

ON DIFFERENTIAL INCLUSIONS OF VELOCITY
HODOGRAPH TYPE WITH CARATHÉODORY
CONDITIONS ON RIEMANNIAN MANIFOLDS *

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Abstract

We investigate velocity hodograph inclusions for the case of right-hand sides satisfying upper Carathéodory conditions. As an application we obtain an existence theorem for a boundary value problem for second-order differential inclusions on complete Riemannian manifolds with Carathéodory right-hand sides.

Keywords: differential inclusions, Carathéodory conditions, velocity hodograph, Riemannian manifold, two-point boundary value problem.

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The velocity hodograph equation is a special integral equation in a tangent space $T_{m_0}M$ to a Riemannian manifold M that can be constructed from a second order differential equation on M so that the solutions of the equations on M , starting at m_0 , are simply represented via the solutions of the velocity hodograph equation. Thus the velocity hodograph is a powerful tool for reducing equations on manifolds to equations in a single linear space.

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This sort of equations was suggested in [4] and applied to the investigation of a boundary-value problem for second order differential equations on Riemannian manifolds. In [3] the construction was generalized to the case of second order differential inclusions with bounded upper semicontinuous right-hand sides on complete Riemannian manifolds and applied to a boundary value problem for mechanical systems with discontinuous forces on nonlinear configuration spaces. There the hodograph equation was replaced by the corresponding inclusion that we call the velocity hodograph inclusion. A detailed description can be found in [5].

Independently the same sort of inclusion was considered in [7] for a particular case of the Euclidean space but with a more general sort of right-hand sides that might not be jointly upper semicontinuous but satisfied upper Carathéodory conditions.

In this paper we present a generalization of both [3] and [7]: we deal with the velocity hodograph inclusions on complete Riemannian manifolds and with right-hand sides satisfying upper Carathéodory conditions. We describe all constructions within the proof of an existence theorem for solutions of the boundary value problem for second order differential inclusions with upper Carathéodory conditions on complete Riemannian manifolds. They have the physical meaning of equations of motion for complicated mechanical systems on nonlinear configuration spaces. Notice that for such spaces the boundary value problem may not be solvable even in the case of single-valued smooth bounded right-hand sides if the points are conjugate along all geodesic curves that join them. For non-conjugate points the solution may not exist on large time intervals (see details in [5]).

Basic facts from the theory of set-valued maps can be found in [2] and [6] and from geometry of manifolds – in [1].

Let $I \subset \mathbb{R}$ be an interval and M be a complete Riemannian manifold. Denote by TM the tangent bundle of M and by $T_m M$ the tangent space at $m \in M$. Consider $m_0 \in M$ and let $v : I \rightarrow T_{m_0} M$ be a continuous curve.

Theorem 1 (see, e.g., [5]). *There exists a unique C^1 -curve $\gamma : I \rightarrow M$ such that $\gamma(0) = m_0$ and the vector $\dot{\gamma}(t)$ is parallel along $\gamma(\cdot)$ to the vector $v(t) \in T_{m_0} M$ for every $t \in I$.*

Indeed, the curve γ is represented as $\gamma(t) = \delta^{-1}(\int_0^t v(\tau) d\tau)$, where δ is Cartan's development (see, e.g., [1]) and δ^{-1} is its inverse map developing C^1 -curves from $T_{m_0} M$ to M .

In what follows we denote by $Sv(\cdot)$ the curve $\gamma(\cdot)$ as above constructed from $v(\cdot)$.

Remark 2. Notice that if M is an Euclidian space, $Sv(t) = \int_0^t v(\tau)d\tau + m_0$.

Consider the Banach space $C^0(I, T_{m_0}M)$ of continuous maps from I to $T_{m_0}M$ and the Banach manifold $C^1(I, M)$ of C^1 -smooth maps from I to M . It follows from Theorem 1 that the operator

$$S : C^0(I, T_{m_0}M) \rightarrow C^1(I, M)$$

is well posed.

It is shown, e.g., in [5] that S is a homeomorphism between $C^0(I, T_{m_0}M)$ and its image $C_{m_0}^1(I, M)$ in $C^1(I, M)$, where the manifold $C_{m_0}^1(I, M)$ consists of all C^1 -curves γ with $\gamma(0) = m_0$.

