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ON ROBUSTNESS OF SET-VALUED MAPS AND MARGINAL VALUE FUNCTIONS

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Abstract

The ideas of robust sets, robust functions and robustness of general set-valued maps were introduced by Chew and Zheng [7, 26], and further developed by Shi, Zheng, Zhuang [18, 19, 20], Phú, Hoffmann and Hichert [8, 9, 10, 17] to weaken up the semi-continuity requirements of certain global optimization algorithms. The robust analysis, along with the measure theory, has well served as the basis for the integral global optimization method (IGOM) (Chew and Zheng [7]). Hence, we have attempted to extend the robust analysis of Zheng *et al.* to that of robustness of set-valued maps with given structures and marginal value functions. We are also strongly convinced that the results of our investigation could open a way to apply the IGOM for the numerical treatment of some class of parametric optimization problems, when global optima are required.

Keywords: robust set; robust function; robust set-valued map; marginal value function; piecewise lower (upper) semi-continuous; approximatable function; approximatable set-valued map; regularity condition; extended Mangasarian-Fromovitz constraint qualification.

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1. Introduction

The concept of robust sets and functions was first initiated by Chew and Zheng [7, 26] as a weakening of the semi-continuity requirements of certain global optimization algorithms. Later on this theory was elaborated and extended by Shi, Zheng and Zhuang [18, 19, 20], Hoffmann, Phú, and Hichert [8, 9, 10, 17]. In fact, Chew and Zehng [7] proposed and developed an integral global optimization method (IGOM) for the computation of the global optima of discontinuous functions based on robustness properties. Depending on these ideas Hichert [8] designed a more general version of the IGOM into a software routine called BARLO, which is found to be computationally efficient for the global optimization problem with *robust* data.

In their paper, Shi, Zheng and Zhuang [20] also introduced robustness of general set-valued maps with the same purpose of weakening set-valued continuity – a concept which is tantamount to an *almost* (*semi-*)*continuity* property. Hence, the major aims of this paper are:

- to give the robust version of some well-known and standard results of set-valued maps; thereby pointing out connections, differences and similarities between robustness and continuity of such maps;
- to provide conditions for robustness of set-valued maps which are defined through parametric systems of functional inequalities; and
- to verify certain robustness properties of marginal value functions.

Such an undertaking is believed to serve a dual purpose, both as an extension of the theory of robust analysis and to throw some light on the possibility of using the IGOM to some class of optimization problems, where global optimality is ardently needed. The theory of robust analysis coupled with the IGOM has been used to solve: integer and mixed integer optimization through robustification (Zheng and Zhaung [28]); non-linear complementarity problems (Kostreva and Zheng [16]), constrained optimization problems with discontinuous penalty functions (Zheng and Zhang [27], Zheng [26]), the determination of the essential infimum and supremum of summable functions (Phu and Hoffmann [17]); layout optimization of analog circuits, minimization of total energy in atomic clusters, and modelling and design analysis of electrical networks (Hichert [8]); statistical computations (Zheng [26]); determination of economic equilibria and fixed points of discontinuous operators (Zheng [26], Zheng and Zhuang [29]), and so on. Furthermore, in the paper (Geletu and Hoffmann [2]) we have indicated how to use robust analysis and the IGOM for the numerical treatment of a certain class of generalized semi-infinite optimization problems (cf. also Geletu [1]). In addition, we are strongly convinced that the theory of robust analysis plays a significant role in fields like parametric optimization, optimal control, computational partial differential equation, stochastic optimization, etc.; where researchers frequently need to deal with discontinuous functions.

In the following, the statements with no citations are from us. Furthermore, to simplify the reading of the article we have compiled the major definitions and results into tables (see Section 8).

2. Preliminaries

We begin with basic definitions and results from robust analysis. The results mentioned here are mainly taken from [7, 19, 26]. At the same time, we also attempt to provide some minor complementary results.

We use the notations: $\mathbf{B}_\varepsilon(x^0)$ to represent an open ball of radius $\varepsilon > 0$ around the point $x^0 \in X$, when X is a metric space. Furthermore, we use $x^0 + \varepsilon\mathbf{B}$ instead of $\mathbf{B}_\varepsilon(x^0)$, when X is a normed linear space; where $\mathbf{B}_\varepsilon$ and $\mathbf{B}$ denote the open balls of radius ε and unit radius around the zero element of X, respectively.

Definition 2.1 (robust set, Zheng [26])**.** Let X be a topological space and let $D \subset X$. Then D is called a *robust set* iff $\mathrm{cl}\, D = \mathrm{cl}\ \mathrm{int} D$, where $\mathrm{cl} D$ and $\mathrm{int} D$ denote the topological *closure* and the *interior* of D, respectively, in the topology of X.

Remark 2.2. In [26] we find that $\emptyset$, X and open sets are robust, the union of an arbitrary collection of robust sets is again robust; but the intersection of two robust sets may not be robust. However, the intersection of an open and a robust set is again robust.

Corollary 2.3 (see also [26])**.** *Let $D \subset X$. If D is convex (or star-shaped) and* $\mathrm{int}\, D \neq \emptyset$*, then D is a robust set.*

Definition 2.4 (robust point, [26])**.** Let $D \subset X$. A point $x \in \mathrm{cl} D$ is said to be a *robust point to D* if $N(x) \cap \mathrm{int} D \neq \emptyset$ for each neighborhood $N(x)$ of x. If, further, $x \in D$, then x is said to be a *robust point of D*.

Proposition 2.5 (Zheng [26]).

1. *D is a robust subset of X if and only if each point $x \in D$ is a robust point of D.*
2. *Any accumulation point of a set of robust points to D is also a robust point to D.*

Moreover, an open set is a neighborhood of each of its points. Hence, robustness of a set is connected with a weaker notion of a neighborhood.

Definition 2.6 (semi-neighborhood (SNH), [26]). A set D is called a *semi-neighborhood* of a point x iff x is a robust point of the set D.

Corollary 2.7 (Zheng [26]). *A robust set D is a semi-neighborhood of each of its points.*

We also have the following properties, which we may frequently be used:

Proposition 2.8 (Zheng [26]).

1. *If D is a semi-neighborhood of x and $\mathrm{int}D \subset A$, then A is also a semi-neighborhood of x.*
2. *If D is a semi-neighborhood of x and $x \in O$, where O is an open set, then $D \cap O$ is also a semi-neighborhood of x.*

Remark 2.9. The union of a family of semi-neighborhoods of x is again a semi-neighborhood of x, whereas the intersection of two semi-neighborhoods of x may not be again a semi-neighborhood of x. Consequently, the collection of all semi-neighborhoods of a point x (or of robust sets) cannot define a topology.

Definition 2.10 (upper robust (u.r.) function, [26]). A function $f : X \to \mathbb{R}$ is called *upper robust (u.r.)* [*upper semi-continuous*] on X iff for all $c \in \mathbb{R}$ the set

$$F_c := \{x \in X \mid f(x) < c\} =: [f < c]$$

is a robust [*open*] set.

The upper robustness of a function can also be defined pointwise in the traditional way.

Definition 2.11 (upper robustness at a point)**.** Let X be a topological space, $f : X \to \mathbb{R}$ and $x^0 \in X$. If for each given $\varepsilon > 0$ there is a semi-neighborhood $SNH_\varepsilon(x^0)$ of x^0 such that

$$f(x) \leq f(x^0) + \varepsilon, \forall x \in SNH_\varepsilon(x^0),$$

then f is said to be upper robust at x^0.

Proposition 2.12. *Let X be a topological space and $f : X \to \mathbb{R}$. The function f is upper robust at each $x \in X$ iff it is an upper robust function.*

Proof. (a) Suppose f is upper robust at each $x \in X$. Let $c \in \mathbb{R}$ be arbitrary. Then we show that $F_c = \{x \in X \mid f(x) < c\}$ is a robust set. If $F_c = \emptyset$, then we are done. Thus, let $F_c \neq \emptyset$ and $x^0 \in F_c$ be any. Then $f(x^0) < c$. Choose ε such that $0 < \varepsilon < c - f(x^0)$. Then, by assumption, there is a semi-neighborhood $SNH_\varepsilon(x^0)$ such that

$$\forall x \in SNH_\varepsilon(x^0) : f(x) < f(x^0) + \varepsilon.$$

This establishes that

$$x^0 \in SNH_\varepsilon(x^0) \subset F_c.$$

Since x^0 is a robust point of $SNH_\varepsilon(x^0)$, then x^0 is a robust point of F_c (cf. Proposition 2.8(1)). Since $x^0 \in F_c$ is arbitrary, we conclude that F_c is a robust set. Hence, f is an upper robust function.

(b) Suppose that f is an upper robust function. Let $x^0 \in X$ and $\varepsilon > 0$ be given. Then the set

$$SNH_\varepsilon(x^0) := \{x \in X \mid f(x) < f(x^0) + \varepsilon\}$$

contains x^0 and, by assumption, $SNH_\varepsilon(x^0)$ is a robust set. Consequently, $SNH_\varepsilon(x^0)$ is a semi-neighborhood of x^0 and

$$\forall x \in SNH_\varepsilon(x^0) : f(x) < f(x^0) + \varepsilon.$$

This entails that

$$\forall x \in SNH_\varepsilon(x^0) : f(x) \leq f(x^0) + \varepsilon.$$

Hence, f is upper robust at x^0. Since $x^0 \in X$ is arbitrary, we conclude that f is upper robust at each $x \in X$. ∎

Among lots of properties of upper robust functions we find the following statements.

Corollary 2.13 (Zheng [26]). *Let $D \subset X$ and $f : D \to \mathbb{R}$. If D is a robust set and f is u.s.c., then f is u.r. on D.*

Definition 2.14 (lower robust (l.r.) function, [26]). A function $f : X \to \mathbb{R}$ is called *lower robust* (*l.r.*) on X iff $-f$ is upper robust on X.

