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SOME NOTES ON ONE OSCILLATORY CONDITION OF NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract

The aim of this paper is to present sufficient conditions for all bounded solutions of the second order neutral differential equations of the form

$$\big(r(t)(x(t) - px(t-\tau))'\big)' - q(t)f(x(\sigma(t))) = 0$$

to be oscillatory and to compare some existing results.

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1. Introduction

We consider the second order neutral differential equation of the form

$$\big(r(t)(x(t) - px(t-\tau))'\big)' - q(t)f(x(\sigma(t))) = 0\ , \quad t \geq t_0 \tag{1}$$

under the following assumptions:

(a) $0 \leq p \leq 1$ and $\tau > 0$ are constants;

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(b) $r, q \in C([t_0, \infty);\ (0, \infty)),\ R(t) = \int_{t_0}^{t} \frac{ds}{r(s)} \ \to \infty\ ,\ t \to \infty$;

(c) $\sigma \in C([t_0, \infty); R)$, $\sigma(t) \le t$, σ is a nondecreasing and $\lim_{t\to\infty} \sigma(t) = \infty$;

(d) $f \in C(R; R)$, $uf(u) > 0$ for $u \ne 0$, f is nondecreasing and

$$\liminf_{u\to 0} \frac{f(u)}{u} > 0.$$

By a solution of (1) we mean a continuous function $x(t)$ defined on an interval $[T_x; \infty), T_x \ge t_0$ such that $r(t)(x(t) - px(t-\tau))'$ is a continuously differentiable and $x(t)$ satisfies (1) for all sufficiently large t. We focus on solutions of (1) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. Such a solution is said to be oscillatory if it has arbitrarily large zeros; otherwise it said to be nonoscillatory.

The problem of oscillation of neutral differential equations has received considerable attention in the last few years. One way to obtain conditions for qualitative properties of solutions of neutral differential equations is to transform known results of ordinary or delay differential equations. The purpose of this paper is to present a generalization of one oscillation condition for second order differential equations. G.S. Ladde, V. Lakshmikantham and B.G. Zhang in [7] proved bounded oscillation criteria for second order differential equations with a deviating argument

$$(r(t)x'(t))' - q(t)x(\sigma(t)) = 0. \tag{2}$$

Theorem A [7, Theorem 4.3.1]**.** *Assume that* (b)*,* (c)*, and* (d) *hold. Further assume that*

$$\limsup_{t\to\infty} \frac{1}{r(t)} \int_{\sigma(t)}^{t} (s - \sigma(t))q(s)\, ds > 1. \tag{3}$$

Then bounded solutions of (2) *are oscillatory.*

The following example points out that assumptions of Theorem A do not guarantee for all bounded solutions to be oscillatory. Some assumptions are missing.

Example 1. Consider the equation

$$\left(\frac{1}{\sqrt{t}}\,x'(t)\right)' - \frac{1}{8\sqrt{t^5}}\,x\left(\frac{t}{64}\right) = 0\ , \quad t \ge 1.$$

It is easy to verify that the condition (3) holds but the equation has the bounded nonoscillatory solution $x(t) = \frac{1}{\sqrt{t}}$ on $[1;\infty)$.

2. Main results

Theorem 2.1. *Assume that* (a)–(d) *hold and* $0 < p < 1$*. Let there exist a positive integer* n *such that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\big(R(s) - R(\sigma(t))\big)\,ds > \frac{1-p}{1-p^{n-1}} \limsup_{u\to 0} \frac{u}{f(u)}. \tag{4}$$

Then every bounded solution of (1) *is oscillatory.*

Proof. Assume the converse and suppose that equation (1) possesses an eventually positive bounded solution $x(t)$. The case $x(t)$ negative can by treated similarly. Let us define

$$z(t) = x(t) - px(t-\tau). \tag{5}$$

We have $(r(t)z'(t))' > 0$ for all large t, say $t \ge t_0$. If $r(t)z'(t) > 0$ eventually, then according to (b) $\lim_{t\to\infty} z(t) = \infty$, which contradicts the boundedness of x. Therefore, $r(t)z'(t) < 0$ for $t \ge t_1 \ge t_0$, which implies that the function z is decreasing. There are two possibilities for z:

(i) $z(t) > 0$ for $t \ge t_2 \ge t_1$,

(ii) $z(t) < 0$ for $t \ge t_2$.

