Discussiones Mathematicae
Differential Inclusions, Control and Optimization 25 (2005) 109–128

A VIABILITY RESULT FOR NONCONVEX SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS

Vasile Lupulescu

"Constantin Brâncuşi" University
Department of Mathematics
Târgu-Jiu, 210152, Romania

e-mail: vasile@utgjiu.ro

and

Mihai Necula

"Al. I. Cuza" University of Iaşi
Faculty of Mathematics
Iaşi, 700506, Romania

e-mail: necula@uaic.ro

Abstract

We establish some sufficient conditions in order that a given locally closed subset of a separable Banach space be a viable domain for a semilinear functional differential inclusion, using a tangency condition involving a semigroup generated by a linear operator.

Keywords: viability, invariance, tangency condition, semilinear differential inclusions.

2000 Mathematics Subject Classification: Primary 34A60, 49K25; Secondary 34A12, 49J27.

1. Introduction

We shall denote by X a separable Banach space and by $\mathcal{C}_\sigma := C([-\sigma, 0], X)$, with $\sigma > 0$, the Banach space of continuous functions from $[-\sigma, 0]$ into X, endowed with the norm $\|\varphi\|_\sigma := \sup\{\|\varphi(s)\|\,; s \in [-\sigma, 0]\}$. For any function $u : [\tau - \sigma, T] \to X$ and any $t \in [\tau, T]$ we shall denote by u_t the function defined as follows:

$$u_t : [-\sigma, 0] \to X,\ \ u_t(s) = u(t + s),$$

for every $s \in [-\sigma, 0]$. Clearly, if u is continuous, then $u_t \in \mathcal{C}_\sigma$ for every $t \in [\tau, T]$.

Let K be a given locally closed subset in X and let $\mathcal{K}_0$ be the following subset of $\mathcal{C}_\sigma$:

$$\mathcal{K}_0 := \{\varphi \in \mathcal{C}_\sigma; \varphi(0) \in K\}.$$

We recall that a subset $K \subset X$ is *locally closed* if for each $\xi \in K$ there exists $r > 0$ such that $K \cap B(\xi, r)$ is closed in X, where, as usual, $B(\xi, r)$ denotes the closed ball with center ξ and radius r.

We consider the following functional differential inclusion

$$u'(t) \in Au(t) + F(t, u_t), \quad t \in [a, b), \tag{1.1}$$

where $F : [a, b) \times \mathcal{C}_\sigma \to 2^X$ is a multifunction with nonempty and closed values and $A : D(A) \subset X \to X$ is the infinitesimal generator of the C_0-semigroup $S(t) : X \to X$, $t \geq 0$, and we are interested in finding sufficient conditions in order that K be a *viable domain* for (1.1), i.e. for each $(\tau, \varphi) \in [a, b) \times \mathcal{K}_0$ there exists at least one solution $u : [\tau - \sigma, T] \to K$ of (1.1) satisfying the initial condition

$$u_\tau = \varphi. \tag{1.2}$$

By a solution to the problem (1.1) and (1.2) we mean a continuous function $u : [\tau{-}\sigma, T] \to X$ for which there exists $f \in L^1([\tau, T], X)$ with $f(t) \in F(t, u_t)$ a.e. on $[\tau, T]$ and such that

$$u(t) = \begin{cases} \varphi(t - \tau) \text{ for } t \in [\tau - \sigma, \tau), \\ S(t - \tau)\varphi(0) + \displaystyle\int_\tau^t S(t - s)f(s)ds \text{ for } t \in [\tau, T]. \end{cases} \tag{1.3}$$

The existence of solutions for functional differential equations governed or not by linear and nonlinear operators in Banach spaces has been studied extensively in many papers (see, for example, [4, 9, 10, 15, 22, 23, 26, 28]).

The first viability results for (1.1) in the case $A = 0$ and F single valued have been proved in the papers [20] and [19]. The case when $A = 0$, X is a finite dimensional space and F is upper semicontinuous and with convex compact values has been studied by Haddad ([13, 14]). Haddad's result has been extended by Syam [25] and Gavioli and Malaguti [11] to the infinite dimensional setting. For results, references and applications in this framework we refer to the monographs: [1, 8, 12, 17, 18] and [24]. The case when A is the infinitesimal generator of C_0-semigroup and F is a continuous single-valued function has been studied by Iacob and Pavel [16].

There are many methods and techniques in the viability theory, but, generally speaking, the viability criteria fall into two classes: those in which the conditions are given in terms of a classical tangent cone (or Bouligand or Dini or contingent cone) and those in which a proximal normal cone is used. We shall use a tangency condition of the same kind as in [16], accordingly adapted. Also, the construction method for a sequence of approximate solutions to (1.1), defined on an apriori given interval, is closed to the one used by Cârjă and Vrabie [7] and the convergence method is the same that we have used [21].

2. Preliminaries and the main result

We assume that the reader is familiar with the basic concepts and results concerning C_0-semigroups, we refer to Vrabie [27] for details.

Let the Banach space X be endowed, with the σ-field $\mathcal{B}(X)$ of Borel subsets and let $\mathcal{I} = [a, b)$ be endowed with the Lebesgue measure and the σ-field $\mathcal{L}(\mathcal{I})$ of Lebesgue measurable subsets.

For nonempty subsets A, B of X and $a \in A$, we denote

$$d(a, B) = \inf\{\|a - b\|; b \in B\}, \quad d(A, B) = \sup\{d(a, B); a \in A\},$$

and by

$$d_{HP}(A, B) = \max\{d(A, B), d(B, A)\}$$

we denote the Hausdorff-Pompeiu distance between A and B.

Let us introduce the following hypotheses which we shall use throughout this paper.

(H_0) X is a separable Banach space, $A : D(A) \subset X \to X$ is the infinitesimal generator of the C_0-semigroup $\{S(t); t \geq 0\}$, K is a locally closed subset in X and $F : \mathcal{I} \times \mathcal{K}_0 \to 2^X$ is a multifunction with nonempty and closed values;

(H_1) For each $(\tau, \varphi) \in \mathcal{I} \times \mathcal{K}_0$ there exist $\rho > 0$, $r > 0$ and an integrable function $\chi \in L^1([\tau, \tau + \rho], \mathbb{R}_+)$ such that

$$\sup\{|F(t, \psi)|; \psi \in \mathcal{K}_0 \times B_\sigma(\varphi, r)\} \leq \chi(t) \tag{2.1}$$

a.e on $[\tau, \tau + \rho]$, where $|F(t, \varphi)| := \sup\{\|y\|; y \in F(t, \psi)\}$ and

$$B_\sigma(\varphi, r) := \{\psi \in \mathcal{C}_\sigma; \|\psi - \varphi\|_\sigma \leq r\};$$

(H_2) For each $(\tau, \varphi) \in \mathcal{I} \times \mathcal{K}_0$ there exist $\rho > 0$, $r > 0$, $\mu \in L^1([\tau, \tau + \rho], \mathbb{R}_+)$ and a negligible subset $\mathcal{Z} \subset [\tau, \tau + \rho]$ such that

$$d_{HP}(F(t, \varphi_1), F(t, \varphi_2)) \leq \mu(t)\|\varphi_1 - \varphi_2\|_\sigma \tag{2.2}$$

for every $t \in [\tau, \tau + \rho] \backslash \mathcal{Z}$ and every $\varphi_1, \varphi_2 \in \mathcal{K}_0 \times B_\sigma(\varphi, r)$;

(H_3) For each $\varphi \in \mathcal{K}_0$ the multifunction $F(\cdot, \varphi) : \mathcal{I} \to 2^X$ is measurable;

(H_4) For every $(\tau, \varphi) \in \mathcal{I} \times \mathcal{K}_0$ the following tangential condition holds:

$$\liminf_{h \downarrow 0} \frac{1}{h} d(S(h)\varphi(0) + \int_\tau^{\tau+h} S(\tau + h - s)F(s, \varphi)ds, K) = 0.$$

Here the integral is in the sense of Aumann [2].

