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WEAKLY PRECOMPACT OPERATORS ON $C_b(X, E)$ WITH THE STRICT TOPOLOGY

Juliusz Stochmal

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
ul. Szafrana 4A, 65–516 Zielona Góra, Poland

e-mail: juliusz.stochmal@gmail.com

Abstract

Let X be a completely regular Hausdorff space, E and F be Banach spaces. Let $C_b(X, E)$ be the space of all E-valued bounded continuous functions on X, equipped with the strict topology β. We study weakly precompact operators $T : C_b(X, E) \to F$. In particular, we show that if X is a paracompact k-space and E contains no isomorphic copy of l^1, then every strongly bounded operator $T : C_b(X, E) \to F$ is weakly precompact.

Keywords: spaces of vector-valued continuous functions, strict topologies, operator measures, strongly bounded operators, weakly precompact operators.

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1. Introduction and terminology

Throughout the paper let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be (real or complex) Banach spaces, and let E' and F' denote the Banach duals of E and F, respectively. Let $i_F : F \to F''$ denote the canonical embedding, i.e., $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ stand for the left inverse of i_F, that is, $j_F \circ i_F = id_F$. By $B_{F'}$ we denote the closed unit ball in F'. By $\mathcal{L}(E, F)$ we denote the space of all bounded linear operators from E to F, equipped with the norm $\|\cdot\|$ of the uniform operator topology. Given a locally convex Hausdorff space (Z, ξ) by $(Z, \xi)'$ or Z'_ξ we will denote its topological dual.

Assume that $(X, \mathcal{T})$ is a completely regular Hausdorff space. By $\mathcal{K}$ we will denote the family of all compact sets in X. Let $C_b(X, E)$ stand for the Banach space of all bounded continuous functions $f : X \to E$. By τ_u we will denote the topology on $C_b(X, E)$ of the uniform norm $\|\cdot\|$. By $C_b(X, E)'$ we denote the Banach dual of $C_b(X, E)$. Let $C_b(X) \otimes E$ denote the linear space spanned by the set of all functions of the form $u \otimes x$, where $u \in C_b(X)$, $x \in E$, and $(u \otimes x)(t) = u(t)x$ for $t \in X$.

By $\mathcal{B}o$ we denote the σ-algebra of Borel sets in X. By $\mathcal{S}(\mathcal{B}o, E)$ we denote the set of all E-valued $\mathcal{B}o$-simple functions on X. Recall that a function $g : X \to E$ is said to be totally $\mathcal{B}o$-measurable if there exists a sequence (s_n) in $\mathcal{S}(\mathcal{B}o, E)$ such that $\sup_{t\in X} \|s_n(t) - g(t)\|_E \to 0$. Let $B(\mathcal{B}o, E)$ stand for the Banach space of all totally $\mathcal{B}o$-measurable functions $g : X \to E$, equipped with the uniform norm $\|\cdot\|$ (see [9, 10]). Then one can show that $C_b(X) \otimes E \subset B(\mathcal{B}o, E)$.

Recall that the *strict topology* β can be characterized as the finest locally convex topology on $C_b(X, E)$ which coincides with the compact-open topology τ_c on τ_u-bounded subsets of $C_b(X, E)$. This means that $(C_b(X, E), \beta)$ is a generalized DF-space (see [22, Corollary]) (equivalently, β coincides with the mixed topology $\gamma[\tau_u, \tau_c]$ in the sense of Wiweger (see [24] for more details). Then β is weaker than τ_u, and β and τ_u have the same bounded sets (see [15, Theorem 3.4])). Note that $\beta = \tau_u$ whenever X is compact and then we will write $C(X, E)$ instead of $C_b(X, E)$.

Recall that a subset P of a Banach space F is said to be *weakly precompact* if every bounded sequence in P contains a weakly Cauchy subsequence. A bounded linear operator $T : E \to F$ is said to be *weakly precompact* if $T(A)$ is a weakly precompact subset of a Banach space F whenever A is a bounded subset of a Banach space E (equivalently, if every bounded sequence (x_n) in E has a subsequence (x_{k_n}) such that $(T(x_{k_n}))$ is weakly Cauchy in F). If X is a compact Hausdorff space, weakly precompact operators $T : C(X, E) \to F$ have been studied by many authors; see Abbott [1], Abbott, Bator, Bilyeu and Lewis [2], Ghenciu and Lewis [13, 12], Song [21], E. Saab and P. Saab [20]. The aim of the present paper is to extend these studies to the setting of operators $T : C_b(X, E) \to F$, where X is a completely regular Hausdorff space. In Section 2 we present the basic concepts and results concerning integral representation of operators on $C_b(X, E)$. In Sections 3 we study weakly precompact operators $T : C_b(X, E) \to F$. In particular, we show that if X is a paracompact k-space and E contains no isomorphic copy of l^1, then every strongly bounded operator $T : C_b(X, E) \to F$ is weakly precompact.