Proposition 2.15 (upper robustness of composition). *Let X be a topological space. If $f : X \to \mathbb{R}$ is u.r. and $r : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, then the composite function $r \circ f$ (i.e., $(r \circ f)(x) = r(f(x))$) is upper robust.*

Proof. From $[f < r^{-1}(c)] = [r \circ f < c]$ follows the statement. ∎

Remark 2.16. It is to be noted that the concept of robustness (like openness) of sets is dependent on the topology of the underlying space. Likewise, robustness of a function also depends on the topologies of both its domain and image spaces. Hence, the definitions of robustness here are with respect to the relative topology on the set X, when X is assumed to be a subset of some topological space. Such issues of *relative robustness* are discussed in [26, *Section* 2.2.4]. We underline that the relative robustness in [26] and in our article are based on robust subsets X of an underlying topological space, say X_0. Hence, the relative robustness on X is the robustness on the topological subspace X with respect to the relative topology of X induced by X_0.

It seems possible to extend the notion of robustness to a different kind of **relative robustness** of a set B closely connected with the well-known relative interior ($\operatorname{rel\,int} B$) or relative closure ($\operatorname{rel\,cl} B$), which is topologically related to the affine hull of the set B under consideration (or more generally to some B contained in a nonlinear manifold). This approach is frequently and successfully used in the finite dimensional convex analysis and optimization. However, one has to be careful in extending results of the relative interior or closure to a similar kind of relative robustness since openness is strongly connected with the topology and neighborhoods, but robustness bases on semi-neighborhoods which fail to satisfy the intersection property. So far, we have not thought about such a generalization and have not found any studies on it.

3. Approximatable functions

Approximatability of functions and the SVM's have been discussed in relation with robustness by Shi, Zheng and Zhuang in [19, 20]. This concept will be seen to offer the possibility of numerical approximation of the values of a robust function (3.1) or a set-valued map (SVM) at a given point. Roughly speaking, when a function is approximatable at a point x^0, then $f(x^0)$ could be approximated by those values of f at which it is continuous. The same holds true of the SVM's (cf. Section 4.4). In fact, the idea of approximatability reveals the practical usability of robustness for computational purposes; especially, when robustness is guaranteed to be equivalent to approximatability. Hence, in this section, the definition of approximatable functions of [19] will be extended to that of *upper approximatable functions.* Furthermore, a statement of equivalence between upper approximatability and upper robustness is also stated and proved.

We proceed by citing relevant definitions and results.

Definition 3.1 (robust function, [19, 26])**.** Let $f : X \to Y$ and $x \in X$. Then f is called robust at x iff for any neighborhood $U \subset Y$ of $y = f(x)$, x is a robust point of $f^{-1}(U)$.

Clearly,

Corollary 3.2. *If $f : X \to Y$ is continuous, then f is robust.*

Remark 3.3. Upper semi-continuity and lower semi-continuity imply continuity; but, upper and lower robustness do not imply robustness (see Remark 2.9 in [26]).

Definition 3.4 (approximatable functions, [19])**.** Let X and Y be topological spaces and $f : X \to Y$ be a function. Suppose that $S \subset X$ is a set of points of continuity of f. Then f is said to be *approximatable* iff

1. S is dense in X,
2. for any $x^0 \in X$, there exists a net (a Moor-Smith sequence†) $\{x_\alpha\}_{\alpha\in\Lambda} \subset S$

†The notion of a sequence $\{x_n\}_{n\in\mathbb{N}}$ of a countable number of elements x_n is generalized to the notion of a function $x : \Lambda \to X$, where Λ is a directed uncountable set of elements α. The family $\{x_\alpha\}_{\alpha\in\Lambda}$ is called a generalized sequence or Moore-Smith sequence. The notion $\lim_{n\to\infty} x_n$ of the limit of a sequence may be extended to the notion $\lim_{\alpha\in\Lambda} x_\alpha$ or, more shortly, $\lim_\alpha x_\alpha$ of the generalized limit of the generalized sequence $\{x_\alpha\}_{\alpha\in\Lambda}$. For more information see e.g. [14, Chapter I. §5.7] or [25, Chapter IV. 2].

such that
$$\lim_{\alpha} x_\alpha = x^0 \quad \textit{and} \quad \lim_{\alpha} f(x_\alpha) = f(x^0).$$

In contrast to its continuity counterpart, Definition 3.4 requires only the existence of a net to guarantee the approximatability. Actually, with respect to continuity, property 2 of Definition 3.4 is expected to be valid for every net $\{x_\alpha\}_{\alpha\in\Lambda}$ converging to x^0.

Theorem 3.5 (Theorem 2.1, Zheng *et al.* [19]). *Any approximatable function is robust.*

In general, the converse of Theorem 3.5 may not be true. However, if X is a Baire space and Y is second countable, then approximatability will be equivalent to robustness.

Theorem 3.6 (Theorem 3.1, [19]). *Let X be a Baire and Y a second countable topological spaces and $f : X \to Y$. Then f is robust iff f is approximatable.*

Remark 3.7. If X is a complete metric space and $Y = \mathbb{R}$, then the assumptions of Theorem 3.6 will be easily satisfied. However, it should be stressed that the result in Theorem 3.6 is based on general topological spaces. That is what has to be used next.

Subsequently, we give a generalization of Definition 3.4 for the case when X is a metric space and $Y = \mathbb{R}$.

Definition 3.8 (upper approximatable functions). Let X be a metric space and $f : X \to \mathbb{R}$. Suppose that $S \subset X$ be the set of points, where f is u.s.c. Then f is upper approximatable (u.a.) iff

1. S is dense in X;
2. for any $x^0 \in X$, there is a sequence $\{x^k\} \subset S$ such that
$$\lim_{k} x^k = x^0 \quad \text{and} \quad \limsup_{k} \; f(x^k) \le f(x^0).$$

We show next that upper approximatability implies upper robustness.

Proposition 3.9. *Let X be a metric space and $f : X \to \mathbb{R}$. If f is upper approximatable, then f is upper robust.*

Proof. Let $x \in \mathbb{R}$ be any and take an arbitrary $x^0 \in [f < c]$; i.e., $f(x^0) < c$. We show that x^0 is a robust point of $[f < c]$. There are two cases to consider:

Case a. If f is u.s.c. at x^0, then, for every $\varepsilon > 0$, there is a neighborhood $N(x^0)$ in X such that

$$f(x) \leq f(x^0) + \varepsilon, \forall x \in N(x^0).$$

In particular, taking ε with $0 < \varepsilon < c - f(x^0)$ we have

$$\forall x \in N(x^0) : f(x) < c.$$

This yields that $x^0 \in \operatorname{int}[f < c]$. Hence, x^0 is a robust point of $[f < c]$.

Case b. If f is not u.s.c. at x^0, then there is a sequence $\{x^k\} \subset S$ such that

$$\lim_k x^k = x^0 \quad \text{and} \quad \limsup_k f(x^k) \leq f(x^0)$$

and $\limsup_k f(x^k)$ implies that there exists $k_0(\varepsilon)$ such that

$$f(x^k) \leq f(x^0) + \varepsilon, \forall k \geq k_0(\varepsilon).$$

Choosing $0 < \varepsilon < c - f(x^0)$ it then follows, by the upper semi-continuity of f at x^k, that there exists $r_k(\varepsilon) > 0$ such that for all $r \in (0, r_k(\varepsilon))$

$$\mathbf{B}_r(x^k) \subset [f \leq f(x^k) + \varepsilon] \subset [f < c], \forall k \geq k_0(\varepsilon).$$

Hence, for all $k \geq k_0(\varepsilon)$, x^k is a robust point of $[f < c]$. Moreover, $\lim_k x^k = x^0$. Consequently, by Proposition 2.5(2), x^0 is a robust point of $[f < c]$. This completes the proof. ■

A statement of equivalence between upper robustness and upper approximatability could be given if X is assumed to be a complete metric space.

Proposition 3.10. *Let X be a complete metric space with topology τ and $f : X \to \mathbb{R}$. Then f is upper robust iff f is upper approximatable.*

Proof. It only remains to show the forward implication (the reverse implication is already contained in Proposition 3.9). Take the following family of subsets of $\mathbb{R}$

$$\sqsupset := \{(-\infty, q)\}_{q\in\mathbb{Q}} \cup \{\mathbb{R}, \emptyset\},$$

where $\mathbb{Q}$ denotes the set of rational numbers. Obviously, $< \mathbb{R}, \sqsupset >$ is a topological space with countable basis of open sets (but it is not Hausdorff). Hence, $< \mathbb{R}, \sqsupset >$ is second countable. We now consider

$$f :< X, \tau > \to < \mathbb{R}, \sqsupset > .$$

Then the upper robustness of f in the usual topology of $\mathbb{R}$ is now the robustness of f with respect to the topology $\sqsupset$ on $\mathbb{R}$ (cf. Definition 3.1). Hence, Theorem 3.6 (Theorem 3.1 in [19]) yields that f is approximatable with respect to $\sqsupset$ in $\mathbb{R}$. That is, there exists a set $S \subset X$ such that

(i) f is continuous at each $x \in S$ with respect to $\sqsupset$ on $\mathbb{R}$,

(ii) S is dense in X, and

(iii) for each $x^0 \in X$, there is a sequence $\{x^k\} \subset S$ such that

$$\lim_k x^k = x^0 \quad \text{and} \quad (\sqsupset) \lim_k f(x^k) = f(x^0)$$

(observe that the limit with respect to $\sqsupset$ is not unique). We next formulate $(\sqsupset) \lim_k f(x^k) = f(x^0)$ in the traditional notation. Thus, for any $\varepsilon > 0$, there exists $(-\infty, q(\varepsilon)) \in \sqsupset$ such that $f(x^0) \in (-\infty, q(\varepsilon)) \subset (-\infty, f(x^0)+\varepsilon)$. Hence, $(\sqsupset) \lim_k f(x^k) = f(x^0)$ implies that there is $k_0(\varepsilon)$ such that

$$f(x^0) < q(\varepsilon) < f(x^0) + \varepsilon, \forall k \geq k_0(\varepsilon).$$