We are now ready to state the main result of this paper.

Theorem 2.1. *If the assumptions* (H_0)–(H_4) *are satisfied, then* K *is a viable domain for* (1.1).

In order to prove our theorem we need the following technical result, concerning a measurable multifunction in Banach spaces, established by Q.I. Zhu [29].

Theorem 2.2. *Let X be a separable Banach space, $\psi : [a,b) \to X$ a measurable function and $G(\cdot) : [a,b) \to 2^X$ a measurable multifunction with nonempty and closed values. Then for any positive measurable function $\nu : [a,b) \to \mathbb{R}_+$ there exists a measurable selection $g(\cdot) \in G(\cdot)$ such that*

$$\|g(t) - \psi(t)\| \leq d(\psi(t), G(t)) + \nu(t)$$

a.e. on $[a,b)$.

In what follows, we recall a general principle on ordered sets due to Brézis and Browder [3]. It will be used in the next section in order to obtain some "maximal" elements in an ordered set.

Theorem 2.3. *Let $\preccurlyeq$ be a given preorder on the nonempty set M and let $\mathcal{S} : M \to \mathbb{R} \cup \{+\infty\}$ be an increasing function. Suppose that each increasing sequence in M is majorated in M. Then, for each $\xi_0 \in M$, there exists $\overline{\xi} \in M$ with $\xi_0 \preccurlyeq \overline{\xi}$ such that $\overline{\xi} \preccurlyeq \xi$ implies $\mathcal{S}(\overline{\xi}) = \mathcal{S}(\xi)$.*

In the paper by Brézis and Browder [3], the function $\mathcal{S}$ is supposed to be finite and bounded from above, but, as remarked in [6], this restriction can be removed by replacing the function $\mathcal{S}$ by $\xi \to \arctan \mathcal{S}(\xi)$.

Finally, let u be a function defined on the interval $\mathcal{J}$ of $\mathbb{R}$ with values into X. For some $\delta > 0$, we denote by $\omega(u, \mathcal{J}_0, \delta)$ the *modulus of continuity* of u on the subinterval $\mathcal{J}_0 \subset \mathcal{J}$, defined by

$$\omega(u, \mathcal{J}_0, \delta) = \sup\{\|u(t) - u(s)\|; t, s \in \mathcal{J}_0, |t - s| \leq \delta\}.$$

It is easy to see that $\omega(u, \cdot, \delta)$ and $\omega(u, \mathcal{J}_0, \cdot)$ are increasing functions and that u is uniformly continuous on $\mathcal{J}_0$ if and only if $\lim_{\delta \downarrow 0} \omega(u, \mathcal{J}_0, \delta) = 0$.

3. Proof of the main result

We shall show that the tangential condition (H_4) along with Brézis-Browder Ordering Principle, i.e. Theorem 2.3 above, imply that for each initial point $(\tau, \varphi) \in \mathcal{I} \times \mathcal{K}_0$, there exist $T \in (\tau, b)$ and one sequence $u^n : [\tau - \sigma, T] \to X$ of "approximate solutions" of (1.1) such that $(u^n)_n$ converges uniformly to a solution $u : [\tau - \sigma, T] \to K$ of (1.1) satisfying (1.2).

We assume that the hypotheses (H_0)–(H_4) are satisfied and we begin by fixing an arbitrary initial data $(\tau, \varphi) \in \mathcal{I} \times \mathcal{K}_0$. Since the hypotheses (H_1)

and (H_2) have a locall character and K is locally closed we can choose $r > 0$, $\rho \in (0, b-\tau)$, χ and μ in $L^1([\tau, \tau+\rho], \mathbb{R}_+)$ such that $K \cap B(\varphi(0), r)$ is closed in X and the relations (2.1) and (2.2) are satisfied on $[\tau, \tau+\rho] \times B_\sigma(\varphi, r)$. We emphasize that this choice of r, ρ, χ and μ will be kept unmodified until the end of this proof.

Remark 3.1. The following statements hold:

(i) If $\alpha \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ then $\alpha(0) \in K \cap B(\varphi(0), r)$,

(ii) If $K \cap B(\varphi(0), r)$ is closed in X then $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ is closed in C_σ.

Indeed, the first statement is obvious. For the second, let us assume that $K \cap B(\varphi(0), r)$ is closed and let us consider a sequence $(\alpha_n)_n$ in $\mathcal{K}_0 \cap B_\sigma(\varphi, r)$ that is convergent (in the norm $\|\cdot\|_\sigma$) to $\alpha \in \mathcal{C}_\sigma$. It readily follows that $\alpha \in B_\sigma(\varphi, r)$, $\alpha_n(0) \to \alpha(0)$ and $\alpha_n(0) \in K \cap B(\varphi(0), r)$, therefore, since $K \cap B(\varphi(0), r)$ is closed, we obtain that $\alpha(0) \in K$ and thus $\alpha \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$.

Since $\{S(t); t \geq 0\}$ is a C_0-semigroup, there exist $M_0 \geq 1$ and $\omega_0 \geq 0$ such that $\|S(t)\xi\| \leq M_0 e^{\omega_0 t}\|\xi\|$ for every $t \geq 0$ and for every $\xi \in X$. We define

$$M = M_0 e^{\omega_0 \rho} \tag{3.1}$$

and we have $\|S(t-\tau)\xi\| \leq M\|\xi\|$ for every $t \in [\tau, \tau+\rho]$ and every $\xi \in X$.

We shall define the "approximate solution" concept.

Definition 3.1. Let $\varepsilon \in (0, 1)$, $\nu \in (\tau, \tau+\rho]$ and $\psi \in L^1([\tau, \tau+\rho], X)$ be arbitrarily fixed.

We shall denote by (θ, β, g, f, u) a 5-tuple composed of the measurable functions $\theta : [\tau, \nu] \to [\tau, \nu]$, $\beta : \Delta_\nu = \{(t, s); \tau \leq s < t \leq \nu\} \to [0, \nu-\tau]$, $g \in L^\infty([\tau, \nu], X)$, $f \in L^1([\tau, \nu], X)$ and by the function $u : [\tau-\sigma, \nu] \to X$ defined by

$$u(t) = \begin{cases} \varphi(t-\tau) \text{ for } t \in [\tau-\sigma, \tau), \\ S(t-\tau)\varphi(0) + \displaystyle\int_\tau^t S(t-s)f(s)ds + \int_\tau^t S(\beta(t,s))g(s)ds, \ t \in [\tau, \nu]. \end{cases} \tag{3.2}$$

The 5-tuple (θ, β, g, f, u) will de called an (ε, ψ)-approximate solution of (1.1) and (1.2) on $[\tau-\sigma, \nu]$ if the following conditions are satisfied:

(A_1) $u_{\theta(t)} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ for every $t \in [\tau, \nu)$;

(A_2) $0 \le t - \theta(t)$ for every $t \in [\tau, \nu]$;

(A_3) $\beta(t, s) \le t - \tau$ for $\tau \le s < t \le \nu$ and $t \to \beta(t, s)$ is nonexpansive on $(s, \nu]$;

(A_4) $\|g(t)\| \le \varepsilon$ a.e. on $[\tau, \nu]$;

(A_5) $f(t) \in F(t, u_{\theta(t)})$ a.e. on $[\tau, \nu]$;

(A_6) $\|f(t) - \psi(t)\| \le d(\psi(t), F(t, u_{\theta(t)})) + \varepsilon\mu(t)$ a.e. on $[\tau, \nu]$;

(A_7) $\|u_t - u_{\theta(t)}\|_\sigma \le \varepsilon$ for every $t \in [\tau, \nu]$;

(A_8) $u_\nu \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$.