2. Integral representation for operators on C$_b$(X,E)

Recall that a countably additive scalar measure ν on $\mathcal{B}o$ is said to be a Radon measure if its variation $|\nu| : \mathcal{B}o \to \mathbb{R}_+$ is regular, i.e., for each $A \in \mathcal{B}o$,

$$|\nu|(A) = \sup\{|\nu|(K) : K \in \mathcal{K}, K \subset A\} = \inf\{|\nu|(O) : O \in \mathcal{T},\ O \supset A\}.$$

By $M(X)$ we denote the space of all Radon measures.

For $\lambda \in M^+(X)$ let $\mathcal{L}^\infty(\lambda, E)$ stand for the vector space of all λ-measurable functions $g : X \to E$ such that $\|g\|_\infty := \text{ess sup}_{t\in X} \|g(t)\|_E < \infty$. By $\mathcal{L}^1(\lambda, E)$ we denote the vector space of all Bochner λ-integrable functions $g : X \to E$, equipped with the seminorm $\|g\|_1 := \int_X \|g(t)\|_E \, d\lambda$. One can show that (see [18, Proposition 5.1])

$$C_b(X,E) \subset \mathcal{L}^\infty(\lambda, E) \subset \mathcal{L}^1(\lambda, E).$$

Let $M(X,E')$ denote the space of all countably additive measures $\mu : \mathcal{B}o \to E'$ of bounded variation ($|\mu|(X) < \infty$) such that for each $x \in E$, $\mu_x \in M(X)$, where $\mu_x(A) := \mu(A)(x)$ for $A \in \mathcal{B}o$. Then $|\mu| \in M(X)$ if $\mu \in M(X,E')$ (see [16, Lemma 2.3]).

It is known that for $\mu \in M(X,E')$, every $f \in C_b(X,E)$ is μ-integrable in the Riemann-Stieltjes sense (see [18, Definition 2.2]).

The following characterization of β-continuous linear functionals on $C_b(X,E)$ will be of importance (see [18, § 2]).

Theorem 1. *For a linear functional Φ on $C_b(X,E)$ the following statements are equivalent:*

(i) *Φ is β-continuous.*

(ii) *There exists a unique $\mu \in M(X,E')$ such that*

$$\Phi(f) = \Phi_\mu(f) = \int_X f d\mu \quad \textit{for} \quad f \in C_b(X,E).$$

Moreover, $\|\Phi_\mu\| = |\mu|(X)$.

Let $C_b(X,E)''_\beta$ denote the bidual of $(C_b(X,E),\beta)$. Since β-bounded subsets of $C_b(X,E)$ are τ_u-bounded, the strong topology $\beta(C_b(X,E)'_\beta, C_b(X,E))$ in $C_b(X,E)'_\beta$ coincides with the norm topology in $C_b(X,E)'$ restricted to $C_b(X,E)'_\beta$. Hence $C_b(X,E)''_\beta = (C_b(X,E)'_\beta, \|\cdot\|)'$. Then one can embed $B(\mathcal{B}o,E)$ into $C_b(X,E)''_\beta$ by the mapping $\pi : B(\mathcal{B}o,E) \to C_b(X,E)''_\beta$, where for $g \in B(\mathcal{B}o,E)$,

$$\pi(g)(\Phi_\mu) = (I)\int_X g\, d\mu \quad \text{for} \quad \mu \in M(X,E'),$$

and $(I)\int_X g\, d\mu$ denotes the immediate integral (see [9, §6], [10, §1]). Then

$$|\pi(g)(\Phi_\mu)| = \left|(I)\int_X g\, d\mu\right| \le \|g\|\; |\mu|(X) = \|g\|\; \|\Phi_\mu\|$$

and hence π is bounded and $\|\pi(g)\| \le \|g\|$.

Assume that $T : C_b(X, E) \to F$ is a $(\beta, \|\cdot\|_F)$-continuous operator. Let $T' : F' \to C_b(X, E)'_\beta$ and $T'' : C_b(X, E)''_\beta \to F''$ stand for the conjugate and the biconjugate operators of T, respectively. Then one can define a bounded operator,

$$\hat{T} := T'' \circ \pi : B(\mathcal{B}o, E) \to F''.$$

Define a measure $m : \mathcal{B}o \to \mathcal{L}(E, F'')$ (called the *representing measure* of T) by

$$m(A)(x) := \hat{T}(\mathbb{1}_A \otimes x) \quad \text{for} \quad A \in \mathcal{B}o, \ x \in E.$$