Subsequently, it follows that

$$\limsup_k f(x^k) \leq f(x^0) + \varepsilon.$$

It remains now to show that S contains the set of points of X, where f is upper semi-continuous with respect to *the usual topology* on $\mathbb{R}$. Thus, let $x^0 \in S$, then the continuity of $f :< X, \tau > \to < \mathbb{R}, \sqsupset >$ at x^0 implies that for any $\varepsilon > 0, \exists q(\varepsilon) \in \mathbb{Q}, U(x^0) \subset X$ such that

$$f(x) \in (-\infty, q(\varepsilon)) \subset (-\infty, f(x^0) + \varepsilon), \forall x \in U(x^0).$$

This concludes that

$$f(x) \le f(x^0) + \varepsilon, \forall x \in U(x^0),$$

which is the usual upper semi-continuity of f at x^0. Hence, the claim is justified. ■

4. Robustness of set-valued maps

Beginning with the basic definition of set-valued maps, it has been indicated by Zheng *et al.* [20] that a set-valued map with a dense set of upper [lower] semi-continuity is upper [lower] robust. Conversely, on a complete metric space, an upper [lower] robust map has a dense set of upper [lower] semi-continuity. This, in fact, contains implicitly the idea of approximatability of set-valued maps. Furthermore, it also cautiously implies that an upper [lower] robust set-valued map is *almost* upper [lower] semi-continuous.

Thus, one may like to find out: *how much of continuity could be re-writable in terms of robustness.* However, here, we give only some representative results to this very broad question. Furthermore, in Section 6 we refine robustness to set-valued maps with given structures.

4.1. Definitions and results

For a set-valued map (SVM) $M : X \rightrightarrows Y$ and $U \subset Y$, we use the notations

$$\begin{aligned} M^{-1}(U) &:= \{x \in X \mid M(x) \cap U \neq \emptyset\}; \\ M^{+}(U) &:= \{x \in X \mid M(x) \subset U\}. \end{aligned}$$

$M^+(U)$ and $M^{-1}(U)$ are known as the *core* and the *inverse-image*; respectively, of the set U with respect to $M(\cdot)$. See Aubin and Frankowska [4] for the concept of semi-continuity of set-valued maps.

Definition 4.1 (lower robust SVM, [20, 26])**.** Let X and Y be topological spaces and $M : X \rightrightarrows Y$ be a set-valued map. Then $M(\cdot)$ is *lower robust* [*l.s.c.*] at $x \in X$ iff for each $y \in M(x)$ and each neighborhood $U(y) \subset Y$ of y, $M^{-1}(U(y))$ is a semi-neighborhood [*neighborhood*] of x in X. Furthermore, $M(\cdot)$ is *lower robust* (*l.r.*) [*l.s.c.*] iff $M(\cdot)$ is lower robust [*l.s.c.*] at x, for all $x \in X$.

The following two statements are trivially implied by Definition 4.1.

Corollary 4.2 ([20, 26]). *$M(\cdot)$ is l.r. iff $M^{-1}(U)$ is a robust set in X for every open set $U \subset Y$.*

Corollary 4.3 ([20, 26]). *If $M : X \rightrightarrows Y$ is l.s.c., then $M(\cdot)$ is l.r.*

However, the converse of Corollary 4.2 is not always true.

Example 4.4. The set-valued map

$$M(x) := \begin{cases} [1,4] & \text{if } x > 0, \\ \{4\} & \text{if } x = 0, \\ [2,3] & \text{if } x < 0. \end{cases}$$

is a simple example of a map which is l.r., but not l.s.c. at $x = 0$.

Definition 4.5 (upper robust SVM, [20, 26]). Let X and Y be topological spaces and $M : X \rightrightarrows Y$ be a set-valued map.

1. $M(\cdot)$ is said to be upper robust (*u.r.*)[*u.s.c.*] at $x \in X$ iff, for any neighborhood U of $M(x)$, $M^+(U)$ is a semi-neighborhood [*neighborhood*] of x. (i.e., x is a robust point of $M^+(U)$).
2. $M(\cdot)$ is said to be upper robust [*u.s.c.*] iff $M(\cdot)$ is upper robust [*u.s.c.*] on X at every $x \in X$.

Correspondingly, we have the statements

Corollary 4.6.

1. *$M(\cdot)$ is u.r. iff for any open set $U \subset Y$, $M^+(U)$ is a robust set in X (cf. [20] and [26]).*
2. *If $M(\cdot)$ is an u.r. SVM, then the set*

$$E := \{x \in X \mid M(x) = \emptyset\}$$

is robust in X.

Corollary 4.7 ([20, 26]). *If $M(\cdot)$ is u.s.c., then $M(\cdot)$ is u.r.*

Example 4.4 demonstrates that there is a l.r. set-valued map which is not l.s.c. A similar example could be set up for upper robustness. Furthermore, the SVM in Example 4.4 is lower robust, but not upper robust. To see this, for $\varepsilon > 0$, we find that

$$M^+((-\varepsilon, \varepsilon) + 4) = \{0\}.$$

This shows that $M(\cdot)$ is not upper robust.

4.2. ε-robustness of set-valued maps

In the following, we would like to see how far the notions of Hausdorff or ε-semi-continuity could be carried over to that of robustness.

Hence, let X and Y be normed linear spaces, let $M : X \rightrightarrows Y$ be a set-valued map, and denote by $\mathbf{B}_\varepsilon$ the open ball of radius ε at the zero element of Y, with $\varepsilon > 0$.

Definition 4.8 (ε-upper robust SVM)**.** We say that $M(\cdot)$ is ε-upper robust [ε-upper semi-continuous] at x^0 if given $\varepsilon > 0$, there exists a semi-neighborhood [neighborhood] $SNH_\varepsilon(x^0)$ such that

$$\forall x \in SNH_\varepsilon(x^0) : M(x) \subset M(x^0) + \varepsilon\mathbf{B}.$$

$M(\cdot)$ is called an ε-upper robust map, if it is ε-upper robust at every $x^0 \in X$.

Proposition 4.9. *If $M(\cdot)$ is u.r., then $M(\cdot)$ is ε-upper robust.*

Proof. Given $\varepsilon > 0$ and $x^0 \in X$, let $U := M(x^0) + \varepsilon\mathbf{B}$ (which is an open set). Hence, $M(x^0) \subset U$. By assumption $M^+(U)$ is a semi-neighborhood of x^0. Set $SNH_\varepsilon(x^0) := M^+(U)$. Thus,

$$\forall x \in SNH_\varepsilon(x^0) : M(x) \subset U = M(x^0) + \varepsilon\mathbf{B}.$$

This concludes the proof. ∎

Proposition 4.10. *If $M(\cdot)$ is compact-valued and ε-upper robust, then $M(\cdot)$ is u.r.*

Proof. Let $U \subset Y$ be an open set. We need to show that $M^+(U)$ is a robust set in X. Let $x^0 \in M^+(U)$. Hence, $M(x^0) \subset U$. This yields that, for each $y \in M(x^0)$, there exists $\varepsilon(y) > 0$ such that $y + \varepsilon\mathbf{B} \subset U$. But, since

$M(x^0)$ is compact, there are $y_1, \ldots, y_m \in M(x^0)$ and

$$M(x^0) \subset \bigcup_{i=1}^{m} (y_i + \varepsilon(y_i)\mathbf{B}) \subset U.$$

Let $\varepsilon_0 := \min_{1 \le i \le m} \varepsilon(y_i)$. From this it follows that

$$M(x^0) + \varepsilon_0 \mathbf{B} \subset U.$$

By ε-upper robustness, there is a semi-neighborhood $SNH(x^0)$ such that

$$\forall x \in SNH(x^0) : M(x) \subset M(x^0) + \varepsilon_0 \mathbf{B}.$$

Consequently,

$$SNH(x^0) \subset M^+(M(x^0) + \varepsilon_0 \mathbf{B}) \subset M^+(U).$$

Therefore, x^0 is a robust point of $M^+(U)$. Since, $x^0 \in M^+(U)$ is arbitrary, we have that $M(\cdot)$ is upper robust. ■

Similarly, we define

Definition 4.11 (ε-lower robust SVM)**.** We say that $M(\cdot)$ is *ε-lower robust* [*ε-lower semi-continuous*] at x^0 iff for any $\varepsilon > 0$ there exists a semi-neighborhood [neighborhood] $SNH_\varepsilon(x^0)$ such that

$$\forall x \in SNH_\varepsilon(x^0) : M(x^0) \subset M(x) + \varepsilon \mathbf{B}.$$

We say that $M(\cdot)$ is ε-lower robust [ε-lower semi-continuous] if it is ε-lower robust [ε-lower semi-continuous] at every $x^0 \in X$.