Lemma 3 (see, e.g., [5]). *Let a point $m_1 \in M$ be not conjugate to m_0 along some geodesic of the Levi-Civita connection on M . Then for any geodesic $a(t)$, ($a(0) = m_0$, $a(1) = m_1$), along which m_0 and m_1 are not conjugate, and for any number $k > 0$ there exists a number $\bar{L}(m_0, m_1, k, a) > 0$ such that for $0 < t_1 < \bar{L}(m_0, m_1, k, a)$ and for any curve $u(t) \in U_k \subset C^0([0, t_1], T_{m_0}M)$ (where U_k is the ball of radius k centered at the origin), in a certain bounded neighbourhood of the vector $t_1^{-1}\dot{a}(0) \in T_{m_0}M$ there exists a unique vector $C_u \in T_{m_0}M$, continuously depending on u , for which the equality $S(u + C_u)(t_1) = m_1$ holds.*

Consider a single-valued force field $\alpha(t, m, X)$ on M , i.e., a vector field such that at any $m \in M$ the tangent vector $\alpha(t, m, X)$ depends on parameters $t \in I$ and $X \in T_mM$. For a differentiable curve $m(t)$ on M the vector $\alpha(t, m(t), \dot{m}(t))$ has the mechanical sense of force acting on the test particle at the time instant t , point $m(t)$ of configuration space M and velocity value $\dot{m}(t)$.

Let $m(t)$, where $t \in I$ and $m(0) = m_0$, be a C^1 -curve in M . Denote by $\Gamma\alpha(t, m(t), \dot{m}(t))$ the curve in $T_{m_0}M$ obtained by parallel translation of vectors $\alpha(t, m(t), \dot{m}(t))$ along $m(\cdot)$ to the point m_0 for all $t \in I$.

Remark 4. Note that the parallel translation operator Γ turns into the identity mapping in an Euclidian space, i.e., if M is an Euclidian space, then $\Gamma\alpha(t, m(t), \dot{m}(t)) = \alpha(t, m(t), \dot{m}(t))$.

Specify a vector C in $T_{m_0}M$ and consider the integral equation

$$(1) \quad m(t) = S \left(\int_0^t \Gamma\alpha(\tau, m(\tau), \dot{m}(\tau)) d\tau + C \right)$$

on $I = [0, l]$. It is shown, e.g., in [5] that (1) is the integral form of the second Newton's law, that is, its solution is a solution of the equation $\frac{D}{dt}\dot{m}(t) = \alpha(t, m(t), \dot{m}(t))$ (where $\frac{D}{dt}$ is the covariant derivative of Levi-Civita connection) having the initial condition $m(0) = m_0$ and $\dot{m}(0) = C$.

Let $m(t)$, $t \in I$, satisfy the above Newton's law, i.e., it is a solution of (1).

Definition 5. The velocity hodograph of the trajectory $m(t)$ is the curve $v : I \rightarrow T_{m_0}M$ such that $v(t)$ is parallel to $\dot{m}(t)$ along $m(\cdot)$ at any $t \in I$.

It is not hard to see that the velocity hodograph of a solution of (1) satisfies the equation

$$(2) \quad v(t) = \int_0^t \Gamma\alpha \left(\tau, Sv(\tau), \frac{d}{d\tau}Sv(\tau) \right) d\tau + C.$$

It is obvious that if v is a solution of (2), then Sv is a solution of (1), i.e., it satisfies the Newton's law (see [5] for details).

Suppose that for all $m \in M$ we have a set-valued mapping $F(m) : I \times T_mM \rightarrow 2^{T_mM}$ with closed, convex and bounded images, i.e., for all $t \in I$ and $X \in T_mM$ a certain set $F(m)(t, X) \subset T_mM$ is given. This family of maps for all $t \in I$, $m \in M$ and $X \in T_mM$ forms the set-valued map $F : I \times TM \rightarrow TM$ that is denoted by $F(t, m, X)$ (the pair (m, X) is a point of the tangent bundle TM , i.e., $X \in T_mM$). This map is a set-valued vector field of special type on M that is called set-valued force field.

Consider the second order differential inclusion

$$(3) \quad \frac{D}{dt}\dot{m}(t) \in F(t, m(t), \dot{m}(t)),$$

where $\frac{D}{dt}$ is the covariant derivative of Levi-Civita connection on M . Inclusion (3) is a geometrically invariant form of the second Newton's law for mechanical system with a set-valued force F .