Assume that (i) holds. The function rz' is increasing so that there exists $\lim_{t\to\infty} r(t)z'(t) = c \le 0$. We shall show that $c = 0$. For the contradiction let us assume that $c < 0$. Then $r(t)z'(t) \le c < 0$ for $t \ge T$, T sufficiently large. Dividing the last inequality by r and integrating from T to ∞ we have got, according to (b), a contradiction with the positivity of z. So $\lim_{t\to\infty} r(t)z'(t) = 0$.

Further, the function z is positive, decreasing. It follows that $\lim_{t\to\infty} z(t) = d \geq 0$. Again for the contradiction let us assume that $d > 0$. Then $z(t) \geq d$ for $t \geq t_2$ and we have from (5)

$$x(t) = z(t) + px(t-\tau) \geq z(t) \geq d.$$

Taking into account the monotonicity of the function f we obtain from equation (1)

$$(r(t)z'(t))' \geq f(d)q(t), \quad t \geq t_3 \geq t_2.$$

Multiplying this inequality by $R(t) - R(t_3)$ and integrating from t_3 to $t \geq t_3$ we get

$$\begin{aligned} -(z(t) - z(t_3)) &\geq r(t)z'(t)\big(R(t) - R(t_3)\big) - (z(t) - z(t_3)) \\ &\geq f(d) \int_{t_3}^{t} q(s)\big(R(s) - R(t_3)\big)\, ds. \end{aligned}$$

From this inequality for $t \to \infty$ we obtain that the integral on the right hand side is convergent which implies

$$\lim_{t_3\to\infty} \int_{t_3}^{\infty} q(s)\big(R(s) - R(t_3)\big)\, ds = 0.$$

This is a contradiction to (4) for $t_3 = \sigma(t)$ and so $\lim\limits_{t\to\infty} z(t) = 0$.

Using (5) we get

$$x(t) = z(t) + px(t-\tau) = z(t) + pz(t-\tau) + p^2 x(t-2\tau).$$

Repeating this procedure because of the monotonicity of z, the positivity of x we obtain

$$x(t) \geq \Big(\sum_{i=0}^{n} p^i\Big) z(t).$$

For simplicity let us denote $k = \sum_{i=0}^{n} p^i$. Then in view of the monotonicity of the function f one gets

$$(r(t)z'(t))' \geq q(t)f(kz(\sigma(t))), \quad t \geq T, \quad T\text{-sufficiently large.} \tag{6}$$

Integration (6) from s to $t \geq s \geq T$ yields

$$-r(s)z'(s) \geq r(t)z'(t) - r(s)z'(s) \geq \int_s^t q(s)f(kz(\sigma(s)))\,ds. \tag{7}$$

Dividing by $r(s)$ and integrating it with respect to s from $\sigma(t)$ to t we see that

$$\begin{aligned} -z(t) + z(\sigma(t)) &\geq \int_{\sigma(t)}^t \frac{1}{r(s)} \int_s^t q(u)f(kz(\sigma(u)))\,du\,ds \\ &= \int_{\sigma(t)}^t q(u)f(kz(\sigma(u)))\big(R(u) - R(\sigma(t))\big)\,du. \end{aligned}$$

Taking into account the monotonicity of the functions f, z, σ we obtain

$$0 \geq f(kz(\sigma(t)))\left[\int_{\sigma(t)}^t q(u)\big(R(u) - R(\sigma(t))\big)\,du - \frac{kz(\sigma(t))}{f(kz(\sigma(t))} \cdot \frac{1}{k}\right] \tag{8}$$

which contradicts the positiveness of z and (4).

In the case (ii) by (5) we have

$$x(t) < px(t-\tau) < p^2x(t-2\tau) < \cdots < p^nx(t-n\tau)$$

for $t \geq t_2 + n\tau$ and we can conclude that $\lim_{t\to\infty} x(t) = 0$. It follows that $\lim_{t\to\infty} z(t) = 0$. This is a contradiction. ∎

Theorem 2.2. *Assume that* (a)–(d) *hold and* $0 < p < 1$. *Let*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^t q(s)\big(R(s) - R(\sigma(t))\big)\,ds > (1-p)\limsup_{u\to 0} \frac{u}{f(u)}. \tag{9}$$

Then every bounded solution of (1) *is oscillatory.*

Proof. Denote $a = \limsup_{t\to\infty} \int_{\sigma(t)}^t q(s)\big(R(s) - R(\sigma(t))\big)\,ds$. Let an integer n be chosen so that $a > \frac{1-p}{1-p^{n-1}} \limsup_{u\to 0} \frac{u}{f(u)}$. Then the assertion of Theorem 2.2 follows immediately from Theorem 2.1. ∎

Remark 1. Theorems 2.1 and 2.2 are true also in the case $p = 0$.