Proposition 4.12. *If $M(\cdot)$ is ε-lower robust, then $M(\cdot)$ is l.r.*

Proof. Let $U \subset Y$ be an open set. We show that $M^{-1}(U)$ is a robust set in X; i.e., we show for arbitrary $x^0 \in M^{-1}(U)$, x^0 is a robust point of $M^{-1}(U)$. But, then $M(x^0) \cap U \neq \emptyset$. This means that there is a point $y^0 \in M(x^0) \cap U$. Hence, for some $\varepsilon > 0$, we have

$$(y^0 + \varepsilon \mathbf{B}) \subset U \text{ and } M(x^0) \cap (y^0 + \varepsilon \mathbf{B}) \neq \emptyset.$$

By ε-lower robustness, there is a semi-neighborhood $SNH_\varepsilon(x^0)$ of x^0 such that
$$\forall x \in SNH_\varepsilon(x^0) : M(x^0) \subset M(x) + \varepsilon\mathbf{B}.$$
Hence, it follows that
$$\forall x \in SNH_\varepsilon(x^0) : y^0 \in M(x) + \varepsilon\mathbf{B}.$$
Consequently,
$$\forall x \in SNH_\varepsilon(x^0) : M(x) \cap \big(y^0 + \varepsilon\mathbf{B}\big) \neq \emptyset.$$
Since $y^0 + \varepsilon\mathbf{B} \subset U$, we have that
$$SNH_\varepsilon(x^0) \subset M^{-1}(U).$$
Hence, $M^{-1}(U)$ is also a semi-neighborhood of x^0. Since x^0 is arbitrary, it follows that $M^{-1}(U)$ is a robust set. Therefore, $M(\cdot)$ is a lower robust map. ∎

However, the converse of Proposition 4.12 may not hold true even if $M(\cdot)$ is compact-valued. Hence, a similar statement of equivalence, as in the case of l.s.c. set valued maps with compact values (see p. 45, paragraph 3 of Aubin and Cellina [3]), fails to exist between lower robust and ε-lower robust set-valued maps. This is one evidence that robustness of a set-valued map is weaker than continuity.

Example 4.13. Consider the set-valued map $M : \mathbb{R} \rightrightarrows \mathbb{R}$ given by
$$M(x) := \begin{cases} [2,5], & \text{if } x < 0 \\ [1,2] \cup [3,5], & \text{if } x = 0 \\ [1,3], & \text{if } x > 0. \end{cases}$$

Let $\varepsilon = \frac{1}{2}$. For any semi-neighborhood $SNH(0)$ and neighborhood $N(0)$ of 0, there is $x \in SNH(0) \cap N(0)$. Hence, if $x > 0$, we have $M(x) = [1,3]$, but then $M(0) = [1,2] \cup [3,5] \not\subset [1,3] + (-\frac{1}{2}, \frac{1}{2})$. Similarly, if $x < 0$, we have $M(x) = [2,5]$, so that $M(0) = [1,2] \cup [3,5] \not\subset [2,5] + (-\frac{1}{2}, \frac{1}{2})$. Consequently, $M(\cdot)$ is both not l.s.c. and not ε-lower robust at $x = 0$.

Obviously, $M(\cdot)$ is lower robust (also l.s.c.) at x, for either $x < 0$ or $x > 0$. And, if $y^0 \in M(0)$, then either $y^0 \in [1,2]$ or $y^0 \in [3,5]$. Hence, for any open ball $\mathbf{B}_\varepsilon(y^0)$, we have

- if $y^0 \in [1,2]$, then $M^{-1}(\mathbf{B}_\varepsilon(y^0)) = [0,\infty)$; or
- if $y^0 \in [3,5]$, then $M^{-1}(\mathbf{B}_\varepsilon(y^0)) = (-\infty,0]$.

In both cases, $M^{-1}(\mathbf{B}_\varepsilon(y^0))$ is a semi-neighborhood of $x = 0$. Consequently, $M(\cdot)$ is a lower robust SVM with compact values.

4.3. Piecewise semi-continuous set-valued maps

Again, following [20] we define piecewise semi-continuity. Analogously, as a sort of generalization, we also consider *piecewise robustness* properties for set-valued maps (and of functions in Section 5.3). Here, we have the property that piecewise robustness implies robustness, which is not true of semi-continuity. Thus, some suitable decomposition of the domain space is possible under the weaker robustness assumptions.

Let X and Y be two topological spaces. We say that $X_1, X_2, \dots, X_r$ is a *partition of* X iff the sets X_i are pairwise disjoint and X is the union of all X_i. The *partition is called robust* iff each X_i is robust with respect to X.

$M : X \rightrightarrows Y$ is said to be *piecewise l.s.c.* (*l.r.*)[*u.s.c.*]{*u.r.*} iff there exists a robust partition $X_1, X_2, \dots, X_r$ of X such that for all $i \in \{1,\dots,r\}$ the restriction of $M(\cdot)$ to X_i is *l.s.c.*(*l.r.*) [*u.s.c.*] {*u.r.*} with respect to the relative topology of X_i induced by the topological space X.

The proofs of the following two theorems are not available in their original source [20]. Hence, they are supplied here for the sake of completeness.

Theorem 4.14 (Zheng *et al.* [20])**.** *If $M(\cdot)$ is piecewise l.s.c., then $M(\cdot)$ is l.r.*

Proof. Let $U \subset Y$ be any open set. We want to show that $M^{-1}(U)$ is a robust set in X. Since, for each $i = 1,\dots,r$, $M|_{X_i} : X_i \rightrightarrows Y$ is l.s.c. with respect to X_i, we have that $M^{-1}(U) \cap X_i$ is an open set in X_i. Hence, there is an open set $V \subset X$ such that $X_i \cap V = M^{-1}(U) \cap X_i$. But, then $X_i \cap V$ is a robust set in X, by Remark 2.2. Hence, for each $i \in \{1,\dots,r\}$, $M^{-1}(U) \cap X_i$ is a robust set. Subsequently, it follows that

$$\bigcup_{i=1}^{r} M^{-1}(U) \cap X_i = M^{-1}(U)$$

is a robust set in X. ■

Similarly we have

Theorem 4.15 (Zheng *et al.* [20])**.** *If $M(\cdot)$ is piecewise u.s.c., then $M(\cdot)$ is u.r.*

Proof. Let $V \subset Y$ be any open set. We show that $M^+(V) := \{x \in X \mid M(x) \subset V\}$ is a robust set in X. Since $M : X_i \rightrightarrows Y$ is u.s.c. in the relative topology of X_i, we have, for each i,

$$(M|_{X_i})^+(V) = \{x \in X_i \mid M(x) \subset V\}$$

is an open set in X_i. Then the rest of the proof is as in Theorem 4.14. ∎

In the following, we represent the interior of a set B with respect to the relative topology induced in A by X, where $B \subset A$, by $\mathrm{int}_A B$.

Lemma 4.16. *Let X be a topological space and A be a non-empty robust subset of X. If $B \subset A$ is such that $\mathrm{int}_A B \neq \emptyset$, then $\mathrm{int}_X B \neq \emptyset$.*

Proof. Clearly, $\mathrm{int}_A B$ is an open set in A. Hence, there exists $O \subset X$ open in X such that $B \supset \mathrm{int}_A B = O \cap A$. Since A is robust in X and $O \cap A \neq \emptyset$ (while $\mathrm{int}_A B \neq \emptyset$ and $A \neq \emptyset$) we have that $O \cap \mathrm{int}_X A \neq \emptyset$. This yields $B \supset \mathrm{int}_A B = O \cap A \supset O \cap \mathrm{int}_X A \neq \emptyset$. From this we obtain that $\mathrm{int}_X B \supset O \cap \mathrm{int}_X A \neq \emptyset$, which completes the proof. ∎

If the set $A \subset X$ is not assumed to be robust, then the above implication fails to be true. We consider two examples.

Example 4.17. Let $X = \mathbb{R}$, $A = B = \mathbb{Q}$ where $\mathbb{Q}$ is the set of rational numbers. Observe that $\mathrm{int}_A B \neq \emptyset$. However, $\mathrm{int}_X B = \emptyset$ and $A = \mathbb{Q}$ is not robust in $X = \mathbb{R}$.

Example 4.18. Let the interval $[a, b] \subset \mathbb{R}^n$ be the convex hull of the points $a, b \in \mathbb{R}^n$ and define $A = [a, b]$. Suppose the set $B \subset A$ is robust in the relative topology of A with respect to $\mathbb{R}^n$. Again, this does not imply that B is a robust set of $\mathbb{R}^n$.

Analogously to open and closed sets in a topological space, we get the following statement for a robust set X_0.

Lemma 4.19. *Let X_0 be a robust subset of a topological space X and assume that $\hat{X} \subset X_0$. If $\hat{X}$ is robust in X_0 in the relative topology of X_0 with respect to X, then $\hat{X}$ is a robust set in X.*

Proof. Let $x \in \hat{X}$ and $N(x)$ be any open neighborhood of x with respect to X. Then $N(x) \cap X_0$ is the neighborhood of x in the relative topology of X_0. Since $\hat{X}$ is robust in the relative topology of X_0 we have

$$\mathrm{int}_{X_0}\left(N(x) \cap X_0 \cap \hat{X}\right) \neq \emptyset$$

(note that $X_0 \cap \hat{X} = \hat{X}$) and

$$\mathrm{int}_{X_0}\left(N(x) \cap X_0 \cap \hat{X}\right) \subset N(x) \cap X_0.$$

Since $N(x)$ is open in X, $N(x) \cap X_0$ is robust in X (cf. Remark 2.2). Hence, we get, by Lemma 4.16, that

$$\mathrm{int}(N(x) \cap X_0 \cap \hat{X}) \neq \emptyset$$

which implies that $N(x) \cap \mathrm{int}(\hat{X}) \neq \emptyset$. Since $N(x)$ is arbitrary, it follows that x is a robust point of $\hat{X}$. Therefore, using Proposition 2.5(1), $\hat{X}$ is a robust set in X. ∎

Theorem 4.20. *If $M(\cdot)$ is piecewise l.r.* [*u.r.*], *then $M(\cdot)$ is l.r.* [*u.r.*].

Proof. Taking $M(\cdot)$ piecewise-l.r., let $U \subset Y$ be any open set. We have to show that

$$M^{-1}(U) = \{x \in X \mid M(x) \cap U \neq \emptyset\}$$

is a robust set. Since, $M : X_i \rightrightarrows Y$ is l.r. in the relative topology of X_i, we have, for each i, that

$$(M|_{X_i})^{-1}(U) = \{x \in X_i \mid M(x) \cap U \neq \emptyset\}$$

is a robust subset of X_i in the relative topology of X_i with respect to X. Then, the rest of the proof follows by similar arguments as in Theorem 4.14 using Lemma 4.19. ∎

4.4. Approximatable set-valued maps

In the same vein as for functions (cf. Section 3) there are similar statements for the approximatability of SVM's.