Definition 6. A C^1 -curve $m(t)$, such that its derivative is absolutely continuous and inclusion (3) holds for $m(t)$ almost everywhere (a.e.), is called a solution of inclusion (3).

Definition 7. A set-valued force field $F(t, m, X)$ satisfies upper Carathéodory conditions if:

- (1) for every $(m, X) \in TM$ the map $F(\cdot, m, X) : I \rightarrow T_m M$ is measurable,
- (2) for almost all $t \in I$ the map $F(t, \cdot, \cdot) : TM \rightarrow TM$ is upper semicontinuous.

Theorem 8. Let a point $m_1 \in M$ be not conjugate with the point $m_0 \in M$ along some geodesic $a(t)$ of the Levi-Civita connection and let the field $F(t, m, X)$ satisfy upper Carathéodory conditions and be uniformly bounded for all t, m, X . There exists a number $L(m_0, m_1, a)$ such that for any $t_0, 0 < t < L(m_0, m_1, a)$ inclusion (3) has a solution $m(t)$ such that $m(0) = m_0$ and $m(t_0) = m_1$.

Proof. We shall construct a set-valued analog of velocity hodograph and apply it to prove the theorem.

Let the set-valued vector field $F(t, m, X)$ be bounded by a number $C > 0$. Evidently for a sufficiently small $t_1 > 0$ the inequality $t_1 < \bar{L}(m_0, m_1, Ct_1, a)$ holds, where $\bar{L}(m_0, m_1, Ct_1, a)$ is the number from Lemma 3. Define the number $L(m_0, m_1, a)$ as the supremum of above t_1 . Let $t_0 < L(m_0, m_1, a)$. Without loss of generality one can assume that $I = [0, t_0]$.

Consider the set-valued vector field $F(t, m(t), \dot{m}(t))$ defined along the C^1 -curve $m(t) = S(v(t))$, $v \in C^0(I, T_{m_0} M)$, and apply the parallel translation along $m(\cdot)$ at the point $m_0 = m(0)$ to all the sets $F(t, m(t), \dot{m}(t))$. Then for any given $v(\cdot) \in C^0(I, T_{m_0} M)$, we obtain the set-valued mapping $\Gamma F(t, S(v(\cdot)), \frac{d}{dt} S(v(\cdot))) : I \rightarrow T_{m_0} M$

Lemma 9. The set-valued mapping

$$(4) \quad \Gamma F \left(t, S(v(\cdot)), \frac{d}{dt} S(v(\cdot)) \right) : C^0(I, T_{m_0} M) \times I \rightarrow T_{m_0} M$$

satisfies upper Carathéodory conditions.

Proof. The first condition – measurability on t at any $v(\cdot)$ specified – follows from the fact, that the composition of measurable F and continuous S and Γ is a measurable map.

Since $F(t, m, X)$ is upper semicontinuous in (m, X) and the operator $S : C^0(I, T_{m_0} M) \rightarrow C^1(I, M)$ is a homeomorphism, for any t the composition $F(t, S(v(t)), \frac{d}{dt} S(v(t)))$ is upper semicontinuous in v . Now the required

statement follows from the fact that the operator Γ of parallel translation continuously depends on a C^1 -curve $Sv(\cdot)$. ■

Denote by

$$P\Gamma F \left(t, S(v(t)), \frac{d}{dt} S(v(t)) \right) = \left\{ y : y(t) \in \Gamma F \left(t, S(v(t)), \frac{d}{dt} S(v(t)) \right) \right\}$$

the set of all measurable selections of the set-valued map

$$\Gamma F \left(t, S(v(\cdot)), \frac{d}{dt} S(v(\cdot)) \right) : I \rightarrow T_{m_0} M$$

and consider the set of integrals with a variable upper limit of those selections, denoted by $\int P\Gamma F(t, S(v(t)), \frac{d}{dt} S(v(t))) dt$. Thus we have constructed the mapping

$$\int P\Gamma F \left(t, S(v(t)), \frac{d}{dt} S(v(t)) \right) dt : C^0(I, T_{m_0} M) \rightarrow C^0(I, T_{m_0} M).$$

Lemma 10. *The mapping $\int P\Gamma F(t, S(v(t)), \frac{d}{dt} S(v(t))) dt$ sends bounded sets of the space $C^0(I, T_{m_0} M)$ into compact sets.*

