{f(x_k^*, y, w(x_k^*, y))} \right]$$

$$(10) \quad - \left[\frac{u(x_k^*, y^*)}{f(x_k^*, y^*, u(x_k^*, y^*))} - \frac{w(x_k^*, y^*)}{f(x_k^*, y^*, w(x_k^*, y^*))} \right] \leq 0;$$

for all $(x, y) \in ([x_k^*, \bar{x}_k] \times \{y^*\}) \cup (\{x_k^*\} \times [y^*, b])$, and

$$(11) \quad \frac{u(x, y)}{f(x, y, u(x, y))} - \frac{w(x, y)}{f(x, y, w(x, y))} > 0;$$

for all $(x, y) \in (x_k^*, \bar{x}_k] \times (y^*, b]$. By the definition of g one has

$$(12) \quad {}^c D_{\theta_k}^r \left(\frac{u(x, y)}{f(x, y, u(x, y))} \right) \in G(x, y, w(x, y)); \text{ for all } (x, y) \in [x_k^*, \bar{x}_k] \times [y^*, b].$$

An integration of (12) on $[x_k^*, x] \times [y^*, y]$ for each $(x, y) \in [x_k^*, \bar{x}_k] \times [y^*, b]$ yields

$$\begin{aligned} & \frac{u(x, y)}{f(x, y, u(x, y))} + \frac{u(x_k^*, y^*)}{f(x_k^*, y^*, u(x_k^*, y^*))} - \frac{u(x, y^*)}{f(x, y^*, u(x, y^*))} - \frac{u(x_k^*, y)}{f(x_k^*, y, u(x_k^*, y))} \\ (13) \quad & = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k^*}^x \int_{y^*}^y (x-s)^{r_1-1} (y-t)^{r_2-1} g(s, t) dt ds, \end{aligned}$$

where $g(x, y) \in G(x, y, w(x, y))$. From (13) and using the fact that w is an upper solution to (2), we get

$$\begin{aligned} & \frac{u(x, y)}{f(x, y, u(x, y))} + \frac{u(x_k^*, y^*)}{f(x_k^*, y^*, u(x_k^*, y^*))} - \frac{u(x, y^*)}{f(x, y^*, u(x, y^*))} - \frac{u(x_k^*, y)}{f(x_k^*, y, u(x_k^*, y))} \\ \leq & \frac{w(x, y)}{f(x, y, w(x, y))} + \frac{w(x_k^*, y^*)}{f(x_k^*, y^*, w(x_k^*, y^*))} - \frac{w(x, y^*)}{f(x, y^*, w(x, y^*))} - \frac{w(x_k^*, y)}{f(x_k^*, y, w(x_k^*, y))}. \end{aligned}$$

Then, we get,

$$\begin{aligned} & \left[\frac{u(x, y)}{f(x, y, u(x, y))} - \frac{w(x, y)}{f(x, y, w(x, y))} \right] \leq \left[\frac{u(x, y^*)}{f(x, y^*, u(x, y^*))} - \frac{w(x, y^*)}{f(x, y^*, w(x, y^*))} \right] \\ & + \left[\frac{u(x_k^*, y)}{f(x_k^*, y, u(x_k^*, y))} - \frac{w(x_k^*, y)}{f(x_k^*, y, w(x_k^*, y))} \right] \\ (14) \quad & - \left[\frac{u(x_k^*, y^*)}{f(x_k^*, y^*, u(x_k^*, y^*))} - \frac{w(x_k^*, y^*)}{f(x_k^*, y^*, w(x_k^*, y^*))} \right]. \end{aligned}$$

Thus, from (10), (11) and (14) we obtain the following contradiction

$$\begin{aligned} 0 & < \left[\frac{u(x, y)}{f(x, y, u(x, y))} - \frac{w(x, y)}{f(x, y, w(x, y))} \right] \\ & \leq \left[\frac{u(x, y^*)}{f(x, y^*, u(x, y^*))} - \frac{w(x, y^*)}{f(x, y^*, w(x, y^*))} \right] \\ & + \left[\frac{u(x_k^*, y)}{f(x_k^*, y, u(x_k^*, y))} - \frac{w(x_k^*, y)}{f(x_k^*, y, w(x_k^*, y))} \right] \\ & - \left[\frac{u(x_k^*, y^*)}{f(x_k^*, y^*, u(x_k^*, y^*))} - \frac{w(x_k^*, y^*)}{f(x_k^*, y^*, w(x_k^*, y^*))} \right] \\ & \leq 0; \text{ for all } (x, y) \in [x_k^*, \bar{x}_k] \times [y^*, b]. \end{aligned}$$