It is well known that $\widehat{T}$ possesses a representation by the so-called immediate integral $(I) \int_X g\, dm$, developed by Dinculeanu [9, §6], [10, §1], that is,

$$\widehat{T}(g) = (I) \int_X g\, dm := \lim_n (I) \int_X s_n\, dm \quad \text{for} \quad g \in B(\mathcal{B}o, E),$$

where (s_n) is a sequence in $S(\mathcal{B}o, E)$ such that $\|s_n - g\| \to 0$. Then $\|\widehat{T}\| = \widetilde{m}(X)$, where $\widetilde{m}(A)$ stands for the semivariation of m on $A \in \mathcal{B}o$. For each $y' \in F'$, let

$$m_{y'}(A)(x) := (m(A)(x))(y') \quad \text{for} \quad A \in \mathcal{B}o, \ x \in E.$$

Then $m_{y'} \in M(X, E')$ for every $y' \in F'$ and we have (see [9, § 4, Proposition 5]),

$$\widetilde{m}(A) = \sup \{|m_{y'}|(A) : y' \in B_{F'}\}.$$

From the general properties of the operator $\widehat{T}$, we have

$$\widehat{T}(C_b(X) \otimes E) \subset i_F(F) \tag{1}$$

and

$$T(h) = j_F\Big((I) \int_X h\, dm\Big) \quad \text{for} \quad h \in C_b(X) \otimes E.$$

Moreover, according to [18, Theorem 3.1] every $f \in C_b(X, E)$ is m-integrable in the Riemann-Stieltjes sens (see [18, Definition 2.2]) with $\int_X f\, dm \in i_F(F)$ and

$$T(f) = j_F\Big(\int_X f\, dm\Big).$$

For every $x \in E$ we can define:

$$T_x(u) := T(u \otimes x) \quad \text{for} \quad u \in C_b(X) \quad \text{and} \quad m_x(A) := m(A)(x) \quad \text{for} \quad A \in \mathcal{B}o.$$

Note that if $T_x : C_b(X) \to F$ is weakly compact for every $x \in E$, then according to [18, Theorem 3.3] $m_x(A) \in i_F(F)$ for $A \in \mathcal{B}o$, $x \in E$ and one can define a measure $m_F : \mathcal{B}o \to \mathcal{L}(E, F)$ by setting

$$m_F(A)(x) := j_F(m_x(A)) \quad \text{for} \quad A \in \mathcal{B}o, x \in E.$$

Then $(m_F)_{y'} = m_{y'} \in M(X,E')$ for $y' \in F'$ and $\widetilde{m_F}(A) = \widetilde{m}(A)$ for $A \in \mathcal{B}o$.

Recall that a completely regular Hausdorff space X is said to be a *k-space* if each set which meets every compact subset in a closed set must be closed. X is a k-space, for instance if X is locally compact or first countable (see [11, Chap. 3, § 3.3]).

Let $B_{C_b(X)\otimes E}$, $B_{S(\mathcal{B}o,E)}$ and $B_{B(\mathcal{B}o,E)}$ denote the closed unit balls in $C_b(X)\otimes E$, $S(\mathcal{B}o,E)$ and $B(\mathcal{B}o,E)$, respectively. Note that $B_{S(\mathcal{B}o,E)}$ is dense in $B_{B(\mathcal{B}o,E)}$ with respect to the uniform norm $\|\cdot\|$.

The following lemma will be useful.

Lemma 2. *Assume that X is a k-space. Let $T : C_b(X,E) \to F$ be a $(\beta, \|\cdot\|_F)$-continuous operator such that $T_x : C_b(X) \to F$ is a weakly compact operator for each $x \in E$. Then the following statements hold:*

(i) $\hat{T}(B_{C_b(X)\otimes E})$ *is* $\|\cdot\|_{F''}$*-dense in* $\hat{T}(B_{B(\mathcal{B}o,E)})$.

(ii) $\hat{T}(B(\mathcal{B}o,E)) \subset i_F(F)$.

(iii) $T(C_b(X)\otimes E))$ *is* $\|\cdot\|_F$*-dense in* $j_F(\hat{T}(B(\mathcal{B}o,E)))$.

