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TOPOLOGICAL PROPERTIES OF SOME SPACES OF CONTINUOUS OPERATORS

Marian Nowak

Faculty of Mathematics, Computer Science and Econometrics
University of Zielona Góra
ul. Szafrana 4A, 65–516 Zielona Góra, Poland

e-mail: M.Nowak@wmie.uz.zgora.pl

Abstract

Let X be a completely regular Hausdorff space, E and F be Banach spaces. Let $C_b(X,E)$ be the space of all E-valued bounded continuous functions on X, equipped with the strict topology β. We study topological properties of the space $\mathcal{L}_\beta(C_b(X,E),F)$ of all $(\beta,\|\cdot\|_F)$-continuous linear operators from $C_b(X,E)$ to F, equipped with the topology τ_s of simple convergence. If X is a locally compact paracompact space (resp. a P-space), we characterize τ_s-compact subsets of $\mathcal{L}_\beta(C_b(X,E),F)$ in terms of properties of the corresponding sets of the representing operator-valued Borel measures. It is shown that the space $(\mathcal{L}_\beta(C_b(X,E),F),\tau_s)$ is sequentially complete if X is a locally compact paracompact space.

Keywords: spaces of vector-valued continuous functions, strict topologies, operator measures, topology of simple convergence, continuous operators.

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1. Introduction and terminology

Throughout the paper let $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$ be (real or complex) Banach spaces, and let E' and F' denote the Banach duals of E and F, respectively. By $B_{F'}$ and B_E we denote the closed unit ball in F' and E, respectively. By $\mathcal{L}(E,F)$ we denote the space of all bounded linear operators E to F. Given a locally convex space (Z,ξ) by $(Z,\xi)'$ or Z'_ξ we denote its topological dual. We denote by $\sigma(Z,Z'_\xi)$ the weak topology on Z with respect to a dual pair $\langle Z,Z'_\xi\rangle$.

Assume that $(X,\mathcal{T})$ is a completely regular Hausdorff space. Let $\mathcal{B}o$ stand for the σ-algebra of Borel sets in X. By $\mathcal{K}$ (resp. $\mathcal{F}$) we denote the family of all compact (resp. finite) sets in X.

Let $C_b(X,E)$ stand for the space of all bounded continuous functions $f : X \to E$. By τ_u we will denote the topology on $C_b(X,E)$ of the uniform norm $\|\cdot\|$.

The strict topology β (denoted also by β_o and β_t) can be characterized as the finest locally convex topology on $C_b(X,E)$ which coincides with the compact-open topology τ_c on τ_u-bounded subsets of $C_b(X,E)$ (see [6, 9, 11, 12]). This means that $(C_b(X,E),\beta)$ is a generalized DF-space (see [15], [17, Corollary]) (equivalently, β coincides with the mixed topology $\gamma[\tau_u,\tau_c]$ in the sense of Wiweger (see [4, 19] for more details)). Then β is weaker than τ_u, and β coincides with τ_u if and only if X is compact (see [3, Theorem 2.3]).

By $\mathcal{L}_\beta(C_b(X,E),F)$ we will denote the family of all $(\beta,\|\cdot\|_F)$-continuous linear operators $T : C_b(X,E) \to F$. The topology τ_s of *simple convergence* in $\mathcal{L}_\beta(C_b(X,E),F)$ is defined by the family of seminorms $\{p_f : f \in C_b(X,E)\}$, where $p_f(T) = \|T(f)\|_F$ for $T \in \mathcal{L}_\beta(C_b(X,E),F)$.

In this paper we study topological properties of the space $(\mathcal{L}_\beta(C_b(X,E),F),\tau_s)$. We characterize τ_s-compact sets in $\mathcal{L}_\beta(C_b(X,E),F)$ in terms of the properties of the corresponding sets of the representing operator-valued Borel measures whenever X is a locally compact paracompact space (resp. X is a P-space) (see Theorem 3.4 below). It is shown that the space $(\mathcal{L}_\beta(C_b(X,E),F),\tau_s)$ is sequentially complete if X is a locally compact paracompact space (see Theorem 4.2 below).