Definition 4.21 (lower approximatable SVM, [20])**.** Let X and Y be topological spaces and $M : X \rightrightarrows Y$ be a SVM. Suppose that S is a set of points of lower semi-continuity of $M(\cdot)$. Then $M(\cdot)$ is called *lower approximatable* iff

1. S is dense in X;
2. for any $x^0 \in X$ and $y^0 \in M(x^0)$, there exist a net $\{x_\alpha\}_{\alpha\in\Lambda} \subset S$ and a net $\{y_\alpha\}_{\alpha\in\Lambda}$ with $y_\alpha \in M(x_\alpha)$ for every $\alpha \in \Lambda$ such that

$$\lim_\alpha x_\alpha = x^0 \quad \text{and} \quad \lim_\alpha y_\alpha = y^0.$$

Proposition 4.22 (Theorem 2.1, [20]). *Any lower approximatable set-valued map is lower robust.*

Definition 4.23 (upper approximatable SVM, [20]). Let X and Y be topological spaces and $M : X \rightrightarrows Y$ be a SVM. Suppose that S is a set of points of upper semi-continuity of $M(\cdot)$. Then $M(\cdot)$ is called *upper approximatable* iff

1. S is dense in X;
2. for any $x^0 \in X$ there is a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in S such that for each neighborhood U of x^0 there is some directed set $\alpha(U), \alpha(U) \subset \Lambda$, such that

$$\forall \alpha \in \alpha(U) : M(x_\alpha) \subset U.$$

Proposition 4.24 cf. [20]). *Any upper approximatable set-valued map is upper robust.*

Proof. The proof is a similar as for Proposition 4.22 (see the proof of Theorem 2.1 in [20]). ■

It has been indicated by Zheng *et al.* [19, 20] that robustness of SVM's is weaker than approximatability. However, if a lower or upper robust map has a dense set of upper or lower semi-continuity, then it will be approximatable (cf. Proposition 2.3 in [20]).

Remark 4.25. Furthermore, if X is a Baire and Y a second countable topological spaces, then both kinds of approximatability of $M(\cdot)$ will be equivalent to their corresponding robustness properties (cf. [20] for details).

5. Marginal value functions

5.1. Upper robustness of infimum

Let us next come to the investigation of the behavior of marginal functions with respect to robustness properties of its defining data. Hence, we first consider the marginal function φ defined by

$$\varphi(x) := \inf_{y \in M(x)} \psi(x,y). \tag{1}$$

Theorem 5.1 (upper robustness of infimum)**.** *Let $\psi : X \times Y \to \mathbb{R}$ be u.s.c. on $\{x^0\} \times M(x^0)$, where $M : X \rightrightarrows Y$ is a l.r. set-valued map, then φ is u.r. at x^0.*

Proof. Let $c \in \mathbb{R}$ and $x^0 \in \Phi_c := \{x \mid \varphi(x) < c\}$. We have to show that Φ_c is a semi-neighborhood of x^0. Let $\varepsilon > 0$ be arbitrary, then there exists $\overline{y}_\varepsilon \in M(x^0)$ such that $\psi(x^0, \overline{y}_\varepsilon) < \varphi(x^0) + \varepsilon$. Since ψ is u.s.c. on $\{x^0\} \times M(x^0)$, there exist open neighborhoods $N(x^0)$ of x^0 and $N(\overline{y}_\varepsilon)$ of $\overline{y}_\varepsilon$ such that

$$\forall x \in N(x^0), \forall y \in N(\overline{y}_\varepsilon) : \psi(x,y) \leq \psi(x^0, \overline{y}_\varepsilon) + \varepsilon.$$

Since $M(\cdot)$ is l.r., $\overline{y}_\varepsilon \in M(x^0)$ and $N(\overline{y}_\varepsilon)$ is a neighborhood of $\overline{y}_\varepsilon$, then $M^{-1}(N(\overline{y}_\varepsilon))$ is a semi-neighborhood of x^0. Hence, $Q := N(x^0) \cap M^{-1}(N(\overline{y}_\varepsilon))$, by Proposition 2.8(2), is a semi-neighborhood of x^0, too. Thus, we have for all $x \in Q$, $\widetilde{y} \in M(x) \cap N(\overline{y}_\varepsilon)$

$$\varphi(x) = \inf_{y \in M(x)} \psi(x,y) \leq \psi(x, \widetilde{y}) \leq \psi(x^0, \overline{y}_\varepsilon) + \varepsilon < \varphi(x^0) + 2\varepsilon.$$

Choosing $\varepsilon > 0$ such that $0 < 2\varepsilon < c - \varphi(x^0)$, we have $Q \subset \Phi_c$. Hence Φ_c is a semi-neighborhood of x^0. ■

Remark 5.2. In Theorem 5.1, the upper semi-continuity assumption on ψ cannot be replaced by upper robustness (see Remark 2.2). But, if ψ is upper robust, then $M(\cdot)$ has to be lower semi-continuous.

Corollary 5.3 (lower robustness of supremum)**.** *Let $\psi : X \times Y \to \mathbb{R}$ be l.s.c. on $\{x^0\} \times M(x^0)$, where $M : X \rightrightarrows Y$ is a l.r. set-valued map, then the marginal value function*

$$\phi(x) := \sup_{y \in M(x)} \psi(x,y)$$

is l.r. at x^0.

Proof. The claim follows trivially, if we write

$$-\phi(x) := \inf_{y\in M(x)} -\psi(x,y)$$

and observe that $-\phi$ is upper robust by Theorem 5.1. From, which follows that ϕ is lower robust. ■

Corollary 5.4. *Let (X,ρ) be a metric space and $M : X \rightrightarrows Y$ be a set-valued map. If $M(\cdot)$ is l.r., $r : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and strictly increasing on $\mathbb{R}_+$, then the function*

$$\varphi(x) = r\left(\operatorname{dist}(x, M(x))\right) := \inf_{\xi\in M(x)} r\left(\rho(x,\xi)\right)$$

is u.r.

Proof. The functions ρ and r are continuous and $\inf_{\xi\in M(x)} r(\rho(x,\xi)) = r(\inf_{\xi\in M(x)} \rho(x,\xi))$. Then, using Theorem 5.1 and Proposition 2.15 the claim follows. ■

This corollary guarantees that, when r is as above, $\psi : X \times X \to \mathbb{R}_+$, $\psi(x,\xi) := r(\|x-\xi\|)$ and the map $M(\cdot)$ is l.r., then the marginal function $\varphi(x) = \operatorname{dist}(x, M(x))$ is u.r..

Hu and Papageorgiou [12] stated and proved the following

Proposition 5.5 (Proposition 2.26, p. 45, in [12])**.** *Let X be a Hausdorff topological space, Y be a metric space, $M : X \rightrightarrows Y$ and $M(x) \neq \emptyset, \forall x \in X$. Then $M(\cdot)$ is l.s.c. if and only if for every fixed $\xi \in Y$, the function $\varphi_\xi : X \to \mathbb{R}$, $\varphi_\xi(x) := \operatorname{dist}(\xi, M(x))$, is u.s.c.,*

where

$$\varphi_\xi(x) = \inf_{y\in M(x)} \rho(\xi, y).$$

A similar statement of equivalence will be

Proposition 5.6. *Let X be a Hausdorff topological space and Y be a metric space, with a metric ρ on Y, $M : X \rightrightarrows Y$ and $M(x) \neq \emptyset, \forall x \in X$. Then $M(\cdot)$ is l.r. if and only if for every fixed $\xi \in Y$, the function $\varphi_\xi : X \to \mathbb{R}$, $\varphi_\xi(x) := \operatorname{dist}(\xi, M(x))$, is u.r..*

Proof. The forward implication follows from Theorem 5.1 with $\psi(x,y) = \rho(x,y)$. The backward implication follows with some modification of the proof of Proposition 5.5, in [12]. Let $V \subset Y$ be any open set and $\mathbf{B}_\varepsilon(y)$ be the open ball around y with radius ε. We need to show that $M^{-1}(V)$ is a robust set in X; i.e., if $\overline{x} \in M^{-1}(V)$, we have to show $\overline{x}$ is a robust point of $M^{-1}(V)$. Hence, $M(\overline{x}) \cap V \neq \emptyset$. Let $\overline{\xi} \in M(\overline{x}) \cap V$ and $B_\epsilon(\overline{\xi}) \subset V$, for some $\epsilon > 0$. Hence, we have

$$\{x \in X \mid \varphi_{\overline{\xi}}(x) < \varphi_{\overline{\xi}}(\overline{x}) + \epsilon\}$$

is a non-empty robust set, since $\overline{x} \in \{x \in X \mid \varphi_{\overline{\xi}}(x) < \varphi_{\overline{\xi}}(\overline{x}) + \epsilon\} = \{x \in X \mid \varphi_{\overline{\xi}}(x) < \epsilon\}$ as $\varphi_{\overline{\xi}}(\overline{x}) = 0$. Thus, for any neighborhood $N(\overline{x})$ of $\overline{x}$ we have

$$N(\overline{x}) \cap \operatorname{int}\{x \in X \mid \varphi_{\overline{\xi}}(x) < \epsilon\} \neq \emptyset.$$