Remark 3.2. We emphasize that although the function u is uniquely determined by β, g and f, for the sake of simplicity, we preferred to consider it as a component of (θ, β, g, f, u).

Remark 3.3. Let $\nu \in (\tau, \tau + \rho]$, $\beta : \Delta_\nu \to [0, \nu - \tau]$, $g \in L^\infty([\tau, \nu], X)$, $f \in L^1([\tau, \nu], X)$ be given and $u : [\tau - \sigma, \nu] \to X$ be defined by (3.2). If β satisfies (A_3) then, using Lebesgue's Theorem, we deduce that the function u is continuous on the whole interval $[\tau - \sigma, \nu]$ and so $u_t \in C_\sigma$, for every $t \in [\tau, \nu]$.

In the sequel we define the operator solution to the problem (1.1) and (1.2), $Qf : L^1([\tau, \nu], X) \to C([\tau - \sigma, \nu], X)$, by

$$(Qf)(t) = \begin{cases} \varphi(t - \tau) \text{ for } t \in [\tau - \sigma, \tau), \\ S(t - \tau)\varphi(0) + \displaystyle\int_\tau^t S(t - s) f(s) ds \text{ for } t \in [\tau, \nu]. \end{cases} \tag{3.3}$$

We notice that u is a solution of (1.1) and (1.2) on $[\tau - \sigma, T]$ if there exists $f \in L^1([\tau, T], X)$ such that $u = Qf$ and $f(t) \in F(t, u_t)$ a.e. on $[\tau, T]$.

Remark 3.4. Let $\nu \in (\tau, \tau + \rho]$, $\beta : \Delta_\nu \to [0, \nu - \tau]$, $g \in L^\infty([\tau, \nu], X)$, $f \in L^1([\tau, \nu], X)$ be given and $u : [\tau - \sigma, \nu] \to X$ be defined by (3.2). If $\|f(t)\| \le \chi(t)$ and $\|g(t)\| \le 1$ a.e. on $[\tau, \nu]$, then we have

$$
\begin{aligned}
\|u_\nu - u_\tau\|_\sigma &\leq \omega(\varphi, [-\sigma, 0], \nu - \tau) + M \sup_{0 \leq h \leq \nu - \tau} \|S(h)\varphi(0) - \varphi(0)\| \\
&\quad + 2M \int_\tau^\nu \chi(s)ds + 2M(\nu - \tau).
\end{aligned} \tag{3.4}
$$

Indeed, for every $t, s \in [\tau, \nu]$ we have

$$
\begin{aligned}
&\|u_t - u_s\|_\sigma = \sup_{\alpha \in [-\sigma, 0]} \|u_t(\alpha) - u_s(\alpha)\| \\
&= \sup_{\alpha \in [-\sigma, 0]} \|u(t + \alpha) - u(s + \alpha)\| \leq \omega(u, [\tau - \sigma, \nu], |t - s|) \\
&\leq \omega(u, [\tau - \sigma, \tau], |t - s|) + \omega(u, [\tau, \nu], |t - s|).
\end{aligned}
$$

Since $u_\tau = \varphi$ we get $\omega(u, [\tau - \sigma, \tau], |t - s|) = \omega(\varphi, [-\sigma, 0], |t - s|)$ and so

$$
\|u_t - u_s\|_\sigma \leq \omega(\varphi, [-\sigma, 0], |t - s|) + \omega(u, [\tau, \nu], |t - s|).
$$

From the definition of u on $[\tau, \nu]$ we obtain

$$
\omega(u, [\tau, \nu], \delta) \leq \omega(Qf, [\tau, \nu], \delta) + 2M(\nu - \tau)\|g\|_\infty
$$

and therefore we have

$$
\|u_t - u_s\|_\sigma \leq \omega(\varphi, [-\sigma, 0], |t - s|) + \omega(Qf, [\tau, \nu], |t - s|) + 2M(\nu - \tau)\|g\|_\infty, \tag{3.5}
$$

for every $t, s \in [\tau, \nu]$. Consequently, using the estimate

$$
\omega(Qf, [\tau, \nu], \delta) \leq M \sup_{0 \leq h \leq \delta} \|S(h)\varphi(0) - \varphi(0)\| + 2M \int_\tau^\nu \chi(s)ds,
$$

we get (3.4).

Remark 3.5. Let us consider $\overline{f} \in L^1([\tau, \overline{\nu}], X)$, $\nu \in (\tau, \overline{\nu})$ and $f = \overline{f}|_{[\tau,\nu]}$. Since $(Qf)(t)$ depends only on the values of f on the interval $[\tau, \nu]$, we deduce that $Qf = (Q\overline{f})|_{[\tau,\nu]}$ and therefore

$$
\omega(Qf, [\tau, \nu], \delta) = \omega(Q\overline{f}, [\tau, \nu], \delta) \leq \omega(Q\overline{f}, [\tau, \overline{\nu}], \delta), \text{ for every } \delta \geq 0.
$$

In the next lemma we show how to choose $T \in (\tau, \tau+\rho]$ and how to construct, for every $\varepsilon \in (0,1)$ and every $\psi \in L^\infty([\tau, \tau+\rho], X)$, an (ε, ψ)-approximate solution on $[\tau-\sigma, T]$.

Lemma 3.1. *Assume that the hypotheses* (H_0)–(H_4) *are satisfied. Then there exists* $T \in (\tau, \tau+\rho]$ *with* $\int_\tau^T \mu(s)ds \le 1/2$ *such that for every* $\varepsilon \in (0,1)$ *and every* $\psi \in L^\infty([\tau, \tau+\rho], X)$ *the problem* (1.1) *and* (1.2) *have at least one* (ε, ψ)*-approximate solution on* $[\tau-\sigma, T]$.