Thus

$$u(x, y) \leq w(x, y) \text{ for all } (x, y) \in I_k, \quad k = 0, \dots, m.$$

Analogously, we can prove that

$$u(x, y) \geq v(x, y), \text{ for all } (x, y) \in I_k, k = 0, \dots, m.$$

From cases 1 and 2 we get

$$v(x, y) \leq u(x, y) \leq w(x, y), \text{ for all } (x, y) \in J.$$

This shows that the problem (5) has a solution u satisfying $v \leq u \leq w$ which is solution of (2). ■

4. EXISTENCE OF EXTREMAL SOLUTIONS

We need the following definitions and preliminary facts for proving the existence of extremal solutions for our problem

Definition 4.1. A multivalued mapping $G(x, y, w)$ is called strictly monotone increasing in w almost everywhere for $(x, y) \in J$ if $G(x, y, w) \leq G(x, y, \bar{w})$ a.e. $(x, y) \in J$, for all $w, \bar{w} \in \mathbb{R}$ with $w < \bar{w}$. Similarly, $G(x, y, w)$ is called strictly monotone decreasing in w almost everywhere for $(x, y) \in J$ if $G(x, y, w) \geq G(x, y, \bar{w})$ a.e. $(x, y) \in J$, for all $w, \bar{w} \in \mathbb{R}$ with $w > \bar{w}$.

Definition 4.2. Let X be an ordered Banach space. A multivalued operator $G : X \rightarrow \mathcal{P}_{cl}(X)$ is called strict monotone increasing if $u, v \in X$ with $u < v$, then we have that $G(u) \leq G(v)$. Similarly, G is called strict monotone decreasing if $G(u) \geq G(v)$ whenever $u < v$.

We equip the space \mathcal{PC} with the order relation \leq with the help of the cone defined by

$$K = \{u \in \mathcal{PC} : u(x, y) \geq 0; \forall (x, y) \in J\}.$$

Thus $u \leq \bar{u}$ if and only if $u(x, y) \leq \bar{u}(x, y)$ for each $(x, y) \in J$.

It is well-known that the cone K is positive and normal in \mathcal{PC} (see [20]). If $\underline{u}, \bar{u} \in C(J)$ and $\underline{u} \leq \bar{u}$, we put

$$[\underline{u}, \bar{u}] = \{u \in \mathcal{PC} : \underline{u} \leq u \leq \bar{u}\}.$$

Definition 4.3. A solution u_M of the problem (2) is said to be maximal if for any other solution u to the problem (2) one has $u(x, y) \leq u_M(x, y)$, for all $(x, y) \in J$. Again a solution u_m of the problem (2) is said to be minimal if $u_m(x, y) \leq u(x, y)$, for all $(x, y) \in J$ where u is any solution of the problem (2) on J .

Lemma 4.4. [18] *Let K be a positive cone in a real Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1 u_2 \leq v_1 v_2$.*

For any $v, w \in X, v \leq w$, the order interval $[v, w]$ is a set in X given by

$$[v, w] = \{u \in X : v \leq u \leq w\}.$$

We use the following fixed point theorem by Dhage [18] for proving the existence of extremal solutions for our problem under certain monotonicity conditions.

Theorem 4.5. *Let K be a cone in a Banach algebra X and let $v, w \in X$. Suppose that $A : [v, w] \rightarrow K$ and $B : [v, w] \rightarrow \mathcal{P}_{cl}(K)$ are two operators such that*

- (a) A is Lipschitz with a Lipschitz constant α ,
- (b) B is completely continuous,
- (c) $Au Bu \subset [v, w]$ for all $u \in [v, w]$, and
- (d) A is nondecreasing and B is strict monotone increasing on $[v, w]$,
- (e) $2M\alpha < 1$, where $M = \|B([v, w])\|$.