It then follows that there is $x^* \in N(\overline{x}) \cap \operatorname{int}\{x \in X \mid \varphi_{\overline{\xi}}(x) < \epsilon\}$ and an open neighborhood $U(x^*) \subset N(\overline{x}) \cap \operatorname{int}\{x \in X \mid \varphi_{\overline{\xi}}(x) < \epsilon\}$ of x^*. Consequently,

$$\forall x \in U(x^*) : \varphi_{\overline{\xi}}(x) < \epsilon,$$

that is

$$\forall x \in U(x^*) : \operatorname{dist}(\overline{\xi}, M(x)) < \epsilon.$$

This implies that

$$\forall x \in U(x^*), \exists y \in M(x) : \|\overline{\xi} - y\| < \epsilon$$

and thus

$$\forall x \in U(x^*) : M(x) \cap \mathbf{B}_\epsilon(\overline{\xi}) \subset M(x) \cap V.$$

Hence, $U(x^*) \subset M^{-1}(V)$. Which yields $N(\overline{x}) \cap \operatorname{int} M^{-1}(V) \neq \emptyset$. Since $N(\overline{x})$ is an arbitrary neighborhood, we conclude that $\overline{x}$ is a robust point of $M^{-1}(V)$. As $\overline{x} \in M^{-1}(V)$ was chosen arbitrarily, we have that $M^{-1}(V)$ is a robust set in X. ■

5.2. Upper robustness of supremum

A similar statement of upper robustness could also be given for supremum marginal value functions. Hence, let

$$\phi(x) := \sup_{y \in M(x)} \psi(x,y).$$

Theorem 5.7 (upper robustness of supremum)**.** *Let X and Y be topological spaces and $x^0 \in X$. If $M : X \rightrightarrows Y$ is compact-valued, u.r. at x^0 and $\psi : X \times Y \to \mathbb{R}$ is u.s.c. on $\{x^0\} \times M(x^0)$, then ϕ is upper robust at x^0.*

Proof. (cf. Theorem 2, p. 52 in [3]). Let $c \in \mathbb{R}$ be arbitrary and let $x^0 \in [\phi < c]$ be any, then we show that x^0 is a robust point of $[\phi < c]$ or, equivalently, $[\phi < c]$ is a semi-neighborhood of x^0. Since, ψ is u.s.c. on $\{x^0\} \times M(x^0)$, we have, for each $\overline{y} \in M(x^0)$ and $\varepsilon > 0$ neighborhoods $N^\varepsilon(\overline{y})$ and $N^\varepsilon_{\overline{y}}(x^0)$ of $\overline{y}$ and x^0, respectively, such that

$$\forall y \in N^\varepsilon(\overline{y}), \forall x \in N^\varepsilon_{\overline{y}}(x^0) : \psi(x, y) \leq \psi(x^0, \overline{y}) + \varepsilon.$$

The compactness of $M(x^0)$ implies the existence of $\{\overline{y}_1, \dots, \overline{y}_{n(\varepsilon)}\} \subset M(x^0)$ such that:

$$M(x^0) \subset \bigcup_{i=1}^{n(\varepsilon)} N^\varepsilon(\overline{y}_i).$$

We define the open set:

$$N := \bigcup_{i=1}^{n(\varepsilon)} N^\varepsilon(\overline{y}_i).$$

Since, $M(\cdot)$ is upper robust at x^0, $M(x^0) \subset N$ and N is open, there is a semi-neighborhood $S(x^0)$ of x^0 such that

$$\forall x \in S(x^0) : M(x) \subset N$$

(As $M(\cdot)$ is u.r. at x^0 and $M(x^0) \subset N$, we may take the set $S(x^0) := \{x \in X \mid M(x) \subset N\}$). Hence, the set

$$N_\varepsilon(x^0) := S(x^0) \cap \bigcap_{i=1}^{n(\varepsilon)} N^\varepsilon_{\overline{y}_i}(x^0)$$

is a semi-neighborhood of x^0 (see Proposition 2.8(2)). If $D(x^0)$ is any open neighborhood of x^0 in X, then $D(x^0) \cap N_\varepsilon(x^0)$ is also a semi-neighborhood of x^0. Let $x \in D(x^0) \cap N_\varepsilon(x^0)$ be arbitrarily chosen. Then it follows that

$$x \in S(x^0), x \in \bigcap_{i=1}^{n(\varepsilon)} N^\varepsilon_{\overline{y}_i}(x^0) \quad \text{and} \quad y \in M(x) \subset N.$$

Hence, for some $i_0, 1 \le i_0 \le n_\varepsilon(x)$, $y \in N^\varepsilon(\overline{y}_{i_0})$ and $x \in \bigcap_{i=1}^{n(\varepsilon)} N^\varepsilon_{\overline{y}_i}(x^0) \subset N^\varepsilon_{\overline{y}_{i_0}}(x^0)$. This implies that $\psi(x,y) \le \psi(x^0, \overline{y}_{i_0}) + \varepsilon$. Moreover, since $x \in D(x^0) \cap N_\varepsilon(x^0)$ and $y \in M(x)$ are arbitrary, we have that

$$\forall x \in D(x^0) \cap N_\varepsilon(x^0) : \sup_{y \in M(x)} \psi(x,y) \le \psi(x^0, \overline{y}_{i_0}) + \varepsilon \le \phi(x^0) + \varepsilon.$$

This yields

$$\forall x \in D(x^0) \cap N_\varepsilon(x^0) : \phi(x) \le \phi(x^0) + \varepsilon.$$

Now, since $x^0 \in [\phi < c]$ and $\varepsilon > 0$ are arbitrary, we can choose $0 < \varepsilon < c - \phi(x^0)$. It then follows that

$$\forall x \in D(x^0) \cap N_\varepsilon(x^0) : \phi(x) \le \phi(x^0) + \varepsilon < \phi(x^0) + c - \phi(x^0) = c.$$

From this we conclude that

$$\forall x \in D(x^0) \cap N_\varepsilon(x^0) : \phi(x) < c.$$

Hence, $D(x^0) \cap N_\varepsilon(x^0) \subset [\phi < c]$. Therefore, $[\phi < c]$ is a semi-neighborhood of x^0 and the claim follows from Proposition 2.8(2). ■

Corollary 5.8 (lower robustness of infimum)**.** *Let X and Y be topological spaces and $x^0 \in X$. If $M : X \rightrightarrows Y$ is compact-valued and u.r. at x^0; and $\psi : X \times Y \to \mathbb{R}$ l.s.c. on $\{x^0\} \times M(x^0)$, then the marginal value function*

$$\varphi(x) = \inf_{y \in M(x)} \psi(x,y)$$

is lower robust at x^0.

Proof. Use

$$-\varphi(x) = \sup_{y \in M(x)} [-\psi(x,y)]$$

and apply Theorem 5.7; i.e., $-\varphi$ will be u.r. at x^0. This concludes that φ is lower robust at x^0. ■

Remark 5.9. For similar reasons as in Remark 5.2, the upper semi-continuity of ψ, in Theorem 5.7, cannot be weakened further. But if ψ is u.r., then $M(\cdot)$ has to be u.s.c.

5.3. Upper robustness over robust partitions

We call $\varphi : X \to \mathbb{R}$ *piecewise u.r.* (*l.r.*) iff there exists a robust partition $X_1, X_2, \ldots, X_r$ of X such that for all $i \in \{1, \ldots, r\}$ the restriction of φ to X_i is u.r. (l.r.) with respect to the relative topology of X_i induced by the topological space X.

Theorem 5.10. *Let X be a topological space and $\varphi : X \to \mathbb{R}$. If φ is piecewise u.r. (l.r.), then φ is u.r. (l.r.).*

Proof. Let $c \in \mathbb{R}$, such that $F_c := \{x \in X \mid \varphi(x) < c\}$. Then $F_c = \bigcup_{i \in I}(X_i \cap \{x \in X \mid \varphi(x) < c\})$. Assume now $x \in F_c$ and $N(x)$ be any open neighborhood of x with respect to X. Then we get $x \in X_i \cap \{x \in X \mid \varphi(x) < c\}$ for some $i \in I$; hence, $N(x) \cap X_i$ is a neighborhood of x relative to X_i. Since φ is u.r. with respect to the relative topology on X_i, we get $\text{int}_{X_i}[X_i \cap \{x \in X \mid \varphi(x) < c\} \cap N(x)] \neq \emptyset$ and $\text{int}_{X_i}[X_i \cap \{x \in X \mid \varphi(x) < c\} \cap N(x)] \subset N(x) \cap X_i$. Since $N(x)$ is open and $N(x) \cap X_i$ is robust in X, Lemma 4.16 yields:

$$\begin{aligned}
&\text{int}_X[\{x \in X \mid \varphi(x) < c\} \cap N(x)] \\
&\supset \text{int}_X[X_i \cap \{x \in X \mid \varphi(x) < c\} \cap N(x)] \neq \emptyset \\
&\Rightarrow N(x) \cap \text{int}\,(\{x \in X_i \mid \varphi(x) < c\}) \neq \emptyset.
\end{aligned}$$

Consequently, it follows that x is a robust point of $\{x \in X_i \mid \varphi(x) < c\}$. Consequently, by Remark 2.2, we have that the set $\{x \in X \mid \varphi(x) < c\}$ is robust. Therefore, φ is u.r. on X. The proof for l.r. follows along the same line of arguments. ∎

Theorem 5.11. *Let $\psi : X \times Y \to \mathbb{R}$ be an u.s.c. function and let $M : X \rightrightarrows Y$ be a piecewise l.s.c. (l.r.) SVM on X. Then the marginal function φ is u.r. on X.*

Proof. Since $M(\cdot)$ is piecewise l.s.c. (piecewise l.r.), there is a robust partition $X_1, ..., X_r$ of X such that for each $i \in I := \{1, ..., r\}$, X_i is robust in X and the restriction of $M(\cdot)$ to X_i is l.s.c. (l.r.). Thus, using [3, Theorem 4, p. 51] (or Theorem 5.1), we see that φ is u.s.c. (φ is u.r.) on X_i, which implies that φ is u.r. on X_i for each $i \in I$. Therefore, by Theorem 5.10, φ is u.r. on X. ∎

Similarly,

Theorem 5.12. *If $\psi : X \times Y \to \mathbb{R}$ is u.s.c. and $M : X \rightrightarrows Y$ is a piecewise u.s.c. (u.r.) compact-valued SVM, then the marginal function ϕ is u.r.*

Furthermore, Corollary 5.3 and Corollary 5.8 could be reformulated to provide lower robustness properties of marginal functions, based on the corresponding piecewise semi-continuity of $M(\cdot)$.