Proof. We fix $T \in (\tau, \tau+\rho]$ such that

$$\begin{aligned} &\omega(\varphi, [-\sigma, 0], T-\tau) + M \sup_{0 \le h \le T-\tau} \|S(h)\varphi(0) - \varphi(0)\| \\ &+ 2M \int_\tau^T \chi(s)ds + 2M(T-\tau) \le r \end{aligned} \tag{3.6}$$

and

$$\int_\tau^T \mu(s)ds \le 1/2. \tag{3.7}$$

We denote by $\mathcal{M}_T$ the set of all (ε, ψ)-approximate solutions (θ, β, g, f, u) on $[\tau-\sigma, \nu] \subset [\tau-\sigma, T]$ and we begin by proving that $\mathcal{M}_T$ is a nonempty set. Applying Theorem 2.2 to $G(\cdot) = F(\cdot, \varphi)$ on $[\tau, T]$ we obtain that there exists a measurable selection $\overline{f} : [\tau, T] \to X$ such that $\overline{f}(t) \in F(t, \varphi)$ a.e. on $[\tau, T]$ and

$$\|\bar{f}(t) - \psi(t)\| \le d(\psi(t), F(t, \varphi)) + \varepsilon\mu(t) \text{ a.e. on } [\tau, T].$$

Moreover, from (H_1) we obtain that $\|\overline{f}(t)\| \le \chi(t)$ a.e. on $[\tau, T]$ and therefore $\bar{f} \in L^1([\tau, T], X)$. Using the tangential condition (H_4) for $(\tau, \varphi) \in \mathcal{I} \times \mathcal{K}_0$ we obtain that there exist $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$ and $(q_n)_n$ in X with $q_n \to 0$ such that

$$S(h_n)\varphi(0) + \int_\tau^{\tau+h_n} S(\tau+h_n-s)\bar{f}(s)ds + h_n q_n \in K, \tag{3.8}$$

for every $n \in \mathbb{N}$. We can fix $n_0 \in \mathbb{N}$ such that $h_{n_0} \in (0, T-\tau]$ and $\|q_{n_0}\| \le \varepsilon$. This choice is possible because

$$\lim_{\delta \downarrow 0} \omega(\varphi, [-\sigma, 0], \delta) = 0 \text{ and} \lim_{\delta \downarrow 0} \omega(Q\bar{f}, [\tau, T], \delta) = 0.$$

For n_0 fixed as above, we define: $\nu_0 := \tau + h_{n_0}$, $\theta(t) := \tau$ for every $t \in [\tau, \nu_0]$, $\beta(t,s) = 0$ for every $(t,s) \in \Delta_{\nu_0}$, $g(t) := q_{n_0}$ and $f(t) := \bar{f}(t)$ a.e. on $[\tau, \nu_0]$ and we show that (θ, β, g, f, u), with u defined by (3.2), is an (ε, ψ)-approximate solution on $[\tau - \sigma, \nu_0] \subset [\tau - \sigma, T]$.

It is easy to see that the conditions (A_1)–(A_6) are fulfilled.

Let us verify the conditions (A_7) and (A_8). Using (3.5), Remark 3.5 and our choice for h_{n_0}, we obtain that

$$\|u_t - u_{\theta(t)}\|_\sigma = \|u_t - u_\tau\|_\sigma \le \omega(\varphi, [-\sigma, 0], t - \tau)$$

$$+ \omega(Qf, [\tau, \nu_0], t - \tau) + 2M(\nu_0 - \tau)\|g\|_\sigma \le \omega(\varphi, [-\sigma, 0], h_{n_0})$$

$$+ \omega(Q\bar{f}, [\tau, T], h_{n_0}) + 2M(T - \tau)\|g\|_\sigma \le \varepsilon$$

for every $t \in [\tau, \nu_0)$ and so (A_7) is fulfilled. Furthermore, from (A_1), (A_4) and (A_5) we get $\|f(t)\| \le \chi(t)$ and $\|g(t)\| \le \varepsilon \le 1$ a.e. on $[\tau, \nu_0]$ and therefore, using (3.4) and (3.6), we have

$$\|u_{\nu_0} - \varphi\|_\sigma = \|u_{\nu_0} - u_\tau\|_\sigma \le r,$$

hence $u_{\nu_0} \in B_\sigma(\varphi, r)$. Since by (3.3) and (3.8) we have

$$u_{\nu_0}(0) = u(\nu_0) = S(h_{n_0})\varphi(0) + \int_\tau^{\tau + h_{n_0}} S(\tau + h_{n_0} - s)\bar{f}(s)ds + h_{n_0} q_{n_0} \in K,$$

it follows that $u_{\nu_0} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$. Hence (A_8) is also satisfied and conseqently $(\theta, \beta, g, f, u) \in \mathcal{M}_T$.

Next, we shall prove that there exists at least one (ε, ψ)-approximate solution of (1.1) and (1.2), defined on the whole interval $[\tau - \sigma, T]$. To this end, we shall apply Theorem 2.3 to the set $\mathcal{M}_T$ endowed with the following preorder:

If $(\theta^1, \beta^1, g^1, f^1, u^1)$ and $(\theta^2, \beta^2, g^2, f^2, u^2)$ are two (ε, ψ)-approximate solution on $[\tau - \sigma, \nu^1]$ and respectively on $[\tau - \sigma, \nu^2]$, then we say that

$$\left(\theta^1, \beta^1, g^1, f^1, u^1\right) \preccurlyeq \left(\theta^2, \beta^2, g^2, f^2, u^2\right)$$

if only if $\nu^1 \le \nu^2$, $\theta^1 = \theta^2|_{[\tau, \nu^1]}$, $\beta^1 = \beta^2|_{[\tau, \nu^1]}$, $g^1 = g^2|_{[\tau, \nu^1]}$, and $f^1 = f^2|_{[\tau, \nu^1]}$.

It is obvious that $\preccurlyeq$ is a preorder on $\mathcal{M}_T$. Moreover, let us notice that $(\theta^1,\beta^1,g^1,f^1,u^1)\preccurlyeq(\theta^2,\beta^2,g^2,f^2,u^2)$ implies, by (3.2), that $u^2|_{[\tau-\sigma,\nu_1]}=u^1$.

We define the function $\mathcal{S}:\mathcal{M}_T\to\mathbb{R}$ by

$$\mathcal{S}((\theta,\beta,g,f,u))=\nu,$$

if (θ,β,g,f,u) is an (ε,ψ)-approximate solution defined on $[\tau-\sigma,\nu]$. It is clear that $\mathcal{S}$ is increasing on $\mathcal{M}_T$. Further on, we shall show that each increasing sequence $((\theta^i,\beta^i,g^i,f^i,u^i))_{i\in\mathbb{N}}$ in $\mathcal{M}_T$ is majorated in $\mathcal{M}_T$. We construct a majorant as follows. We define

$$\nu^*=\lim_i \nu^i$$

and we have $\nu^*\in(\tau,T]$. For each $i\in\mathbb{N}$, we define $\theta^*(t)=\theta^i(t)$ if $t\in[\tau,\nu^i]$ and $\theta^*(\nu^*)=\nu^*$, $\beta^*(t,s)=\beta^i(t,s)$ for every $(t,s)\in\Delta_{\nu^i}$, $g^*(t)=g^i(t)$ and $f^*(t)=f^i(t)$ if $t\in[\tau,\nu^i]$. Since $((\theta^i,\beta^i,g^i,f^i,u^i))_{i\in\mathbb{N}}$ is an increasing sequence in $\mathcal{M}_T$, the functions θ^*,β^*,g^*, and f^* are well defined. Moreover, for every $i\in\mathbb{N}$ we have that $\|f^i(t)\|\le\chi(t)$ and $\|g^i(t)\|\le\varepsilon$ a.e. on $[\tau,\nu^i]$, which yields

$$\|f^*(t)\|\le\chi(t) \text{ and } \|g^*(t)\|\le\varepsilon \text{ a.e. on } [\tau,\nu^i] \tag{3.9}$$

and therefore $g^*\in L^\infty([\tau,\nu^*],X)$ and $f^*\in L^1([\tau,\nu^*],X)$. It is obvious that $\theta^*:[\tau,\nu^*]\to[\tau,\nu^*]$ and thus we can consider the 5-tuple $(\theta^*,\beta^*,g^*,f^*,u^*)$ with the function u^* defined by (3.2) . Since $t\to\beta^*(t,s)$ is nonexpansive on (s,ν^*), in view of Remark 3.3 we infer that u^* is continuous on $[t-\tau,\nu^*]$.