Remark 2. In the linear case, $f(u) = u$ and $p = 0$, Theorem 2.2 gives the sufficient condition for the oscillation of all bounded solutions of equation (2) in the form

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\big(R(s) - R(\sigma(t))\big)\, ds > 1. \tag{10}$$

It is easy to verify that for the equation from the Example 1 the condition (10) does not hold.

Using the theory of Rieman-Stielties integral, Theorem 2.2 can be expressed as a modification of Theorem A.

Theorem 2.3. *Assume that* (a)–(d) *hold with* $0 < p < 1$ *and* r *is an increasing function. If there exists a positive integer* n *such that*

$$\limsup_{t\to\infty} \frac{1}{r(t)} \int_{\sigma(t)}^{t} (s - \sigma(t))q(s)\, ds > \frac{1-p}{1-p^{n-1}} \limsup_{u\to 0} \frac{u}{f(u)}, \tag{11}$$

then every bounded solution of (1) *is oscillatory.*

Proof. The proof is similar to the proof of Theorem 2.1 until (7). Integrating this inequality from $\sigma(t)$ to t we see that

$$0 \ge \int_{\sigma(t)}^{t} r(s)\, dz(s) + \int_{\sigma(t)}^{t} (u - \sigma(t))q(u)f(kz(\sigma(u)))\, du.$$

Using the monotonicity of the functions r, f, z, σ we obtain

$$\begin{aligned}
0 \;&\ge\; r(t)z(t) - r(\sigma(t))z(\sigma(t)) \\
&\quad - \int_{\sigma(t)}^{t} z(s)\, dr(s) + \int_{\sigma(t)}^{t} (u - \sigma(t))q(u)f(kz(\sigma(u)))\, du \\
&\ge\; r(t)(z(t) - z(\sigma(t))) - z(\sigma(t))(r(t) - r(\sigma(t))) \\
&\quad + f(kz(\sigma(t))) \int_{\sigma(t)}^{t} (u - \sigma(t))q(u))\, du \\
&\ge\; r(t)(z(t) - z(\sigma(t))) + f(kz(\sigma(t))) \int_{\sigma(t)}^{t} (u - \sigma(t))q(u))\, du,
\end{aligned}$$

or

$$0 \geq z(t) - z(\sigma(t)) + \frac{f(kz(\sigma(t)))}{r(t)} \int_{\sigma(t)}^{t} (u - \sigma(t))q(u))\,du.$$

As it is customary, all functional inequalities are assumed to hold eventually, that is they are satisfied for all sufficiently large t. Dividing the above inequality by $z(\sigma(t))$ and using the monotonicity of z, σ we get

$$0 \geq \frac{z(t)}{z(\sigma(t))} + \frac{f(kz(\sigma(t)))}{kz(\sigma(t))}\left[\frac{k}{r(t)} \int_{\sigma(t)}^{t} (u - \sigma(t))q(u))\,du - \frac{kz(\sigma(t))}{f(kz(\sigma(t)))}\right].$$

Because of (11) we have arrived at a contradiction.

In the case $z(\sigma(t)) < 0$ the proof of the theorem continues as the proof of Theorem 2.1. ∎

Analogously as Theorem 2.2 we can obtain the next theorem

Theorem 2.4. *Assume that* (a)–(d) *hold with* $0 < p < 1$ *and* r *is an increasing function. Let*

$$\limsup_{t\to\infty} \frac{1}{r(t)} \int_{\sigma(t)}^{t} (s - \sigma(t))q(s)\,ds > (1 - p) \limsup_{u\to 0} \frac{u}{f(u)}. \tag{12}$$

Then every bounded solution of (1) *is oscillatory.*

Remark 3. Theorems 2.3 and 2.4 hold for the case $p = 0$ also.

Remark 4. If $p = 0$, $f(u) = u$ we have that equation (2) and condition (12) are equivalent to (3) but only under an additional assumption on the function r. So it means that Theorem 2.4 cannot be used on the equation from the Example 1 because the function $r(t) = \frac{1}{\sqrt{t}}$ is decreasing on the interval $[1; \infty)$.

Example 2. Consider the equation

$$(\sqrt{t}y'(t))' - \frac{1}{5\sqrt{t}^3} y(\sqrt{t}) = 0, \quad t \geq 1.$$

Theorem 2.4 cannot be used, because the assumption (12) fails, but by Theorem 2.2 every bounded solution of this equation is oscillatory.