Proof. (i) Let $y'' \in \hat{T}(B_{B(\mathcal{B}o,E)})$, i.e., $y'' = \hat{T}(g)$ for some $g \in B(\mathcal{B}o,E)$ with $\|g\| \le 1$. Let $\varepsilon > 0$ be given. Then one can choose $s = \sum_{i=1}^n (1\!\!1_{A_i} \otimes x_i) \in \mathcal{S}(\mathcal{B}o,E)$ with $\|s\| \le 1$ and $\|s-g\| \le \frac{\varepsilon}{2\|\hat{T}\|}$. Hence

$$\|\hat{T}(s) - y''\|_{F''} = \|\hat{T}(s-g)\|_{F''} \le \|\hat{T}\| \cdot \|s-g\| \le \frac{\varepsilon}{2}. \tag{2}$$

Let m stand for the representing measure of T. In view of [18, Theorem 2.3] for each $i = 1,\ldots,n$ the family $\{m_{x_i,y'} : y' \in B_{F'}\}$ is uniformly countably additive, and hence by [18, Theorem 3.4] $\{m_{x_i,y'} : y' \in B_{F'}\}$ is a uniformly regular set in $M(X)$. Hence for each $i = 1,\ldots,n$ one can choose $K_i \in \mathcal{K}$ and $O_i \in \mathcal{T}$ with $K_i \subset A_i \subset O_i$ such that

$$\sup_{y' \in B_{F'}} |m_{x_i,y'}|\,(O_i \setminus K_i) \le \frac{\varepsilon}{2n}.$$

Choose $u_i \in C_b(X)$ with $0 \le u_i \le 1\!\!1_X$ such that $u_i|_{K_i} \equiv 1$ and $u_i|_{X\setminus O_i} \equiv 0$ for $i = 1,\ldots,n$. Then $\sum_{i=1}^n (u_i \otimes x_i) \in B_{C_b(X)\otimes E}$ and for $y' \in B_{F'}$ we have,

$$\begin{aligned} &\Big|\hat{T}\Big(\sum_{i=1}^n (u_i \otimes x_i)\Big)(y') - \hat{T}(s)(y')\Big| \\ &= \Big|\sum_{i=1}^n (I)\int_X (u_i \otimes x_i)\,dm_{y'} - \sum_{i=1}^n m_{x_i,y'}(A_i)\Big| \end{aligned}$$

$$\le \sum_{i=1}^{n} \left| \int_X (u_i - \mathbb{1}_{A_i})\, dm_{x_i,y'} \right| \le \sum_{i=1}^{n} \int_X |u_i - \mathbb{1}_{A_i}|\, d\,|m_{x_i,y'}|$$
$$\le \sum_{i=1}^{n} \int_{O_i \setminus K_i} \mathbb{1}_X d\,|m_{x_i,y'}| \le \sum_{i=1}^{n} |m_{x_i,y'}|\,(O_i \setminus K_i) \le \frac{\varepsilon}{2}.$$

It follows that

$$\left\| \hat{T}\Big(\sum_{i=1}^{n} (u_i \otimes x_i) \Big) - \hat{T}(s) \right\|_{F''} \le \frac{\varepsilon}{2},$$

and hence by (2) we get $\|\hat{T}(\sum_{i=1}^{n}(u_i \otimes x_i)) - y''\|_{F''} \le \varepsilon$. This means that (i) holds.

(ii) Since $\hat{T}(h) \in i_F(F)$ for each $h \in C_b(X) \otimes E$ (see (1)), in view of (i) we conclude that $\hat{T}(B(\mathcal{B}o, E)) \subset i_F(F)$.

(iii) It follows from (i) and (ii). ■

Recall that a $(\beta, \|\cdot\|_F)$-continuous linear operator $T : C_b(X, E) \to F$ is said to be *strongly bounded* if its representing measure m has the strongly bounded semivariation, i.e., $\widetilde{m}(A_n) \to 0$ whenever (A_n) is a pairwise disjoint sequence in $\mathcal{B}o$. Then $\widetilde{m}$ is strongly bounded if and only if $\{|m_{y'}| : y' \in B_{F'}\}$ is uniformly countably additive, i.e., $\sup\{|m_{y'}|(A_n) : y' \in B_{F'}\} \to 0$ whenever $A_n \downarrow \emptyset$, $(A_n) \subset \mathcal{B}o$ (see [7, Theorem 10, pp. 88–89]), i.e., $\widetilde{m}$ is *continuous at* $\emptyset$. Note that for all $x \in E$ we have, $\|m_x(A)\|_{F''} \le \widetilde{m}(A)\|x\|_E$ for $A \in \mathcal{B}o$. Hence, if $T : C_b(X, E) \to F$ is strongly bounded, then for each $x \in E$, $m_x : \mathcal{B}o \to F''$ is strongly bounded. In view of [18, Theorem 3.3], $m_x(A) \in i_F(F)$ for $A \in \mathcal{B}o$ and $T_x : C_b(X) \to F$ is a weakly compact operator.

By $M(X, \mathcal{L}(E, F))$ we denote the set of all measures $m : \mathcal{B}o \to \mathcal{L}(E, F)$ such that $\widetilde{m}(X) < \infty$ and $m_{y'} \in M(X, E')$ for $y' \in F'$.