Now, we show that $(\theta^*,\beta^*,g^*,f^*,u^*)\in\mathcal{M}_T$. To this end, we fix an arbitrary $i\in\mathbb{N}$ and we observe that for every $t\in[\tau-\sigma,\tau]$ we have $u^*(t)=\varphi(t-\tau)=u^i(t)$ and for every $t\in[\tau,\nu^i]$ we have

$$\begin{aligned}u^*(t)&=S(t-\tau)\varphi(0)+\int_\tau^t S(t-s)f^*(s)ds+\int_\tau^t S(\beta^*(t,s))g^*(s)ds\\&=S(t-\tau)\varphi(0)+\int_\tau^t S(t-s)f^i(s)ds+\int_\tau^t S(\beta^i(t,s))g^i(s)ds=u^i(t).\end{aligned}$$

Consequently, $u^*(t)=u^i(t)$ for every $t\in[\tau-\sigma,\nu^i]$. Moreover, for every $t\in[\tau,\nu^i]$ and every $s\in[-\sigma,0]$ we have

$$\tau-\sigma\le\theta^*(t)+s=\theta^i(t)+s\le t+s\le t\le\nu^i,$$

and

$$u^*_{\theta^*(t)}(s) = u^*(\theta^*(t) + s) = u^*(\theta^i(t) + s) = u^i(\theta^i(t) + s) = {u^i}_{\theta^i(t)}(s).$$

Therefore

$$u^*_{\theta^*(t)} = u^i_{\theta^i(t)}, \tag{3.10}$$

for every $t \in [\tau, \nu^i]$. Taking into account the above relations, it readily follows that $(\theta^*, \beta^*, g^*, f^*, u^*)$ satisfies (A_2)–(A_7).

Let us verify the conditions (A_1) and (A_8). For any $t \in [\tau, \nu^*)$ there exists $i \in \mathbb{N}$ such that $t \in [\tau, \nu^i]$ so, by (3.10) we get $u^*_{\theta^*(t)} = {u^i}_{\theta^i(t)} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$. For $t = \nu^*$ we have $\theta^*(\nu^*) = \nu^*$ and $u^*_{\theta^*(\nu^*)} = u^*_{\nu^*}$. Then, by (3.9), we can use the relation (3.4) which, together with (3.6), yields $\|u^*_{\nu^*} - \varphi\|_\sigma \le r$ and thus $u^*_{\theta^*(\nu^*)} = u^*_{\nu^*} \in B_\sigma(\varphi, r)$. By the continuity of u^* we have

$$u^*_{\nu^*}(0) = u^*(\nu^*) = \lim_i u^*(\nu^i) = \lim_i u^i(\nu^i).$$

Since $u^i_{\nu^i} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ we have that $u^i(\nu^i) = u^i_{\nu^i}(0) \in K \cap B(\varphi(0), r)$, for every $i \in \mathbb{N}$. The set $K \cap B(\varphi(0), r)$ is closed, hence $u^*_{\nu^*}(0) \in K \cap B(\varphi(0), r)$ and therefore $u^*_{\theta^*(\nu^*)} = u^*_{\nu^*} \in \mathcal{K}_0$. It follows that $(\theta^*, \beta^*, g^*, f^*, u^*) \in \mathcal{M}_T$. In addition, $(\theta^i, \beta^i, g^i, f^i, u^i) \preccurlyeq (\theta^*, \beta^*, g^*, f^*, u^*)$ for each $i \in \mathbb{N}$ and thus the sequence $((\theta^i, \beta^i, g^i, f^i, u^i))_{i \in \mathbb{N}}$ is majorated in $\mathcal{M}_T$. Therefore, the set $\mathcal{M}_T$, endowed with the preorder $\preccurlyeq$ and the function $\mathcal{S}$, satisfies the hypotheses of Theorem 2.3.

Before using the conclusion of Theorem 2.3, we shall show that any element $(\theta, \beta, g, f, u) \in \mathcal{M}_T$ with $\mathcal{S}((\theta, \beta, g, f, u)) < T$ is majorated in $\mathcal{M}_T$ by an element $(\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u}) \in \mathcal{M}_T$ with $\mathcal{S}((\theta, \beta, g, f, u)) < \mathcal{S}((\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u}))$. To this aim let us consider that (θ, β, g, f, u) is an (ε, ψ)-approximate solution defined $[\tau - \sigma, \nu]$ with $\nu \in (\tau, T)$. Since $u_\nu \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ we can apply Theorem 2.2 on $[\nu, T]$ for $G(\cdot) = F(\cdot, u_\nu)$. It follows that there exists a measurable function $\overline{f} : [\nu, T] \to X$ such that $\overline{f}(t) \in F(t, u_\nu)$ a.e. on $[\nu, T]$ and

$$\|\bar{f}(t) - \psi(t)\| \le d(\psi(t), F(t, u_\nu)) + \varepsilon\mu(t) \text{ a.e. on } [\nu, T].$$

By (H_1) we have $\|\bar{f}(t)\| \le \chi(t)$ a.e. on $[\nu, T]$ and hence $\bar{f} \in L^1([\nu, T], X)$. Since $(\nu, u_\nu) \in \mathcal{I} \times \mathcal{K}_0$ we can apply the tangency condition (H_4) at (ν, u_ν).

Therefore, there exist $(h_n)_n$ in $\mathbb{R}_+$ with $h_n \downarrow 0$ and $(q_n)_n$ in X with $q_n \to 0$ such that

$$S(h_{n_0})u_\nu(0) + \int_\nu^{\nu+h_n} S(\nu + h_n - s)\bar{f}(s)ds + h_n q_n \in K, \tag{3.11}$$

for every $n \in \mathbb{N}$.

We define

$$\hat{f}(t) := \begin{cases} f(t) & \text{if } t \in [\tau, \nu], \\ \bar{f}(t) & \text{if } t \in (\nu, T]. \end{cases}$$

Since

$$\lim_{\delta\downarrow 0} \omega(\varphi, [-\sigma, 0], \delta) = 0 \text{ and } \lim_{\delta\downarrow 0} \omega(Q\hat{f}, [\tau, T], \delta) = 0$$

we can fix $\widetilde{n} \in \mathbb{N}$ such that $h_{\widetilde{n}} \in (0, T - \nu]$, $\|q_{\widetilde{n}}\| \le \varepsilon$ and

$$\omega(\varphi, [-\sigma, 0], h_{\widetilde{n}}) + \omega(Q\widehat{f}, [\tau, T], h_{\widetilde{n}}) + 2M(T - \tau)\|q_{\widetilde{n}}\| \le \varepsilon.$$