2. Integral representation of operators on $C_b(X,E)$

Recall that a countably additive measure scalar measure ν on $\mathcal{B}o$ is said to be a Radon measure if its variation $|\nu| : \mathcal{B}o \to \mathbb{R}_+$ is regular, i.e., for each $A \in \mathcal{B}o$,

$$|\nu|(A) = \sup\{|\nu|(K) : K \in \mathcal{K}, K \subset A\},\ |\nu|(A) = \inf\{|\nu|(O) : O \in \mathcal{T},\ O \supset A\}.$$

By $M(X)$ we denote the space of all Radon measures.

Let $M(X,E')$ denote the space of all countably additive measures $\mu : \mathcal{B}o \to E'$ of bounded variation ($|\mu|(X) < \infty$) such that for each $x \in E$, $\mu_x \in M(X)$, where $\mu_x(A) := \mu(A)(x)$ for $A \in \mathcal{B}o$. Then $|\mu| \in M(X)$ (see [10, Lemma 2.3]).

It is known that for $\mu \in M(X,E')$, every $f \in C_b(X,E)$ is μ-integrable in the Riemann-Stjeltjes sense (see [7, Definition 2], [14, Definition 2.2]).

The following characterization of β-continuous linear functionals on $C_b(X,E)$ will be of importance (see [14, § 2]).

Theorem 2.1. *For a linear functional Φ on $C_b(X,E)$ the following statements are equivalent:*

(i) *Φ is β-continuous.*

(ii) *There exists a unique* $\mu \in M(X, E')$ *such that*

$$\Phi(f) = \Phi_\mu(f) = \int_X f d\mu \quad \textit{for} \quad f \in C_b(X, E).$$

Moreover, $\|\Phi_\mu\| = |\mu|(X)$.

The following result will be useful (see [12, Lemma 2]).

Lemma 2.2. *For a subset* $\mathcal{M}$ *of* $M(X, E')$ *the following statements are equivalent:*

(i) $\sup_{\mu \in \mathcal{M}} |\mu|(X) < \infty$ *and* $\mathcal{M}$ *is uniformly tight, that is, for each* $\varepsilon > 0$ *there exists* $K \in \mathcal{K}$ *such that* $\sup_{\mu \in \mathcal{M}} |\mu|(X \setminus K) \leq \varepsilon$.

(ii) *The family* $\{\Phi_\mu : \mu \in \mathcal{M}\}$ *in* $C_b(X, E)'_\beta$ *is* β*-equicontinuous.*

Let $i_F : F \to F''$ denote the canonical embedding, i.e., $i_F(y)(y') = y'(y)$ for $y \in F$, $y' \in F'$. Moreover, let $j_F : i_F(F) \to F$ stand for the left inverse of i_F, that is, $j_F \circ i_F = id_F$.

Assume that $T : C_b(X, E) \to F$ is a $(\beta, \|\cdot\|_F)$-continuous linear operator. Then according to [14, Theorem 3.1] there exists a unique measure $m_T : \mathcal{B}o \to \mathcal{L}(E, F'')$ (called the *representing measure of* T) such that the following statements hold:

(2.1) For every $y' \in F'$, $(m_T)_{y'} \in M(X, E')$, where

$$(m_T)_{y'}(A)(x) := (m_T(A)(x))(y') \quad \text{for} \quad A \in \mathcal{B}o,\ x \in E.$$

(2.2) The mapping $F' \ni y' \mapsto (m_T)_{y'} \in M(X, E')$ is $(\sigma(F', F), \sigma(M, E'), C_b(X, E))$-continuous.

(2.3) $\widetilde{m}_T(X) < \infty$ and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that $\widetilde{m}_T(X \setminus K) \leq \varepsilon$ (here $\widetilde{m}_T(A)$ stands for the semivariation of m_T on $A \in \mathcal{B}o$).

(2.4) $\|T\| = \widetilde{m}_T(X)$.

(2.5) Every $f \in C_b(X, E)$ is m-integrable in the Riemann-Stjeltjes sense and $\int_X f\, dm \in i_F(F)$ (here $\int_X f\, dm$ denotes the Riemann-Stjeltjes integral) and $T(f) = j_F(\int_X f\, dm)$.