We say that $m \in M(X, \mathcal{L}(E, F))$ has the *regular semivariation*, if for every $A \in \mathcal{B}o$ and $\varepsilon > 0$ there exist $K \in \mathcal{K}$ and $O \in \mathcal{T}$ such that $K \subset A \subset O$ and $\widetilde{m}(O \setminus K) \le \varepsilon$.

Assume that $m \in M(X, \mathcal{L}(E, F))$ has the regular semivariation and let $\lambda \in M^+(X)$ be a control measure for $\{|m_{y'}| : y' \in B_{F'}\}$. We can assume that λ to be complete (if necessary we can take the completion $(X, \overline{\mathcal{B}o}, \overline{\lambda})$ of $(X, \mathcal{B}o, \lambda)$, and for each $y' \in B_{F'}$, we extend $|m_{y'}|$ to $\overline{\mathcal{B}o}$). It is known that if $g \in \mathcal{L}^\infty(\lambda, E)$, then one can define the Radon-type integral of g with respect to m by the equation:

$$(R) \int_X g\, dm := \lim_n\, (R) \int_X s_n\, dm,$$

where (s_n) is a sequence of E-valued $\mathcal{B}o$-simple functions on X such that $\|s_n(t) - g(t)\|_E \to 0$ λ-a.e. on X and $\|s_n(t)\|_E \le \|g(t)\|_E$ λ-a.e. on X (see [17, § 1]). Then the *Radon integration operator* $T_m : \mathcal{L}^\infty(\lambda, E) \to F$ defined by the equation:

$$T_m(g) := (R) \int_X g \, dm$$

is *λ-σ-smooth*, i.e., $T_m(g_n) \to 0$ whenever (g_n) is a sequence in $\mathcal{L}^\infty(\lambda, E)$ such that $\sup_n \|g_n\|_\infty < \infty$ and $\|g_n(t)\|_E \to 0$ λ-a.e. on X (see [17, Proposition 3.5]). Note that for $y' \in F'$ and $g \in \mathcal{L}^\infty(\lambda, E)$, we have

$$y'(T_m(g)) = (R) \int_X g \, dm_{y'} \quad \text{and} \quad \left| (R) \int_X g \, dm_{y'} \right| \le \int_X \|g(t)\|_E \, d|m_{y'}|.$$

The following Riesz Representation Theorem will be of importance in the study of strongly bounded operators on $C_b(X,E)$ (see [18, Corollary 4.3 and Theorem 5.2]).

Theorem 3. *Assume that X is a k-space. Let $T : C_b(X,E) \to F$ be a strongly bounded operator with representing measure m. Then the following statements hold:*

(i) *m_F has the regular semivariation.*

(ii) *Every $f \in C_b(X,E)$ is m_F-integrable in the Riemann-Stieltjes sens and*

$$T(f) = \int_X f \, dm_F \quad \text{for} \quad f \in C_b(X,E).$$

(iii) *If $\lambda \in M^+(X)$ is a control measure for $\{|m_{y'}| : y' \in B_{F'}\}$, then T possesses a λ-σ-smooth Radon extension $T_{m_F} : \mathcal{L}^\infty(\lambda, E) \to F$ such that*

$$T(f) = T_{m_F}(f) = (R) \int_X f \, dm_F \quad \text{for } f \in C_b(X,E).$$

3. WEAKLY PRECOMPACT OPERATORS ON C_b(X,E)

Since β-bounded and τ_u-bounded sets coincide, we can formulate the definition of weakly precompact operators on $C_b(X,E)$ in the following way:

Definition. We say that a $(\beta, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ is *weakly precompact* if T maps τ_u-bounded sets in $C_b(X,E)$ onto weakly precompact sets in a Banach space F.

It is known that if X is compact, then the representing measure of a weakly precompact operator on $C(X,E)$ need not have weakly precompact values (see [1, Example 2]). We show that if a weakly precompact operator $T : C_b(X,E) \to F$ satisfies an additional condition (in particular if T is strongly bounded), then $m(A)$ is a weakly precompact operator for $A \in \mathcal{B}o$. Let us start with the following result.