Example 3. Consider the equation

$$(\sqrt{t}y'(t))' - \frac{1}{4\sqrt{t}}y((\sqrt{t}-\pi)^2) = 0, \quad t \geq \frac{\pi^2}{4}.$$

For this equation we can apply any of the Theorems 2.2 or 2.4. They say that all bounded solutions are oscillatory. One such solution is $y(t) = \sin\sqrt{t}$.

Remark 5. In the case $f(u) = u$ and $r(t) = 1$ Theorem 2.4 (Theorem 2.2) gives the result obtained in [2].

Theorem 2.5. *Assume that* (a)–(d) *hold and* $p = 1$. *Let*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\big(R(s) - R(\sigma(t))\big)\, ds > 0. \tag{13}$$

Then every bounded solution of (1) *is oscillatory.*

Proof. Assume that x is an eventually positive bonded solution of equation (1). We can proceed exactly as in the proof of Theorem 2.1 to see that there are two possibilities for z:

(i) $z(t) > 0$, $z'(t) < 0$ for $t \geq t_2 \geq t_1$,

(ii) $z(t) < 0$, $z'(t) < 0$ for $t \geq t_2$.

Assume that (i) holds. Denote $a = \limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\big(R(s) - R(\sigma(t))\big)\, ds$. Let an integer n be chosen so that

$$a > \frac{1}{n} \limsup_{u\to 0} \frac{u}{f(u)} > 0. \tag{14}$$

Analogously as in the proof of Theorem 2.1 we are led to (8) with constant $k = n$, which contradicts (14).

In the case (ii) we have $\lim_{t\to\infty} z(t) = -\alpha$, where $\alpha > 0$ is a finite number. So there exists $t_3 \geq t_2$ such that $-\alpha < z(t) < -\frac{\alpha}{2}$, $t \geq t_3$. Thus

$$-\alpha < x(t) - x(t-\tau) < -\frac{\alpha}{2}, \quad t \geq t_3.$$

Consequently,

$$x(t) < -\frac{\alpha}{2} + x(t-\tau) < 2\frac{\alpha}{2} + x(t-2\tau) < \cdots < -n\frac{\alpha}{2} + x(t-n\tau)$$

for $t \geq t_3 + n\tau$. Choose a sequence $\{t_n\}$ such that $t_n = t_3 + n\tau$. Then

$$x(t_3 + n\tau) < -n\frac{\alpha}{2} + x(t_3)$$

and therefore $\lim_{t\to\infty} x(t_n) = -\infty$. This is a contradiction to the boundedness of x. ■

Combining our previous results we have

Corollary 2.1. *Assume that* (a)–(d) *hold and* $0 \leq p \leq 1$. *Further assume that*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\big(R(s) - R(\sigma(t))\big)\, ds > (1-p) \limsup_{u\to 0} \frac{u}{f(u)}.$$

Then every bounded solution of (1) *is oscillatory.*

Corollary 2.2. *Assume that* (a)–(d) *hold with* $0 \leq p \leq 1$ *and* r *is an increasing function. Let*

$$\limsup_{t\to\infty} \frac{1}{r(t)} \int_{\sigma(t)}^{t} (s - \sigma(t))q(s)\, ds > (1-p) \limsup_{u\to 0} \frac{u}{f(u)}.$$

Then every bounded solution of (1) *is oscillatory.*

REFERENCES

[1] T.A. Čanturia and R.G. Koplatadze, *On oscillatory properties of differential equations with deviating arguments*, Tbilisi, Univ. Press, Tbilisi, 1977 (in Russian).

[2] J. Džurina, *On unstable neutral differential equations of the second order*, Czech. Math. J. **52** (127) (2002), 739–747.

[3] J. Džurina and D. Lacková, *Oscillation results for second order nonlinear differential equations*, CEJM **2** (1) (2004), 57–66.

[4] L.H. Erbe, Q. Kong and B.G. Zhang, *Oscillation theory for functional differential equations*, Dekker, New York 1995.

[5] G.B. Gustafson, *Bounded oscillation of linear and nonlinear delay differential equations of even order*, J. Math. Anal. Appl. **46** (1974), 175–189.

[6] Š. Kulcsár, *On the asymptotic behaviour of solutions of the second order neutral differential equations*, Publ. Math. Debrecen **57** (1–2) (2000), 153–161.

[7] G.S. Ladde, V. Lakshmikanthan and B.G. Zhang, *Oscillation theory of differential equations with deviating arguments*, Dekker, New York 1987.

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