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SELECTION THEOREM IN L^1

Andrzej Nowak

Silesian University
Institute of Mathematics
Bankowa 14, 40–007 Katowice, Poland

e-mail: anowak@ux2.math.us.edu.pl

and

Celina Rom

University of Bielsko-Biała
Department of Mathematics
Willowa 2, 43–309 Bielsko-Biała, Poland

Abstract

Let F be a multifunction from a metric space X into L^1, and B a subset of X. We give sufficient conditions for the existence of a measurable selector of F which is continuous at every point of B. Among other assumptions, we require the decomposability of $F(x)$ for $x \in B$.

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Let (X, ϱ) be a metric space and Y a separable Banach space. If $A \subset X$, then $\bar{A}$ denotes the closure of A. By 2^X we mean the family of all nonempty subsets of X and $\mathcal{B}(X)$ stands for the Borel σ-field on X. We assume that X is endowed with a σ-finite measure μ, and denote by $\mathcal{B}_\mu(X)$ the completion of $\mathcal{B}(X)$ with respect to μ.

A mapping $F : X \to 2^Y$ is called a *multifunction* from X to Y. By the *graph* of F we mean $\operatorname{Gr} F = \{(x, y) \in X \times Y : y \in F(x)\}$. A function $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$ is a *selector* of F. We say that F has a *Castaing representation* if there exists a countable family $\mathcal{F}$ of measurable selectors of F such that $F(x) = \operatorname{cl}\{f(x) : f \in \mathcal{F}\}$, $x \in X$.

Let $F : X \to 2^Y$ be a multifunction. For $V \subset Y$ we define $F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$. The multifunction F is $\mathcal{B}_\mu(X)$-*measurable* if $F^{-1}(V) \in \mathcal{B}_\mu(X)$ for each open $V \subset Y$. If F is $\mathcal{B}_\mu(X)$-measurable and closed-valued, then its graph $\operatorname{Gr} F$ belongs to the product σ-field $\mathcal{B}_\mu(X) \otimes \mathcal{B}(Y)$. We say that a multifunction $F : X \to 2^Y$ is *lower semicontinuous* if for each open $V \subset Y$ the preimage $F^{-1}(V)$ is open in X.

A.V. Arutyunov [2] posed the following problem: Let F be a $\mathcal{B}_\mu(X)$-measurable and closed-valued multifunction from X to Y. It is well known that such an F has a $\mathcal{B}_\mu(X)$-measurable selector. Let B be a subset of X and suppose that $F|_B$ satisfies assumptions of a continuous selection theorem. Does there exist a measurable selector of F which is continuous at any point of B ?

Of course, if B is open, then there is no problem. In [2] A.V. Arutyunov obtained the following result (in fact, we cite a simplified version of his theorem):

Theorem A. *Suppose $F : X \to 2^Y$ is $\mathcal{B}_\mu(X)$-measurable and closed-valued, F is lower semicontinuous on $\bar{B}$, $F(x)$ is convex for each $x \in B$ and the measure μ is regular. Then F has a $\mathcal{B}_\mu(X)$-measurable selector which is continuous at every point of B.*

The proof is based on the Michael continuous selection theorem and the Lusin theorem.

In this note, we give an analogous result for the case when $Y = L^1$ and $F(x)$ need not be convex for $x \in B$.

Let $(T, \mathcal{A}, P)$ be a probability space with a non-atomic measure P, E a Banach space, and $L^1(T, E)$ (abbreviated to L^1) the space of equivalence classes of Bochner integrable functions $u : T \to E$. Then L^1 is a Banach space with the norm $\|u\| = \int_T \|u(t)\|_E P(dt)$. We say that the subset $Z \subset L^1$ is *decomposable* if for each $u, v \in Z$, $A \in \mathcal{A}$ we have $\chi_A u + \chi_{T\setminus A} v \in Z$ (see [9], [5]). If $Z \subset L^1$, then *cldec* Z stands for the *closed decomposable hull* of Z, i.e., the least closed and decomposable set containing Z.

The main result of this note is the following

Theorem. *Let $F : X \to 2^{L^1}$ be $\mathcal{B}_\mu(X)$-measurable and closed-valued, and B a separable subset of X. Assume that F is lower semicontinuous on $\bar{B}$, $F(x)$ are decomposable for $x \in B$ and L^1 is separable. Then F has a $\mathcal{B}_\mu(X)$-measurable selector f which is continuous at every point of B.*

In the proof we shall use the following continuous selection theorem of A. Bressan and G. Colombo [3], a generalization of A. Fryszkowski's result [4]:

Theorem B. *Let X be a separable metric space, and $\Phi : X \to 2^{L^1}$ a lower semicontinuous multifunction with closed and decomposable values. Then Φ has a continuous selector.*

We adopt the same method of the proof as A.V. Arutyunov [2], but unlike this author we do not apply the Lusin theorem, but use the following graph-conditioned measurable selection theorem:

Theorem C ([7])**.** *Suppose $(M, \mathcal{M})$ is a complete measurable space, Y is a Polish space, and $\Phi : M \to 2^Y$ is a multifunction such that* $\operatorname{Gr}\Phi \in \mathcal{M} \otimes \mathcal{B}(Y)$*. Then Φ has a Castaing representation.*

We shall omit these parts of the proof of our Theorem, which are the same as in Arutyunov [2].