Proof. From lemma 3.1 of [5] and from completeness of the manifold it follows that for any ball $U_K \subset C^0(I, T_{m_0} M)$ with radius K and center at the origin the set of curves $\{(m(\cdot), \dot{m}(\cdot)) | m \in SU_K\}$ lays in a compact set of the manifold TM . Then from boundedness of $F(t, m, X)$ it follows that all sets $F(t, m(\cdot), \dot{m}(\cdot))$, $m(\cdot) \in SU_K$, are uniformly bounded. Since the parallel translation preserves the norm of a vector, all sets of curves $\Gamma F(t, S(v(t)), \frac{d}{dt} S(v(t)))$ are also uniformly bounded as well as the sets of their measurable selections $P\Gamma F(t, S(v(t)), \frac{d}{dt} S(v(t)))$. Hence, all continuous curves

$$u(\cdot) \in \cup_{v \in U_K} \left(\int P\Gamma F \left(t, S(v(t)), \frac{d}{dt} S(v(t)) \right) dt \right)$$

are uniformly bounded and equicontinuous. ■

Lemma 11. *The mapping $\int P\Gamma F(t, S(v(t)), \frac{d}{dt} S(v(t))) dt$ is upper semicontinuous.*

Proof. Since the mapping

$$\int P\Gamma F \left(t, S(v(t)), \frac{d}{dt}S(v(t)) \right) dt : C^0(I, T_{m_0}M) \rightarrow C^0(I, T_{m_0}M)$$

satisfies upper Carathéodory conditions and is uniformly bounded and since (by Lemma 9) it sends all bounded sets into compact ones, by statement 1.5.23 of [2] it is closed. Then by theorem 1.2.15 of [2] this mapping is upper semicontinuous. ■

By the construction the following set-valued operator

$$Bu = \int P\Gamma F \left(t, S(u(t) + C_u), \frac{d}{dt}S(u(t) + C_u) \right) dt,$$

where C_u is the vector from Lemma 3, is well-posed on the ball $U_{C_{t_0}} \subset C^0(I, T_{m_0}M)$. By Lemma 3 the vector C_u is continuous in $u(\cdot)$ and bounded. Hence, setting $v(\cdot) = u(\cdot) + C_u$, we obtain from Lemma 10 and Lemma 11 that B is upper semicontinuous and compact. Since the parallel translation preserves the norm of a vector, one can easily see that B maps $U_{C_{t_0}}$ into itself and therefore it has a fixed point $u_0 \in Bu_0$ in $U_{C_{t_0}}$:

$$(5) \quad u_0(\cdot) \in \int P\Gamma F \left(t, S(u_0(t) + C_{u_0}), \frac{d}{dt}S(u_0(t) + C_{u_0}) \right) dt.$$

Now we are in the position to demonstrate that $m(t) = S(u_0(t) + C_{u_0})$ is a solution in question of (3), i.e., that (5) plays the role of velocity hodograph for (3). By the construction $m(0) = m_0$, $m(t_0) = m_1$, $m(t)$ is a C^1 -curve, $\dot{m}(t)$ is absolutely continuous. Since u_0 is a fixed point of B , \dot{u}_0 is a selection of $\Gamma F(t, S(u_0 + C_{u_0}), \frac{d}{dt}S(u_0 + C_{u_0}))$, i.e., at points t , where \dot{u}_0 exists, we have the inclusion $\dot{u}_0 \in \Gamma F(t, S(u_0 + C_{u_0}), \frac{d}{dt}S(u_0 + C_{u_0}))$. Using the properties of covariant derivatives one can easily derive from the construction that after parallel translation of $\dot{u}_0(t)$ and $\Gamma F(t, S(u_0 + C_{u_0}), \frac{d}{dt}S(u_0 + C_{u_0}))dt$ along $m(\cdot)$ to the point $m(t)$ we obtain $\frac{D}{dt}\dot{m}$ and $F(t, m(t), \dot{m}(t))$, respectively. Thus we get

$$\frac{D}{dt}\dot{m}(t) \in F(t, m(t), \dot{m}(t)).$$

The theorem is proved. ■

Remark 12. Taking into account remarks 2 and 4, one can easily see that in the case of an Euclidian space inclusion (5) transforms into an inclusion of type (2) from [7].

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