(2.6) For every $y' \in F'$,

$$y'(T(f)) = \Big(\int_X f\, dm_T\Big)(y') = \int_X f\, d(m_T)_{y'} \quad \text{for} \quad f \in C_b(X, E).$$

Note that (see [5, §4, Proposition 5]),

(2.7) $\widetilde{m}_T(A) = \sup\{|(m_T)_{y'}|(A) : y' \in B_{F'}\}$ for $A \in \mathcal{B}o$.

Let $B_{C_b(X,E)} := \{f \in C_b(X,E) : \|f\| \leq 1\}$.

We will need the following result.

Lemma 2.3. *Assume that $T : C_b(X,E) \to F$ be $(\beta, \|\cdot\|_F)$-continuous linear operator and m_T is its representing measure. Then for $y' \in F'$ and $K \in \mathcal{K}$, we have:*

$$\text{(i)} \quad |(m_T)_{y'}|(X \smallsetminus K) = \sup\left\{\left|\int_X f\, d(m_T)_{y'}\right| : f \in B_{C_b(X,E)} \textit{ with } f \equiv 0 \textit{ on } K\right\}$$
$$= \sup\{|y'(T(f))| : f \in B_{C_b(X,E)} \textit{ with } f \equiv 0 \textit{ on } K\}$$

$$\text{(ii)} \quad \widetilde{m}_T(X \smallsetminus K) = \sup\left\{\left\|\int_X f\, dm_T\right\|_{F''} : f \in B_{C_b(X,E)} \textit{ with } f \equiv 0 \textit{ on } K\right\}$$
$$= \sup\{\|T(f)\|_F : f \in B_{C_b(X,E)} \textit{ with } f \equiv 0 \textit{ on } K\}.$$

Proof. (i) It follows from [14, Lemma 2.3] and (2.6).
(ii) Using (i), (2.7), (2.6) and (2.5), we get

$$\begin{aligned}
\widetilde{m}_T(X \smallsetminus K) &= \sup\left\{\left|\left(\int_X f\, dm_T\right)(y')\right| : y' \in B_{F'}, f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K\right\} \\
&= \sup\left\{\left\|\int_X f\, dm_T\right\|_{F''} : f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K\right\} \\
&= \sup\{\|T(f)\|_F : f \in B_{C_b(X,E)} \text{ with } f \equiv 0 \text{ on } K\}.
\end{aligned}$$
■

3. Relative compactness in $(\mathcal{L}_\beta(C_b(X,E),F), \tau_s)$

We start with the following characterization of $(\beta, \|\cdot\|_F)$-equicontinuous subsets of $\mathcal{L}_\beta(C_b(X,E),F)$.

Proposition 3.1. *For a subset $\mathcal{A}$ of $\mathcal{L}_\beta(C_b(X,E),F)$ the following statements are equivalent:*

(i) *$\mathcal{A}$ is $(\beta, \|\cdot\|_F)$-equicontinuous.*

(ii) *$\sup_{T\in\mathcal{A}} \widetilde{m}_T(X) < \infty$ and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that $\sup_{T\in\mathcal{A}} \widetilde{m}_T(X \smallsetminus K) \leq \varepsilon$.*

(iii) *$\sup_{T\in\mathcal{A}} \|T\| < \infty$ and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that $\sup_{T\in\mathcal{A}} \|\int_X f\, dm_T\|_{F''} \leq \varepsilon$ whenever $f \in C_b(X,E), \|f\| \leq 1$ with $f \equiv 0$ on K.*

Proof. (i)$\Rightarrow$(ii) Assume that $\mathcal{A}$ is $(\beta, \|\cdot\|_F)$-equicontinuous. This means that the set $\{y' \circ T : T \in \mathcal{A}, y' \in B_{F'}\}$ is β-equicontinuous in $C_b(X,E)'_\beta$. Hence by (2.6), (2.7) and Lemma 2.2, we get

$$\sup_{T\in\mathcal{A}} \widetilde{m}_T(X) = \sup\{|(m_T)_{y'}|(X) : T \in \mathcal{A}, y' \in B_{F'}\} < \infty,$$

and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that

$$\sup_{T\in\mathcal{A}} \widetilde{m}_T(X \setminus K) = \sup\{|(m_T)_{y'}|(X \setminus K) : T \in \mathcal{A}, y' \in B_{F'}\} \le \varepsilon.$$

(ii)$\Rightarrow$(i) Assume that (ii) holds. Then by Lemma 2.2 and (2.6), (2.7), we obtain that the family $\{y' \circ T : T \in \mathcal{A}, y' \in B_{F'}\}$ is β-equicontinuous in $C_b(X,E)'_\beta$, and it follows that $\mathcal{A}$ is $(\beta, \|\cdot\|_F)$-equicontinuous.