Proposition 4. *Assume that X is a k-space. Let $T : C_b(X,E) \to F$ be a weakly precompact operator such that $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$. Then $\hat{T}(B(\mathcal{B}o,E)) \subset i_F(F)$ and the operator $j_F \circ \hat{T} : B(\mathcal{B}o,E) \to F$ is weakly precompact.*

Proof. Note that $\hat{T}(B(\mathcal{B}o,E)) \subset i_F(F)$ (see Lemma 2). Let (g_n) be a uniformly bounded sequence in $B(\mathcal{B}o,E)$. According to Lemma 2 and (1) we can choose a uniformly bounded sequence (h_n) in $C_b(X) \otimes E$ such that

$$\begin{aligned}\|T(h_n) - j_F(\hat{T}(g_n))\|_F &= \|j_F(\hat{T}(h_n)) - j_F(\hat{T}(g_n))\|_F \\ &= \|\hat{T}(h_n) - \hat{T}(g_n)\|_{F''} \leq \frac{1}{n}.\end{aligned}$$

Since T is weakly precompact, we can choose a subsequence (h_{k_n}) of (h_n) such that $(T(h_{k_n}))$ is a $\sigma(F,F')$-Cauchy sequence. We will show that $(j_F(\hat{T}(g_{k_n})))$ is a $\sigma(F,F')$-Cauchy sequence. Indeed, let $y' \in F'$ and $\varepsilon > 0$ be given. Then $y'(T(h_{k_n})) \to a_{y'}$ for some $a_{y'} \in \mathbb{R}$. Choose $n_o \in \mathbb{N}$ such that $\frac{1}{n_o}\|y'\|_{F'} \leq \frac{\varepsilon}{2}$ and $|y'(T(h_{k_n})) - a_{y'}| \leq \frac{\varepsilon}{2}$ for $n \geq n_o$. Then for $n \geq n_o$,

$$\begin{aligned}&|y'(j_F(\hat{T}(g_{k_n}))) - a_{y'}| \\ &\leq |y'(T(h_{k_n})) - a_{y'}| + \|j_F(\hat{T}(g_{k_n})) - j_F(\hat{T}(h_{k_n}))\|_F \cdot \|y'\|_{F'} \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This means that $j_F \circ \hat{T}$ is weakly precompact, as desired. ∎

Corollary 5. *Assume that X is a k-space. Let $T : C_b(X,E) \to F$ be a weakly precompact operator and m be its representing measure. If $T_x : C_b(X) \to F$ is weakly compact for each $x \in E$, then for each $A \in \mathcal{B}o$, $m_F(A) : E \to F$ is a weakly precompact operator.*

Proof. It follows from Proposition 4 because $m_F(A)(x) = (j_F \circ \hat{T})(\mathbb{1}_A \otimes x)$ for $A \in \mathcal{B}o$, $x \in E$. ∎

It is known that $C_b(X,E)'_\beta$ is equal to the closure of $C_b(X,E)'_{\tau_c}$ in the Banach space $(C_b(X,E)', \|\cdot\|)$ (see [6, Proposition 1]), and it follows that $(C_b(X,E)'_\beta, \|\cdot\|)$ is a Banach space. Hence the weak topology in $C_b(X,E)'_\beta$ coincides with the weak topology in $C_b(X,E)'$ restricted to $C_b(X,E)'_\beta$ (see [14, Chap. 3, §3, Corollary 3]).

Corollary 6. *Assume that X is a k-space. Let $T : C_b(X,E) \to F$ be a $(\beta, \|\cdot\|_F)$-continuous operator and m be its representing measure. If $T' : F' \to C_b(X,E)'_\beta$ is weakly precompact, then the following statements hold:*

(i) *T is strongly bounded, i.e., $\widetilde{m}$ is continuous at $\emptyset$.*

(ii) *For each $A \in \mathcal{B}o$, $m_F(A) : E \to F$ is a weakly precompact operator.*

Proof. Since $T'(B_{F'}) = \{\Phi_{m_{y'}} : y' \in B_{F'}\}$ is a weakly precompact subset of the Banach space $C_b(X,E)'_\beta$, in view of Theorem 1 $\{m_{y'} : y' \in B_{F'}\}$ is a weakly precompact subset of the Banach space $M(X,E')$. Making use of [1, Lemma 9], [12, Lemma 12], we obtain that the set $\{|m_{y'}| : y' \in B_{F'}\}$ is uniformly countably additive, i.e., (i) holds. Note that $T' : F' \to C_b(X,E)'$ is weakly precompact, and hence T is weakly precompact (see [1, Lemma 10]). According to Corollary 5 the condition (ii) is satisfied. ■

Recall that a $(\beta, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ is said to be *completely continuous* if T maps $\sigma(C_b(X,E), M(X,E'))$-Cauchy sequences in $C_b(X,E)$ onto norm convergent sequences in F (see [19]).

As a consequence of Corollary 6, using [19, Corollary 4.3], we obtain a related result to [13, Corollary 11 (iv)].