Further on, we define $\widetilde{\nu} := \nu + h_{\widetilde{n}}$, $\widetilde{f}(t) := \hat{f}(t)$ for $t \in [\tau, \widetilde{\nu}]$ and

$$\widetilde{\theta}(t) := \begin{cases} \theta(t) & \text{if } t \in [\tau, \nu], \\ \nu & \text{if } t \in (\nu, \widetilde{\nu}]; \end{cases}$$

$$\widetilde{g}(t) := \begin{cases} g(t) & \text{if } t \in [\tau, \nu] \\ q_{n_0} & \text{if } t \in (\nu, \widetilde{\nu}] \end{cases}$$

$$\widetilde{\beta}(t,s) := \begin{cases} \beta(t,s) & \text{if } \tau \le s < t \le \nu, \\ \tau - \nu + \beta(\nu, s) & \text{if } \tau \le s < \nu < t \le \widetilde{\nu}, \\ 0 & \text{if } \nu \le s < t \le \widetilde{\nu}. \end{cases}$$

We show that $(\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$, with $\widetilde{u}$ given by (3.2), is an (ε, ψ)-approximate solution defined on $[\tau - \sigma, \widetilde{\nu}] \subset [\tau - \sigma, T]$. First, we notice that $\widetilde{\beta}$ satisfies (A_3), $\widetilde{g} \in L^\infty([\tau, \widetilde{\nu}], X)$ and $\widetilde{f} \in L^1([\tau, \widetilde{\nu}], X)$ so, in view of Remark 3.3, it follows that $\widetilde{u} \in C([\tau - \sigma, \widetilde{\nu}], X)$. Moreover,

$$\widetilde{u}(t) = u(t) \text{ for every } t \in [\tau - \sigma, \nu]$$

and

$$\begin{aligned}\tilde{u}(t) &= S(t-\tau)\varphi(0) + \int_{\tau}^{t} S(t-s)\tilde{f}(s)ds + \int_{\tau}^{t} S(\widetilde{\beta}(t,s))\tilde{g}(s)ds \\ &= S(t-\nu)\tilde{u}(\nu) + \int_{\nu}^{t} S(t-s)\tilde{f}(s)ds + \int_{\nu}^{t} S(\widetilde{\beta}(t,s))\tilde{g}(s)ds \\ &= S(t-\nu)u_{\nu}(0) + \int_{\nu}^{t} S(t-s)\hat{f}(s)ds + (t-\nu)q_{\widetilde{n}},\end{aligned}$$

for every $t \in [\nu, \tilde{\nu}]$. Also, it is obvious that $(\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$ satisfies (A_2)–(A_6).

Since for every $t \in [\tau, \nu]$ we have $\tilde{\theta}(t) = \theta(t)$ and

$$\tilde{u}_{\tilde{\theta}(t)} = \tilde{u}_{\theta(t)} = u_{\theta(t)} \in \mathcal{K}_0 \cap B_{\sigma}(\varphi, r)$$

and for every $t \in (\nu, \tilde{\nu}]$ we have

$$\tilde{u}_{\tilde{\theta}(t)} = \tilde{u}_{\nu} = u_{\nu} \in \mathcal{K}_0 \cap B_{\sigma}(\varphi, r),$$

we deduce that (A_1) is fulfilled.

Let us verify the conditions (A_7) and (A_8). For every $t \in [\tau, \nu]$ we have

$$\|\tilde{u}_t - u_{\tilde{\theta}(t)}\|_{\sigma} = \|\tilde{u}_t - \tilde{u}_{\theta(t)}\|_{\sigma} = \|u_t - u_{\theta(t)}\|_{\sigma} \le \varepsilon$$

and for $t \in (\nu, \tilde{\nu}]$, using (3.5), Remark 3.5 and our choice of $h_{\widetilde{n}}$, we obtain

$$\begin{aligned}&\|\widetilde{u}_t - \widetilde{u}_{\widetilde{\theta}(t)}\|_{\sigma} = \|\widetilde{u}_t - \widetilde{u}_{\nu}\|_{\sigma} \le \omega(\varphi, [-\sigma, 0], t-\nu) \\ &+ \omega(Q\widetilde{f}, [\tau, \widetilde{\nu}], t-\nu) + 2M(\widetilde{\nu} - \tau)\|g\|_{\infty} \le \omega(\varphi, [-\sigma, 0], h_{\widetilde{n}}) \\ &+ \omega(Q\widehat{f}, [\tau, T], h_{\widetilde{n}}) + 2M(T-\tau)\|q_{\widetilde{n}}\| \le \varepsilon\end{aligned}$$

and so (A_7) is fulfilled.

By (A_1), (A_4) and (A_5) we have that $\|f(t)\| \le \chi(t)$ and $\|g(t)\| \le \varepsilon \le 1$ a.e. on $[\tau, \widetilde{\nu}]$ and therefore we can use (3.4) which, together with (3.6), yields

$$\|\tilde{u}_{\tilde{\nu}} - \varphi\|_{\sigma} = \|\tilde{u}_{\tilde{\nu}} - \tilde{u}_{\tau}\|_{\sigma} \le r$$

and thus $\tilde{u}_{\tilde{\nu}} \in B_{\sigma}(\varphi, r)$. Since, by (3.2) and (3.11), we have

$$\tilde{u}_{\tilde{\nu}}(0) = S(h_{\widetilde{n}})u_\nu(0) + \int_\nu^{\nu+h_{\widetilde{n}}} S(\nu + h_{\widetilde{n}} - s)\widehat{f}(s)ds + h_{\widetilde{n}} q_{\widetilde{n}} \in K,$$

it follows that $\tilde{u}_{\tilde{\nu}} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$. Therefore, $(\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$ is an (ε, ψ)-approximate solution defined on $[\tau - \sigma, \widetilde{\nu}]$. Moreover, by construction, we have $(\theta, \beta, g, f, u) \preccurlyeq (\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$ and $\mathcal{S}((\theta, \beta, g, f, u)) = \nu < \widetilde{\nu} = \mathcal{S}((\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u}))$.

Now, from Theorem 2.3 we infer that there exists $(\underset{\sim}{\theta}, \underset{\sim}{\beta}, g, \underset{\sim}{f}, u) \in \mathcal{M}_T$ such that $\mathcal{S}((\underset{\sim}{\theta}, \underset{\sim}{\beta}, \underset{\sim}{g}, \underset{\sim}{f}, \underset{\sim}{u})) = \mathcal{S}((\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u}))$, for each $(\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u}) \in \mathcal{M}_T$ with $(\underset{\sim}{\theta}, \underset{\sim}{\beta}, \underset{\sim}{g}, \underset{\sim}{f}, \underset{\sim}{u}) \preccurlyeq (\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$. If $\mathcal{S}((\underset{\sim}{\theta}, \underset{\sim}{\beta}, \underset{\sim}{g}, \underset{\sim}{f}, \underset{\sim}{u})) < T$ then, by the last step, there exists $(\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u}) \in \mathcal{M}_T$ with $(\underset{\sim}{\theta}, \underset{\sim}{\beta}, \underset{\sim}{g}, \underset{\sim}{f}, \underset{\sim}{u}) \preccurlyeq (\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u})$ and such that $\mathcal{S}((\underset{\sim}{\theta}, \underset{\sim}{\beta}, \underset{\sim}{g}, \underset{\sim}{f}, \underset{\sim}{u})) < \mathcal{S}((\widetilde{\theta}, \widetilde{\beta}, \widetilde{g}, \widetilde{f}, \widetilde{u}))$. We conclude that $\mathcal{S}((\underset{\sim}{\theta}, \underset{\sim}{\beta}, \underset{\sim}{g}, \underset{\sim}{f}, \underset{\sim}{u})) = T$ and this completes the proof of Lemma 3.1. ■

We are now prepared to complete the proof of Theorem 2.1.