Remark 5.13. Obviously, to verify the upper robustness of φ and ϕ in Theorems 5.11 and 5.12 we need only the upper semi-continuity of ψ on $X_i \times Y$ with respect to relative topology for $i = 1, \ldots, r$.

5.4. Approximatable marginal functions

Here, we stress the fact that X does not need to be a complete metric space. In this case approximatability can be a sharper assumption than robustness.

Remark 5.14. If X is a complete metric space, then the upper approximatability of φ follows immediately from its upper robustness by Proposition 3.9, with $M(\cdot)$ taken as lower robust.

To guarantee the upper approximatability of a supremum we need the following lemma.

Lemma 5.15. *Let X be a metric space, $\phi : X \to \mathbb{R}$ be a function and $S \subset X$ be the set of points of upper semi-continuity of ϕ. If ϕ is upper robust and S is dense in X, then ϕ is upper approximatable.*

Proof. (The proof follows Zheng *et al.* (Proposition 3.4, [19]) for a robust function.) Take an arbitrary $x^0 \in X$. For each $n \in \mathbb{N}$ consider the set

$$\Phi_n := \left\{ x \in X \mid \phi(x) < \phi(x^0) + \frac{1}{n} \right\}.$$

Then for all $n \in \mathbb{N}, x^0 \in \Phi_n$ and Φ_n is a robust set in X. Moreover, if $\mathbf{B}_{\frac{1}{n}}(x^0)$ is any open ball around x^0, then $x^0 \in \mathbf{B}_{\frac{1}{n}}(x^0) \cap \Phi_n, \forall n \in \mathbb{N}$, and $\mathbf{B}_{\frac{1}{n}}(x^0) \cap \Phi_n$ is a robust set in X. Hence, $\mathrm{int}(\mathbf{B}_{\frac{1}{n}}(x^0) \cap \Phi_n) \neq \emptyset$. Since, S is dense in X, for each $n \in \mathbb{N}$, there is $x^n \in S \cap (\mathbf{B}_{\frac{1}{n}}(x^0) \cap \Phi_n)$. Consequently, there is a sequence $\{x^n\} \subset S$ such that

$$\lim_{n\to\infty} x^n = x^0, \text{ and}$$
$$\phi(x^n) < \phi(x^0) + \frac{1}{n}, \forall n \in \mathbb{N}.$$

The latter implies that $\limsup_n \phi(x^n) \leq \phi(x^0)$. Therefore, ϕ is upper approximatable by Definition 3.8. ■

Theorem 5.16 (upper approximatability of supremum)**.** *Let X and Y be metric spaces. If $M : X \rightrightarrows Y$ is upper approximatable and compact-valued; and $\psi : X \times Y \to \mathbb{R}$ is u.s.c., then φ is upper approximatable.*

Proof.

(i) Since $M(\cdot)$ is upper approximatable, $M(\cdot)$ is upper robust. Moreover, since $M(\cdot)$ is compact-valued, Theorem 5.7 assures that ϕ is upper robust.

(ii) At the same time, since $M(\cdot)$ is upper approximatable then $M(\cdot)$ has a dense set S of upper semi-continuity. From this it follows that $\phi(x) = \sup_{y\in M(x)} \psi(x, y)$ is u.s.c. on S (Theorem 5, [3]); i.e., ϕ has a dense set of upper semi-continuity.

Consequently, using (i) and (ii), the claim follows from Lemma 5.15. ■

Proposition 5.17 (upper approximatability of infimum)**.** *Let X and Y be metric spaces. Suppose that the function $\psi : X \times Y \to \mathbb{R}$ is u.s.c. and $M : X \rightrightarrows Y$ is a lower approximatable SVM, then the function φ is upper approximatable.*

Proof. It follows from a similar argument as in Theorem 5.16 using Theorem 5.1 and Lemma 5.15. ■

6. Robustness of SVM's with given structures

Subsequently, we consider the robustness of set-valued maps defined by using systems of functional inequalities. For issues related with the continuity properties of such set-valued maps one finds the book of Bank *et al.* [5] and the paper of Hogan [11] indispensable.

6.1. The finite parametric case

6.1.1. Lower robustness

To begin with, let X and T be metric spaces and $B : X \rightrightarrows T$ be a SVM given by

$$B(x) := \{t \in T \mid h_i(x,t) \leq 0, i \in I\},$$

where $h_i : X \times T \to \mathbb{R}, i \in I := \{1, \ldots, p\}$. We would like to characterize robustness properties of $B(\cdot)$ through the functions $h_i, i \in I$.

Proposition 6.1. *Let $B(\cdot)$ be as given above and suppose that for each fixed $t \in T$ the set*

$$\triangle(t) := \bigcap_{i=1}^{p} \{x \in X \mid h_i(x,t) \leq 0\}$$

is non-empty and robust in X. Then for every open set $U \subset T$, the set $B^{-1}(U)$ is robust; i.e., $B(\cdot)$ is a lower robust SVM.

Proof. Let $U \subset T$ be any open set. It suffices to show that $B^{-1}(U)$ is a robust set in X. Then

$$\begin{aligned} B^{-1}(U) &= \bigcup_{t \in U} B^{-1}(t) = \bigcup_{t \in U} \{x \in X \mid t \in B(x)\} \\ &= \bigcup_{t \in U} \{x \in X \mid h_i(x,t) \leq 0, i \in I\} \\ &= \bigcup_{t \in U} \bigcap_{i=1}^{p} \{x \in X \mid h_i(x,t) \leq 0\}. \end{aligned}$$

By assumption $\triangle(t) = \bigcap_{i=1}^{p} \{x \in X \mid h_i(x,t) \leq 0\}$ is robust; hence, $B^{-1}(U)$ is a union of robust sets. Therefore, by Remark 2.2, $B^{-1}(U)$ is a robust set. ∎

Remark 6.2. Let $B(\cdot)$ and $\triangle(t)$ be as given in Proposition 6.1. If for every fixed $t \in T$ and for each fixed $i \in I$, the functions $h_i(\cdot, t)$ are quasi-convex, then this is equivalent to that the set

$$\triangle_i(t) := \{x \in X \mid h_i(x,t) \leq 0\}$$

is convex. Furthermore, if $\text{int}(\triangle(t)) \neq \emptyset$, then $\triangle(t)$ will be a convex set with a non-empty interior, which is a robust set (cf. Corollary 2.3). Hence, in such a case to guarantee that $\operatorname{int}(\triangle(t)) \neq \emptyset$, we may need to assume the satisfaction of some Slater condition.

Proposition 6.3. *Let X be a topological space, T be a normed linear space, and $B: X \rightrightarrows T$ be given according to*

$$B(x) = \{t \in T \mid h_i(x,t) \leq 0, \forall i \in I\},$$

where $I = \{1, \dots, p\}$. If

1. *for every pair $(x^0, t^0) \in X \times T$, and every neighborhood $V(t^0)$ of t^0, there exists $\widetilde{t} \in V(t^0)$ such that*
$$h_i(x^0, t^0) \leq 0 \quad \textit{implies} \quad h_i(x^0, \widetilde{t}) < 0, \forall i \in I;$$
2. *$h_{i_0}(\cdot, t)$ is u.r. on X for each $t \in T$;*
3. *$h_i(\cdot, t)$ u.s.c. on X for each $t \in T, i \in I \setminus \{i_0\}$;*

then $B(\cdot)$ is l.r. on X.

Proof. Let $x^0 \in X$ and $t^0 \in B(x^0)$ and $V(t^0)$ be a neighborhood of t^0. Then we want to show that $B^{-1}(V)$ is a semi-neighborhood of x^0; i.e., x^0 is a robust point of $B^{-1}(V)$.

By (i) we have some $\widetilde{t} \in V(t^0)$ with $h_i(x^0, \widetilde{t}) < 0$. Using the upper semicontinuity, there is some neighborhood $U(x^0)$ such that for all $x \in U(x^0)$

$$h_i(x, \widetilde{t}) < 0, i \in I \setminus \{i_0\}$$

and we know, by the upper robustness of $h_{i_0}(\cdot, \widetilde{t})$, that $x^0 \in \{x \in X \mid h_{i_0}(x, \widetilde{t}) < 0\} =: H$ and that H is robust. Hence, $U(x^0) \cap H$ is a robust set containing x^0 (cf. Remark 2.2). Furthermore, $B^{-1}(V) \supset U(x^0) \cap H \ni x^0$; i.e., $B^{-1}(V)$ is a semi-neighborhood of x^0. ■

In the following corollary we use some well-known generalization of convexity ensuring Slater's condition, whenever level sets are not a singleton. For convenience, we repeat here its definition.

Definition 6.4 (see e.g., [6]). A function h of the normed linear space T in $\mathbb{R}$ is called **strictly quasi-convex** iff $h(\lambda s + (1-\lambda)t) < \max\{h(s), h(t)\}$ for all $\lambda \in (0,1)$ and all $s, t \in T$, $s \neq t$.

Corollary 6.5 (cf. also Theorem 3.1.6, p. 41, [5])**.** *Let X be a topological space, T be a normed linear space and $B : X \rightrightarrows T$ be given by*

$$B(x) = \{t \in T \mid h_i(x,t) \leq 0, i \in I\}.$$

If the following conditions hold true:

1. *for each fixed $x^0 \in X$, $B(x^0) \neq \emptyset$ and is not a singleton;*
2. *for each $i \in I$, $h_i(x,\cdot) : T \to \mathbb{R}$ is strictly quasi-convex, for every fixed $x \in X$,*
3. *for one $i_0 \in I$, $h_{i_0}(\cdot,t)$ is upper robust on X, for every fixed $t \in T$;*
4. *for each $i \in I \setminus \{i_0\}$, $h_i(\cdot,t)$ is u.s.c., for each fixed $t \in T$;*

then $B(\cdot)$ is lower robust on X.