Corollary 7. *Assume that X is a k-space and E is a Schur space. If $T : C_b(X,E) \to F$ is a $(\beta, \|\cdot\|_F)$-continuous operator such that $T' : F' \to C_b(X,E)'_\beta$ is weakly precompact, then T is completely continuous.*

Recall that a $(\beta, \|\cdot\|_F)$-continuous linear operator $T : C_b(X,E) \to F$ is said to be *unconditionally converging* if the series $\sum_{n=1}^{\infty} T(f_n)$ converges unconditionally in F whenever $\sum_{n=1}^{\infty} |\int_X f_n \, d\mu| < \infty$ for all $\mu \in M(X,E')$ (see [18, § 7]).

Corollary 8. *Assume that X is a k-space. Let $T : C_b(X,E) \to F$ be a $(\beta, \|\cdot\|_F)$-continuous operator. If E' contains no isomorphic copy of l^1, then the following statements are equivalent:*

(i) *T is unconditionally converging.*

(ii) *T is strongly bounded.*

(iii) *T' is weakly precompact.*

Proof. (i)⇒(ii) See [18, Theorem 7.3].

(ii)⇒(iii) Recall that $T' : F' \to C_b(X,E)'_\beta$. Therefore $T'(y') = m_{y'} \in M(X,E')$ for all $y' \in F'$. Since T' is bounded operator, the set $\{m_{y'} : y' \in B_{F'}\}$ is bounded. On the other hand, if T is strongly bounded, then $\{|m_{y'}| : y' \in B_{F'}\}$ is uniformly countable additive. Hence, using [12, Corollary 17] we get $T'(B_{F'}) = \{\Phi_{m_{y'}} : y' \in B_{F'}\}$ is weakly precompact in $C_b(X,E)'_\beta$.

(iii)⇒(ii) See Corollary 6.

(ii)⇒(i) It is well known that if E' contains no isomorphic copy of l^1, then E'' contains no isomorphic copy of c_0. Hence, E contains no isomorphic copy of c_0 and using [18, Theorem 7.4] we get T is unconditionally converging. ■

Remark 9. If X is a compact Hausdorff space, then related results to Corollary 5 can be found in [13, Corollary 9], [20, Corollary 5], [21, Theorem 7].

We will need the following version of the Borsuk-Dugundji theorem (see [19, Theorem 3.8]).

Theorem 10. *Let K be a compact subset of a paracompact space X and let H be a separable closed subspace of the Banach space $C(K,E)$. Then there exists a linear isometric operator $S : H \to C_b(X,E)$ such that $S(h)$ is an extension of $h \in H$.*

Lemma 11. *Assume that X is a paracompact k-space. Let $m : \mathcal{B}o \to \mathcal{L}(E,F)$ have the regular semivariation $\widetilde{m}$ and let $\lambda \in M^+(X)$ be a control measure for $\{|m_{y'}| : y' \in B_{F'}\}$. Then the following statements hold:*

(i) *If (h_n) is a sequence in $\mathcal{L}^\infty(\lambda,E)$ such that $\sup_n \|h_n\|_\infty < \infty$ and $(h_n(t))$ is a $\sigma(E,E')$-Cauchy sequence for each $t \in X$, then $((R)\int_X h_n\,dm)$ is a $\sigma(F,F')$-Cauchy sequence.*

(ii) *If (g_n) is a sequence in a Banach space $L^1(\lambda,E)$ such that $\sup_n \|g_n\|_\infty < \infty$ and $g_n \to 0$ weakly in $L^1(\lambda,E)$, then $(R)\int_X g_n\,dm \to 0$ for $\sigma(F,F')$.*

Proof. (i) Let $\varepsilon > 0$ and $y'_o \in F'$ be fixed. Denote $M = \sup_n \|h_n\|_\infty$. Then there exists $\delta > 0$ such that

$$\widetilde{m}(A) = \sup\{|m_{y'}|(A) : y' \in B_{F'}\} \leq \frac{\varepsilon}{6M\|y'_o\|_{F'}} \quad \text{if} \quad \lambda(A) \leq \delta \quad \text{for} \quad A \in \mathcal{B}o.$$

Using the Luzin Type Theorem (see [5, Theorem 1.2]) for every $n \in \mathbb{N}$ there exists $K_n \in \mathcal{K}$ such that $h_n|_{K_n}$ is continuous and $\lambda(X \setminus K_n) \leq \frac{\delta}{2^n}$. Define the set $K = \bigcap_{n=1}^\infty K_n$ and the functions $g_n = h_n|_K$ for $n \in \mathbb{N}$. Since we have $\lambda(X \setminus K) \leq \delta$, then $\widetilde{m}(X \setminus K) \leq \frac{\varepsilon}{6M\|y'_o\|_{F'}}$. Let $H = [g_n]$ be the closed linear span of (g_n) in the Banach space $C(K,E)$. By Theorem 10 we can define a linear isometric extension operator $S : H \to C_b(X,E)$. Note that $(g_n(t))$ is a $\sigma(E,E')$-Cauchy sequence for each $t \in K$. Hence (g_n) is weakly Cauchy in $C(K,E)$ (see [2, Lemma 3.2]). Since S is $(\|\cdot\|,\beta)$-continuous, the sequence $(S(g_n))$ is weakly Cauchy in $(C_b(X,E),\beta)$. It follows that, there exists $n_o \in \mathbb{N}$ such that for every $n, m \in \mathbb{N}$ with $n, m \geq n_o$, we get