(ii)$\Leftrightarrow$(iii) It follows from Lemma 2.3. ∎

In view of [16, Theorem 2] we have the following useful result.

Theorem 3.2. *Let $\mathcal{A}$ be a τ_s-compact subset of $\mathcal{L}_\beta(C_b(X,E),F)$. Then the set $\{y' \circ T : T \in \mathcal{A}, y' \in B_{F'}\}$ is a $\sigma(C_b(X,E)'_\beta, C_b(X,E))$-compact subset of $C_b(X,E)'_\beta$.*

Assume that X is a locally compact space. Then $\beta = \beta_\tau$ and β is the topology defined by Buck [2] (see [6, p. 844]).

Recall that X is a P-space if every G_δ set in X is open (see [8]). Then every compact set in X is finite and $\beta = \beta_\tau$ on $C_b(X)$ (see [18, Theorem 2.2]) and it follows that $\beta = \beta_\tau$ on $C_b(X,E)$.

Note that if X is a locally compact paracompact space (resp. a P-space), then $(C_b(X,E),\beta)$ is a strongly Mackey space, that is, every relatively $\sigma(C_b(X,E)'_\beta,$ $C_b(X,E))$-countably compact subset of $C_b(X,E)'_\beta$ is β-equicontinuous (see [11, Theorem 6.1], [12, theorem 5]).

Corollary 3.3. *Assume that X is a locally compact paracompact space (resp. a P-space). Let $\mathcal{A}$ be a τ_s-compact subset of $\mathcal{L}_\beta(C_b(X,E),F)$. Then $\mathcal{A}$ is $(\beta, \|\cdot\|_F)$-equicontinuous.*

Proof. Since $(C_b(X,E),\beta)$ is a strongly Mackey space, by Theorem 3.2 $\{y' \circ T : T \in \mathcal{A}, y' \in B_{F'}\}$ is a β-equicontinuous subset of $C_b(X,E)'_\beta$, and it follows that $\mathcal{A}$ is $(\beta, \|\cdot\|_F)$-equicontinuous. ∎

Now we can state a characterization of τ_s-compact sets in $\mathcal{L}_\beta(C_b(X,E),F)$ in terms of the properties of the corresponding sets of representing operator-valued Borel measures.

Theorem 3.4. *Assume that X is a locally compact paracompact space (resp. a P-space). Then for a subset $\mathcal{A}$ of $\mathcal{L}_\beta(C_b(X,E),F)$, the following statements are equivalent:*

(i) *$\mathcal{A}$ is relatively τ_s-compact.*

(ii) *$\mathcal{A}$ is $(\beta, \|\cdot\|_F)$-equicontinuous and for every $f \in C_b(X,E)$, the set $\{T(f) : T \in \mathcal{A}\}$ is relatively compact in F.*

(iii) *The following statements hold:*

  (a) *$\sup_{T\in\mathcal{A}} \widetilde{m}_T(X) < \infty$ and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ (resp. $M \in \mathcal{F}$) such that $\sup_{T\in\mathcal{A}} \widetilde{m}_T(X \setminus K) \le \varepsilon$ (resp. $\sup_{T\in\mathcal{A}} \widetilde{m}_T(X \setminus M) \le \varepsilon$).*

  (b) *For every $f \in C_b(X,E)$, the set $\{\int_X f\, dm_T : T \in \mathcal{A}\}$ is relatively compact in F''.*

(iv) *The following statements hold:*

  (a) *$\sup_{T\in\mathcal{A}} \|T\| < \infty$ and for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ (resp. $M \in \mathcal{F}$) such that $\sup_{T\in\mathcal{A}} \|\int_X f\, dm_T\|_{F''} \le \varepsilon$ whenever $f \in C_b(X,E), \|f\| \le 1$ and $f \equiv 0$ on K (resp. M).*

  (b) *For every $f \in C_b(X,E)$, the set $\{\int_X f\, dm_T : T \in \mathcal{A}\}$ is relatively compact in F''.*

Proof. (i)$\Rightarrow$(ii) Assume that (i) holds. Then by Corollary 3.3 the set $\mathcal{A}$ is $(\beta, \|\cdot\|_F)$-equicontinuous. Clearly for each $f \in C_b(X,E)$, the set $\{T(f) : T \in \mathcal{A}\}$ is relatively compact in F.