Further if the cone K is positive and normal, then the operator inclusion $u \in Au Bu$ has a least and a greatest positive solutions in $[v, w]$.

The following hypotheses will be used in the sequel.

(H₅) $f : J \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^*$, $\frac{\varphi(y)}{f(0, y, \varphi(y))} \geq 0$ on $[0, b]$ and

$$\frac{\varphi(x)}{f(x, 0, \varphi(x))} \geq \frac{\varphi(0)}{f(0, 0, \varphi(0))} \quad \text{for all } x \in [0, a].$$

(H₆) $f(x, y, w)$ is nondecreasing in w almost everywhere for $(x, y) \in J$,

(H₇) $G : J \times \mathbb{R}_+ \rightarrow \mathcal{P}_{cp}(\mathbb{R}_+)$,

(H₈) $G(x, y, w)$ is strictly monotone increasing in w almost everywhere for $(x, y) \in J$,

(H₉) The problem (2) has a lower solution \underline{u} and an upper solution \bar{u} with $\underline{u} \leq \bar{u}$.

Theorem 4.6. *Assume that hypotheses (H₃), (H₄) and (H₅)–(H₉) hold. If the condition (6) is satisfied, then (2) has a minimal and a maximal positive solution on J .*

Proof. Consider a closed interval $[\underline{u}, \bar{u}]$ in \mathcal{PC} . Define the operator $A : [\underline{u}, \bar{u}] \rightarrow \mathcal{PC}$ and the multivalued operator $B : [\underline{u}, \bar{u}] \rightarrow \mathcal{P}_{cp, cv}(\mathcal{PC})$ by (7) and (8), respectively. We show that operators A and B satisfy all the assumptions of Theorem 4.5. As in Theorem 3.5 we can prove that A is Lipschitz with a Lipschitz constant $\|\alpha\|_\infty$ and B is completely continuous operator on $[\underline{u}, \bar{u}]$. We shall show that A

is nondecreasing and B strictly monotone increasing on $[\underline{u}, \bar{u}]$. To see this, let $u_1, u_2 \in [\underline{u}, \bar{u}]$ be such that $u_1 \leq u_2$. Then by (H_6) , we have

$$(Au_1)(x, y) = f(x, y, u_1(x, y)) \leq f(x, y, u_2(x, y)) = (Au_2)(x, y); \text{ for all } (x, y) \in J,$$

and by (H_3) and (H_6) , we get

$$(Bu_1)(x, y) \leq (Bu_2)(x, y); \quad \text{for all } (x, y) \in J.$$

So A is nondecreasing and B strictly monotone increasing on $[\underline{u}, \bar{u}]$. By Lemma 4.4, we get

$$Au(x, y)Bu(x, y) \leq A\bar{u}(x, y)B\bar{u}(x, y) \leq \bar{u}(x, y),$$

and

$$\underline{u}(x, y) \leq A\underline{u}(x, y)B\underline{u}(x, y) \leq A\underline{u}(x, y)B\underline{u}(x, y),$$

for all $(x, y) \in J$ and $u \in [\underline{u}, \bar{u}]$. As a result

$$\underline{u}(x, y) \leq Au(x, y)Bu(x, y) \leq \bar{u}(x, y), \quad \forall (x, y) \in J \text{ and } u \in [\underline{u}, \bar{u}].$$

Hence $AuBu \in [\underline{u}, \bar{u}]$, for all $u \in [\underline{u}, \bar{u}]$. Notice for any $u \in [\underline{u}, \bar{u}]$,

$$\ell = \|B([\underline{u}, \bar{u}])\| \leq \|\mu\|_\infty + 2\beta + \frac{a^{r_1}b^{r_2}\|h\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)}.$$

Then

$$\|\alpha\|_\infty \ell \leq M < \frac{1}{2}.$$

Thus the operators A and B satisfy all the conditions of Theorem 4.5 and so the inclusion (9) has a least and a greatest solution in $[\underline{u}, \bar{u}]$. This further implies that the problem (2) has a minimal and a maximal positive solution on J . ■

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