Proof. Let $T \in (\tau, \tau + \rho]$ be given by Lemma 3.1 and let $(\varepsilon_n)_n$ be a decreasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$ and $\varepsilon_n \in (0, 1)$ for every $n \in \mathbb{N}$.

Starting with one measurable selection $f_0(\cdot) \in F(\cdot, \varphi)$, in view of Lemma 3.1 we can define inductively the sequence $((\theta^n, \beta^n, g^n, f^n, u^n))_{n\in\mathbb{N}}$ such that $(\theta^n, \beta^n, g^n, f^n, u^n)$ is an (ε_n, f^n)-approximate solution on $[\tau - \sigma, T]$ for every $n \in \mathbb{N}$.

Thus, for every $n \in \mathbb{N}$ we have

$$(3.12) \quad u^n(t) = (Qf^n)(t) + \begin{cases} 0, & \text{for } t \in [\tau - \sigma, \tau], \\ \displaystyle\int_\tau^t S(\beta^n(t, s))g^n(s)ds, & \text{for } t \in (\tau, T] \end{cases}$$

and

(B_1) $u^n_{\theta^n(t)} \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ for every $t \in [\tau, T)$;

(B_2) $0 \leq t - \theta^n(t)$ for every $t \in [\tau, T]$;

(B_3) $0 \leq \beta^n(t, s) \leq t - \tau$ for $\tau \leq s < t \leq T$;

(B_4) $\|g^n(t)\| \leq \varepsilon_n$ a.e. on $[\tau, T]$;

(B_5) $f^n(t) \in F(t, u^n_{\theta^n(t)})$ a.e. on $[\tau, T]$;

(B_6) $\|f^n(t) - f^{n-1}(t)\| \le d(f^{n-1}(t), F(t, u^n_{\theta^n(t)})) + \varepsilon_n \mu(t)$ a.e. on $[\tau, T]$;

(B_7) $\|u^n_t - u^n_{\theta^n(t)}\|_\sigma \le \varepsilon_n$ for every $t \in [\tau, T]$;

(B_8) $u^n_T \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$.

We shall prove that $(u^n)_n$ converges uniformly to a function $u : [\tau - \sigma, T] \to X$ that is a solution of (1.1) and (1.2).

For this, we first show that for every $n \in \mathbb{N}$ we have

(C_1) $\|f^n(t)\| \le \chi(t)$ a.e. on $[\tau, T]$;

(C_2) $\|u^n(t) - (Qf^n)(t)\| \le M(T - \tau)\varepsilon_n$ for every $t \in [\tau, T]$;

(C_3) $\|u^{n+1}_{\theta^{n+1}(t)} - u^n_{\theta^n(t)}\|_\sigma \le 2\varepsilon_n + \|u^{n+1} - u^n\|_T$, for every $t \in [\tau, T]$,

where $\|\cdot\|_T$ is the norm in $C([\tau - \sigma, T]; X)$;

(C_4) $\|f^{n+1}(t) - f^n(t)\| \le \mu(t)(\|u^{n+1} - u^n\|_T + 3\varepsilon_n)$ a.e. on $[\tau, T]$.

Indeed, (C_1) follows from (H_1) and (B_5), (C_2) follows from (3.3), (3.12) and (B_4). In order to prove (C_3) we observe that

$$\|u^{n+1}_t - u^n_t\|_\sigma = \sup_{-\sigma \le s \le 0} \|u^{n+1}(t+s) - u^n(t+s)\|$$

$$\le \sup_{\tau - \sigma \le \nu \le T} \|u^{n+1}(\nu) - u^n(\nu)\| = \|u^{n+1} - u^n\|_T$$

and thus by (B_7) we obtain that

$$\begin{aligned}\|u^{n+1}_{\theta^{n+1}(t)} - u^n_{\theta^n(t)}\|_\sigma &\le \|u^{n+1}_{\theta^{n+1}(t)} - u^{n+1}_t\|_\sigma + \|u^{n+1}_t - u^n_t\|_\sigma + \|u^n_t - u^n_{\theta^n(t)}\|_\sigma \\ &\le \varepsilon_{n+1} + \|u^{n+1}_t - u^n_t\|_\sigma + \varepsilon_n \le 2\varepsilon_n + \|u^{n+1} - u^n\|_T\end{aligned}$$

for every $t \in [\tau, T]$. Finally, by (H_2), (B_5) and (C_3) we have

$$\begin{aligned}\|f^{n+1}(t)-f^n(t)\| &\le d\left(f^n(t), F\left(t, u^{n+1}_{\theta^{n+1}(t)}\right)\right)+\varepsilon_{n+1}\mu(t)\\ &\le d_{HP}\left(F\left(t, u^{n}_{\theta^{n}(t)}\right), F\left(t, u^{n+1}_{\theta^{n+1}(t)}\right)\right)+\varepsilon_{n+1}\mu(t)\\ &\le \mu(t)\left(\|u^{n}_{\theta^{n}(t)}-u^{n+1}_{\theta^{n+1}(t)}\|_\sigma+\varepsilon_{n+1}\right)\\ &\le \mu(t)\left(\|u^{n+1}-u^n\|_T+3\varepsilon_n\right)\end{aligned}$$

a.e. on $[\tau, T]$ and hence (C_4) is also checked. Further on, for every $t \in [\tau, T]$, by (3.6), (3.11), (C_3) and (C_5) we have

$$\|u^{n+1}(t)-u^n(t)\| \le \|u^{n+1}(t)-(Qf^{n+1})(t)\|+\|(Qf^{n+1})(t)-(Qf^n)(t)\|$$

$$+\|(Qf^n)(t)-u^n(t)\| \le M(T-\tau)(\varepsilon_{n+1}+\varepsilon_n)+\int_\tau^T \|f^{n+1}(s)-f^n(s)\|ds$$

$$\le 2M(T-\tau)\varepsilon_n + M(3\varepsilon_n+\|u^{n+1}-u^n\|_T)\int_\tau^T \mu(s)\,ds$$

$$\le M\left(2(T-\tau)+\frac{3}{2}\right)\varepsilon_n+\frac{1}{2}\|u^{n+1}-u^n\|_T.$$

Therefore, since $\|u^{n+1}(t)-u^n(t)\|=0$ for every $t \in [\tau-\sigma, \tau]$, we obtain

$$\|u^{n+1}(t)-u^n(t)\| \le M\left(2(T-\tau)+\frac{3}{2}\right)\varepsilon_n+\frac{1}{2}\|u^{n+1}-u^n\|_T,$$

for every $t \in [\tau-\sigma, T]$, and thus

$$\|u^{n+1}-u^n\|_T \le M\left(2(T-\tau)+\frac{3}{2}\right)\varepsilon_n+\frac{1}{2}\|u^{n+1}-u^n\|_T,$$

for every $n \in \mathbb{N}$. It follows that

$$\|u^{n+1}-u^n\|_T \le M\left(4(T-\tau)+3\right)\varepsilon_n \tag{3.13}$$

for every $n \in \mathbb{N}^*$ with $\sum_{n=1}^\infty \varepsilon_n < +\infty$ and so we deduce that $(u^n)_n$ converge uniformly to a function $u : [\tau-\sigma, T] \to X$.