(ii)$\Rightarrow$(ii) It follows from [1, Chap. 3, §3.4, Corollary 1].

(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) It follows from Proposition 3.1. ∎

4. Sequential completeness of $(\mathcal{L}_\beta(C_b(X,E),F), \tau_s)$

It is known that if X is a paracompact space, then X is metacompact and normal. It follows that if X is a locally compact paracompact space, then $\beta = \beta_\tau$ on $C_b(X,E)$ and the space $(C_b(X,E)'_\beta, \sigma(C_b(X,E)'_\beta, C_b(X,E)))$ is sequentially complete (see [13, Theorem 3]).

Now we can state a Banach-Steinhaus type theorem for $(\beta, \|\cdot\|_F)$-continuous operators $T : C_b(X,E) \to F$.

Theorem 4.1. *Assume that X is a locally compact paracompact space. Let $T_k : C_b(X,E) \to F$ be a $(\beta, \|\cdot\|_F)$-continuous linear operator for $k \in \mathbb{N}$. Assume that $T(f) := \lim_k T_k(f)$ exists in F for every $f \in C_b(X,E)$. Then T is a $(\beta, \|\cdot\|_F)$-continuous linear operator and the set $\{T_k : k \in \mathbb{N}\}$ is $(\beta, \|\cdot\|_F)$-equicontinuous.*

Proof. In view of the Banach-Steinhaus theorem $T : C_b(X,E) \to F$ is a bounded linear operator. Then for each $y' \in F'$ $(y' \circ T)(f) = \lim(y' \circ T_k)(f)$ for all $f \in C_b(X,E)$, where $y' \circ T_k \in C_b(X,E)'_\beta$ for $k \in \mathbb{N}$ and $y' \circ T \in C_b(X,E)'$. It follows that $(y' \circ T_k)$ is a $\sigma(C_b(X,E)'_\beta, C_b(X,E))$-Cauchy sequence in $C_b(X,E)'_\beta$. Note that under the assumptions on X, we have that $\beta = \beta_\tau$ on $C_b(X,E)$ and hence by [13, Theorem 3] the space $(C_b(X,E)'_\beta, \sigma(C_b(X,E)'_\beta, C_b(X,E)))$ is sequentially complete. Hence for each $y' \in F'$ there exists $\Phi_{y'} \in C_b(X,E)'_\beta$ such that $\Phi_{y'}(f) = \lim(y' \circ T_k)(f)$ for all $f \in C_b(X,E)$. Then $y' \circ T = \Phi_{y'} \in C_b(X,E)'_\beta$. Since β is a Mackey topology, we derive that T is $(\beta, \|\cdot\|_F)$-continuous. Thus $T_k \to T$ for $\mathcal{T}_s$ in $\mathcal{L}_\beta(C_b(X,E),F)$, so $\{T_k : k \in \mathbb{N}\} \cup \{T\}$ is a $\mathcal{T}_s$-compact subset of $\mathcal{L}_\beta(C_b(X,E),F)$. Hence by Corollary 3.3 the set $\{T_k : k \in \mathbb{N}\}$ is $(\beta, \|\cdot\|_F)$-equicontinuous. ■

As a consequence of theorem 4.1 we get:

Corollary 4.2. *Assume that X is a locally compact paracompact space. Then the space $(\mathcal{L}_\beta(C_b(X,E),F), \tau_s)$ is sequentially complete.*

Proof. Let (T_k) be a τ_s-Cauchy sequence in $\mathcal{L}_\beta(C_b(X,E),F)$. Then for each $f \in C_b(X,E)$, $(T_k(f))$ is a Cauchy sequence in F, so $T(f) := \lim_k T_k(f)$ exists in F. By Theorem 4.1 T is $(\beta, \|\cdot\|_F)$-continuous and $T_k \to T$ for τ_s. ■

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