Proof of Theorem. First, we consider the case when the set B is closed. By Theorem B, $F|_B$ has a continuous selector $\tilde{f} : B \to L^1$. For a fixed $x \in X$, let $d(x)$ denote the gap between the graph of $\tilde{f}$ and $\{x\} \times F(x)$, i.e.,

$$d(x) = \inf\{\|y - \tilde{f}(\xi)\| + \varrho(x, \xi) : y \in F(x),\ \xi \in B\}.$$

It is immediate that $d(x) = 0$ iff $x \in B$.

Let $\hat{F} : X \setminus B \to 2^{L^1}$ be defined by

$$\hat{F}(x) = \{y \in F(x) : \bigvee_{\xi \in B} \|y - \tilde{f}(\xi)\| + \varrho(x, \xi) < 2d(x)\}.$$

We shall see that $\hat{F}$ has a measurable graph. Note that

$$\operatorname{Gr}\hat{F} = \operatorname{Gr}F \cap \bigcup_{\xi \in D} \{(x, y) \in X \setminus B \times L^1 : \|y - \tilde{f}(\xi)\| + \varrho(x, \xi) < 2d(x)\},$$

where D is a countable dense subset of B. Since F is measurable and closed-valued, $\operatorname{Gr}F \in \mathcal{B}_\mu(X) \otimes \mathcal{B}(L^1)$, and F has a Castaing representation $\{f_i : i \in \mathbb{N}\}$. Now $d(x) = \inf\{\|f_i(x) - \tilde{f}(\xi)\| + \varrho(x, \xi) : i \in \mathbb{N}, \xi \in D\}$ and, consequently, d is measurable. Hence, $\operatorname{Gr}\hat{F} \in \mathcal{B}_\mu(X) \otimes \mathcal{B}(L^1)$. By Theorem C, $\hat{F}$ has a $\mathcal{B}_\mu(X)$-measurable selector $\hat{f}$. Let $f : X \to L^1$ be defined by

$f(x) = \tilde{f}(x)$ for $x \in B$ and $f(x) = \hat{f}(x)$ for $x \in X \setminus B$. Then f is a measurable selector of F with required properties. We omit the proof that f is continuous at any point of B, which is the same as in Arutyunov [2, Lemma 1].

Now let B be arbitrary. Define the new multifunction $H : X \to 2^{L^1}$ by $H(x) = \operatorname{cldec} F(x)$. It is measurable and lower semicontinuous at each point of $\bar{B}$ (cf. [8], [10]). By the first part of the proof, H has a measurable selector h which is continuous on $\bar{B}$. Let $r(x) = d(h(x), F(x))$, $x \in X$. It is immediate that r is $\mathcal{B}_\mu(X)$-measurable, and $r(x) = 0$ for all $x \in B$. Let $C = \{x \in X : r(x) = 0\}$, and $\hat{F} : X \setminus C \to 2^{L^1}$ be defined by

$$\hat{F}(x) = \{y \in F(x) : \|y - h(x)\| < 2r(x)\}.$$

We have

$$\operatorname{Gr} \hat{F} = \operatorname{Gr} F \cap \{(x, y) \in X \setminus C \times L^1 : v(x, y) < 0\},$$

where $v(x, y) = \|y - h(x)\| - 2r(x)$. Being measurable in x and continuous in y, v is $\mathcal{B}_\mu(X) \otimes \mathcal{B}(L^1)$-measurable. Consequently, the graph of $\hat{F}$ belongs to $\mathcal{B}_\mu(X) \otimes \mathcal{B}(L^1)$. By Theorem C, $\hat{F}$ has a measurable selector $\hat{f}$. Define $f : X \to L^1$ as $f(x) = h(x)$ for $x \in C$ and $f(x) = \hat{f}(x)$ for $x \in X \setminus C$. Such an f is a measurable selector of F which is continuous at any point of B (see the final part of the proof of Theorem 1 in [2]). It completes the proof.

Remark. It follows from the proof that in the case of a closed B any continuous selector $\tilde{f}$ of $F|_B$ can be extended to a $\mathcal{B}_\mu(X)$-measurable selector f, which is continuous at any point of B.

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