From (C_4) and (3.13) we deduce that, for almost all $t \in [\tau, T]$, we have

$$\begin{aligned} \|f^{n+1}(t) - f^n(t)\| &\le \mu(t)(\|u^{n+1} - u^n\|_T + 3\varepsilon_n) \\ &\le \mu(t)(4M(T-\tau) + 3M + 3)\varepsilon_n \end{aligned}$$

for every $n \in \mathbb{N}^*$. This implies that $(f^n)_n$ converge pointwise almost everywhere to a measurable function f. For any fixed $t \in [\tau - \sigma, T]$, by (C_1) and Lebesgue's Theorem, we obtain that $\lim\limits_{n\to\infty} (Qf^n)(t) = (Qf)(t)$. Consequently, by (C_2), we conclude that $u(t) = (Qf)(t)$ for every $t \in [\tau - \sigma, T]$.

For every $t \in [\tau, T]$ and every $n \in \mathbb{N}^*$, by (B_7) and (C_3), we have

$$\|u^n_{\theta^n(t)} - u_t\|_\sigma \le \|u^n_{\theta^n(t)} - u^n_t\|_\sigma + \|u^n_t - u_t\|_\sigma \le \varepsilon_n + \|u^n_t - u_t\|_T$$

and thus $u^n_{\theta^n(t)} \to u_t$ in $\|\cdot\|_\sigma$ as $n \to \infty$. In view of (B_1) and Remark 3.1 it follows that $u_t \in \mathcal{K}_0 \cap B_\sigma(\varphi, r)$ and hence $u(t) \in K \cap B(\varphi(0), r)$, for every $t \in [\tau, T]$.

Now, let us observe that, a.e. on $[\tau, T]$, we have

$$\begin{aligned} d(f(t), F(t, u_t)) &\le \|f(t) - f^n(t)\| + d\left(F\left(t, u^n_{\theta^n(t)}\right), F(t, u_t)\right) \\ &\le \|f(t) - f^n(t)\| + \mu(t)\|u^n_{\theta^n(t)} - u_t\|_\sigma, \end{aligned}$$

for every $n \in \mathbb{N}^*$. It follows, by letting $n \to \infty$, that $d(f(t), F(t, u_t)) = 0$ and thus, since F has closed values, $f(t) \in F(t, u_t)$ a.e on $[\tau, T]$.

We have proved that $u : [\tau - \sigma, T] \to X$ is a solution of (1.1) and (1.2), with $u(t) \in K$ for every $t \in [\tau, T]$, and so, (τ, φ) being arbitrarily fixed in $\mathcal{I} \times \mathcal{K}_0$, we have shown that K is a viable domain for (1.1). ∎

Acknowledgment

The authors would like to express their gratitude to the referee for the careful reading of this manuscript and for constructive suggestions that considerably improved this paper.

References

[1] J.P. Aubin, Viability Theory. Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA 1991.

[2] R.J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl. **12** (1965), 1–12.

[3] H. Brézis and F.E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Advances in Math. **21** (3) (1976), 355–364.

[4] C. Castaing and M.D.P. Monteiro Marques, *Topological properties of solution sets for sweeping processes with delay*, Portugal. Math. **54** (4) (1997), 485–507.

[5] O. Cârjă and M.D.P. Monteiro Marques, *Viability for nonautonomous semilinear differential equations*, J. Differential Equations **166** (2) (2000), 328–346.

[6] O. Cârjă and C. Ursescu, *The characteristics method for a first order partial differential equation*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. **39** (4) (1993), 367–396.

[7] O. Cârjă and I.I. Vrabie, *Some new viability results for semilinear differential inclusions*, NoDEA Nonlinear Differential Equations Appl. **4** (3) (1997), 401–424.

[8] K. Deimling, Multivalued differential equations, De Gruyter Series in Nonlinear Analysis and Applications, 1. Walter de Gruyter & Co., Berlin 1992.

[9] W.E. Fitzgibbon, *Semilinear functional differential equations in Banach space*, J. Differential Equations **29** (1) (1978), 1–14.

[10] A. Fryszkowski, *Existence of solutions of functional-differential inclusion in nonconvex case*, Ann. Polon. Math. **45** (2) (1985), 121–124.

[11] A. Gavioli and L. Malaguti, *Viable solutions of differential inclusions with memory in Banach spaces*, Portugal. Math. **57** (2) (2000), 203–217.

[12] L. Górniewicz, Topological fixed point theory of multivalued mappings, Mathematics and its Applications, 495. Kluwer Academic Publishers, Dordrecht 1999.

[13] G. Haddad, *Monotone trajectories of differential inclusions and functional-differential inclusions with memory*, Israel J. Math. **39** (1–2) (1981), 83–100.

[14] G. Haddad, *Monotone viable trajectories for functional-differential inclusions*, J. Differential Equations **42** (1) (1981), 1–24.

[15] J.K. Hale and S.M. Verduyn Lunel, Introduction to functional-differential equations, Applied Mathematical Sciences, 99. Springer-Verlag, New York 1993.

[16] F. Iacob and N.H. Pavel, *Invariant sets for a class of perturbed differential equations of retarded type*, Israel J. Math. **28** (3) (1977), 254–264.

[17] M. Kisielewicz, Differential inclusions and optimal control, Mathematics and its Applications (East European Series), 44, Kluwer Academic Publishers Group, Dordrecht; PWN—Polish Scientific Publishers, Warsaw 1991.

[18] V. Lakshmikantham and S. Leela, Nonlinear differential equations in abstract spaces, International Series in Nonlinear Mathematics: Theory, Methods and Applications, 2, Pergamon Press, Oxford-New York 1981.

[19] V. Lakshmikantham, S. Leela and V. Moauro, *Existence and uniqueness of solutions of delay differential equations on a closed subset of a Banach space*, Nonlinear Anal. **2** (3) (1978), 311–327.

[20] S. Leela and V. Moauro, *Existence of solutions in a closed set for delay differential equations in Banach spaces*, Nonlinear Anal. **2** (1) (1978), 47–58.

[21] V. Lupulescu and M. Necula, *Viability and local invariance for non-convex semilinear differential inclusions*, Nonlinear Funct. Anal. Appl. **9** (3) (2004), 495–512.

[22] E. Mitidieri and I.I. Vrabie, *A class of strongly nonlinear functional differential equations, Universita degli Studi di Trieste*, Instituto di Matematica, Quaderni Matematici II Serie **122** (1986), 1–19.

[23] E. Mitidieri and I.I. Vrabie, *Existence for nonlinear functional differential equations*, Hiroshima Math. J. **17** (3) (1987), 627–649.

[24] N.H. Pavel, Differential equations, flow invariance and applications, Research Notes in Mathematics, 113, Pitman (Advanced Publishing Program), Boston, MA 1984.

[25] A. Syam, Contribution Aux Inclusions Différentielles, Doctoral thesis, Université Montpellier II, 1993.

[26] A.A. Tolstonogov and I.A. Finogenko, *On functional differential inclusions in Banach space with a nonconvex right-hand side*, Soviet. Math. Dokl. **22** (1980), 320–324.

[27] I.I. Vrabie, C_0-semigroups and applications, North-Holland Mathematics Studies, 191, North-Holland Publishing Co., Amsterdam 2003.

[28] G.F. Webb, *Autonomous nonlinear functional differential equations and nonlinear semigroups*, J. Math. Anal. Appl. **46** (1974), 1–12.

[29] Q.J. Zhu, *On the solution set of differential inclusions in Banach space*, J. Differential Equations **93** (2) (1991), 213–237.

Received 20 October 2004