$$\Big|y'_o\Big((R)\int_X (S(g_n) - S(g_m))\,dm\Big)\Big| = \Big|\int_X (S(g_n) - S(g_m))\,dm_{y'_o}\Big| \leq \frac{\varepsilon}{3}.$$

Therefore,

$$\begin{aligned}
&\Big|y'_o\Big((R)\int_X h_n\,dm - (R)\int_X h_m\,dm\Big)\Big| \\
&\leq \Big|y'_o\Big((R)\int_X h_n\,dm - (R)\int_X S(g_n)\,dm\Big)\Big| \\
&+ \frac{\varepsilon}{3} + \Big|y'_o\Big((R)\int_X S(g_m)\,dm - (R)\int_X h_m\,dm\Big)\Big| \\
&\leq 4\|y'_o\|_{F'} \sup_n \|h_n\|_\infty \widetilde{m}(X \setminus K) + \frac{\varepsilon}{3} \leq \varepsilon.
\end{aligned}$$

Hence $((R)\int_X h_n\,dm)$ is a $\sigma(F,F')$-Cauchy sequence.
(ii) See [13, Theorem 1 (i)]. ∎

Now we can state our main result.

Theorem 12. *Assume that X is a paracompact k-space. If E contains no isomorphic copy of l^1, then every strongly bounded operator $T : C_b(X,E) \to F$ is weakly precompact.*

Proof. Since T is strongly bounded operator, in view of Theorem 3, for each $f \in C_b(X,E)$ we have

$$T(f) = (R)\int_X f\,dm_F = \int_X f\,dm_F,$$

where m_F has the regular semivariation. Let $\lambda \in M^+(X)$ be a control measure for $\{|m_{y'}| : y' \in B_{F'}\}$. Assume that (f_n) is a sequence in $C_b(X,E)$ such that $\sup_n \|f_n\| = M < \infty$. Then $f_n \in \mathcal{L}^\infty(\lambda,E) \subset \mathcal{L}^1(\lambda,E)$ for each $n \in \mathbb{N}$. Moreover, $\{f_n : n \in \mathbb{N}\}$ is a uniformly integrable subset of a Banach space $L^1(\lambda,E)$ because

$$\sup_n \Big\| \int_A f_n\,d\lambda \Big\|_E \le \sup_n \int_A \|f_n(t)\|_E\,d\lambda \le M\lambda(A) \quad \text{for} \quad A \in \mathcal{B}o.$$

Hence $\{f_n : n \in \mathbb{N}\}$ is a weakly precompact subset of $L^1(\lambda,E)$ (see [4]). Without loss of generality, we can suppose that (f_n) is a weakly Cauchy sequence in $L^1(\lambda,E)$. Using [23, Theorem 11], for each $n \in \mathbb{N}$ we get

$$f_n = g_n + h_n \quad \lambda - \text{a.e.},$$

where $g_n \to 0$ weakly in $L^1(\lambda,E)$ and $(h_n(t))$ is a $\sigma(E,E')$-Cauchy sequence for each $t \in X$. Moreover, from the proof of [23, Theorem 11] it follows that (g_n) and (h_n) are uniformly bounded. Then by Theorem 3,

$$T(f_n) = (R)\int_X f_n\,dm_F = (R)\int_X g_n\,dm_F + (R)\int_X h_n\,dm_F.$$

Hence, by Lemma 11, $(\,(R)\int_X f_n\,dm\,)$ is a $\sigma(F,F')$-Cauchy sequence and it follows that T is weakly precompact. ∎

Remark 13. For X being a compact Hausdorff space, a related result to Theorem 12 can be found in [13, Corollary 2].

It is well known that if E has the separable dual E', then E' has the RNP and it follows that E has no isomorphic copy of l^1.

Corollary 14. *Assume that X is a paracompact k-space. If E' is separable, then every strongly bounded operator $T : C_b(X, E) \to F$ is weakly precompact.*

As a consequence of Theorem 12 and [18, Corollary 4.5], we get:

Corollary 15. *Assume that X is a paracompact k-space. If E contains no isomorphic copy of l^1 and F contains no isomorphic copy of c_0, then every $(\beta, \|\cdot\|_F)$-continuous operator $T : C_b(X, E) \to F$ is weakly precompact.*

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