Proof. Assumptions 1 and 2 imply assumption 1 of Proposition 6.3 (see Theorem 3.1.6 in [5]). The rest is as in the proof of Proposition 6.3. ■

Proposition 6.6. *Let X be a topological space, T be a normed linear space, and $x^0 \in X$ and $t^0 \in T$. If $h_i(x^0,\cdot) : T \to \mathbb{R}$, $i \in I$, are convex and there is some $\widetilde{t} \in T \setminus \{t^0\}$ such that for all $i \in I$:*

$$h_i(x^0,t^0) \leq 0 \Rightarrow h_i(x^0,\widetilde{t}) < 0 \text{ (\textit{Slater's Condition})},$$

then condition (i) *of Proposition* 6.3 *is satisfied at x^0.*

Proof. Let $h_i(x^0,t^0) \leq 0, \forall i \in I$. Hence, by assumption, there is $\widetilde{t} \neq t^0$ such that $h_i(x^0,\widetilde{t}) < 0, \forall i \in I$. This implies

$$t_n = \frac{1}{n}\widetilde{t} + \left(1 - \frac{1}{n}\right)t^0 \to t^0 \quad for \quad n \to \infty.$$

Subsequently, for a given neighborhood $V(t^0)$ and a sufficiently large n we have that $t_n \in V(t^0)$. Furthermore,

$$h_i(x^0,t_n) \leq \frac{1}{n}h_i(x^0,\bar{t}) + \left(1 - \frac{1}{n}\right) h_i(x^0,\widetilde{t}) < 0.$$

■

To relate lower robustness to a well-known result of Bank *et al.* [5], we consider a function $h : X \to \mathbb{R}$ and define its level set map as

$$\mathcal{L}_{h,X}(\alpha) := \{x \in X \mid h(x) \leq \alpha\}.$$

Theorem 3.1.7 of Bank *et al.* [5] claims that $\mathcal{L}_{h,X}(\cdot)$ is l.s.c. on X if and only if h is continuous and has only global minima on X. However, we give here a general statement indicating that lower robustness of $\mathcal{L}_{h,X}(\cdot)$ does not preclude the existence of local minima of h.

Proposition 6.7. *Let $X \subset \mathbb{R}^n$, $T \subset \mathbb{R}^m$, X be a robust set in $\mathbb{R}^n$ and $B(x) := \{t \in T \mid h_i(t) \le x_i,$ for all $i,\ 1 \le i \le n\}$, where $h_i : T \to \mathbb{R}, 1 \le i \le n$, are functions. If, for each fixed $t \in T$,* $\operatorname{int}\{x \in X \mid h_i(t) \le x_i,$ *for all $i,\ 1 \le i \le n\} \ne \emptyset$, then the set-valued map $B : X \rightrightarrows T$ is lower robust on X.*

Proof. Given $x^0 \in X$ and a $t^0 \in B(x^0)$, observe that

$$\prod_{i=1}^{n} \Big[h_i(t^0), +\infty\Big) \cap X = B^{-1}(t^0) = \{x \in X \mid h_i(t^0) \le x_i, i = 1, \dots, n\}.$$

Hence, $x^0 \in \prod_{i=1}^n [h_i(t^0), +\infty) \cap X$ and $\operatorname{int}[\prod_{i=1}^n [h_i(t^0), +\infty) \cap X] \ne \emptyset$. This shows that x^0 is a robust point of $\prod_{i=1}^n [h_i(t^0), +\infty) \cap X$. Consequently, $\operatorname{int} B^{-1}(t^0) \ne \emptyset$ and x^0 is a robust point of $B^{-1}(t^0)$ (cf. Proposition 2.8(1)). Since x^0 is arbitrary, $B(\cdot)$ will be a lower robust SVM. ■

In the special cases when $X = \mathbb{R}^n$, the assumption $\operatorname{int}\{x \in X \mid h_i(t) \le x_i$, for all $i,\ 1 \le i \le n\} \ne \emptyset$ of Proposition 6.7 is obviously satisfied. Note also that, in Proposition 6.7, the functions $h_i, i \in I$, are not required to possess any topological property like robustness or continuity.

6.1.2. Upper robustness

Once again, reiterating Definition 4.5, we have that $B : X \rightrightarrows T$ is upper robust at $x^0 \in X$ if for each neighborhood U of $B(x^0)$ there is a semi-neighborhood $SNH(x^0)$ of x^0 such that

$$\forall x \in SNH(x^0) : B(x) \subset U.$$

In contrast to the lower robustness of $B(\cdot)$, its upper robustness could follow from relatively weaker assumptions. One standard result has been given below.

Proposition 6.8 (see Theorem 3.1.2 [5] and Theorem 3 [11])**.** *Let X and T be Hausdorff topological spaces and let T be compact. Assume further that*

$h_i : X \times T \to \mathbb{R}$ are lower semi-continuous, for all $i \in I$ of the finite index set I. Then $B(\cdot)$ with

$$B(x) = \{t \in T \mid h_i(x,t) \le 0, i \in I\}$$

is u.s.c. on dom(B)[‡] *with respect to the relative topology on* dom(B)*, hence, it is u.r. on X. In addition, if* dom(B) *is closed, then $B(\cdot)$ is u.s.c. on X.*

To give further results of upper robustness, we introduce the following definition:

Definition 6.9. Let X and T be topological spaces and $h : X \times T \to \mathbb{R}^p$. Then $h(\cdot,t)$ is called *lower robust* [*l.s.c.*] *at* x^0 *uniformly* for all $t \in T$ iff for all $\varepsilon > 0$ there exists a semi-neighborhood $SNH_\varepsilon(x^0)$ [a neighborhood $U(x^0)$] of x^0 such that

$$\begin{aligned} &h_i(x,t) > h_i(x^0,t) - \varepsilon, \forall x \in SNH_\varepsilon(x^0), \forall t \in T, \forall i \in I \\ &\left[h_i(x,t) > h_i(x^0,t) - \varepsilon, \forall x \in U(x^0), \forall t \in T, \forall i \in I\right]. \end{aligned}$$

Moreover, we need some regularity condition given by

Definition 6.10. Let X be a topological space and T be a metric space. The function $h : X \times T \to \mathbb{R}^p$ is called **strictly r-regular** at $x^0 \in X$, for all $t \in T$, iff there is a strictly increasing function $r : \mathbb{R}_+ \to \mathbb{R}_+,\ r(0) = 0$, such that

$$\operatorname{dist}(t, B(x^0)) \le r\left(\max_{i=1,\dots,p}\left[h_i(x^0,t)^+\right]\right).$$

Proposition 6.11. *Let X be a topological space, T be a compact metric space, $h : X \times T \to \mathbb{R}^p$, $h := (h_1,\dots,h_p)$, and let $B : X \rightrightarrows T$ be a set-valued map, such that for each $x \in X, B(x)$ is given by*

$$B(x) = \{t \in T \mid h(x,t) \le 0\}.$$

If

1. *$h(\cdot,t)$ is lower robust [l.s.c.] at x^0 uniformly for all $t \in T$; and*
2. *h is strictly r-regular at x_0 for all $t \in T$,*

then $B(\cdot)$ is upper robust [u.s.c.] at x^0.

[‡]dom$(B) = \{x \in X \mid B(x) \neq \emptyset\}$

Proof. (In the following, to prove the upper semi-continuity, replace SNH by neighborhood.) Thus, for any neighborhood U of $B(x^0)$ we have to find some semi-neighborhood $SNH_U(x^0)$ such that

$$\forall x \in SNH_U(x^0) : \{t \in T \mid h(x,t) \leq 0\} \subset U.$$

But, since $B(x^0)$ is bounded, there is $\varepsilon > 0$ such that

$$U_\varepsilon := \{t \in T \mid \mathrm{dist}(t, B(x^0)) < \varepsilon\} \subset U.$$

Consequently, we need only to show that there is a semi-neighborhood SNH_{U_ε} of x^0 such that

$$\forall x \in SNH_{U_\varepsilon} : \{t \in T \mid h(x,t) \leq 0\} \subset \{t \in T \mid \mathrm{dist}(t, B(x^0)) < \varepsilon\}.$$

From the lower robustness of $h(\cdot, t)$ at x^0 uniformly for $t \in T$, we get, for any $\sigma > 0$, a semi-neighborhood $SNH_\sigma(x_0)$ such that, for each $x \in SNH_\sigma(x_0)$ and each $t \in \{t \in T \mid h_i(x,t) \leq 0, i = 1, \ldots, p\}$, the following holds

$$h_i(x,t) > h_i(x^0,t) - \sigma.$$

Hence,

$$h_i(x^0,t) < \sigma, \qquad \forall i = 1, \ldots, p.$$

This implies

$$\max_{i=1,\ldots,p} \Big[h_i(x^0,t)\Big]^+ < \sigma.$$

Using strict r-regularity and the monotonicity of r, we obtain

$$\mathrm{dist}(t, B(x^0)) < r(\sigma).$$

Taking $SNH_{U_\varepsilon} := SNH_{r^{-1}(\varepsilon)}(x^0)$ (i.e., $\varepsilon := r(\sigma)$, by choosing $\sigma = r^{-1}(\varepsilon)$) the proof is complete. ∎

6.2. A semi-infinite case

In the marginal analysis of the generalized semi-infinite optimization, the following parametric problem (with a parameter x) is considered (see e.g. Geletu [1], Jongen *et al.* [13], Stein [21, 22], Weber [24], etc.):

$$f(\xi, x) \to \min$$

subject to the constraint $\xi \in M(x)$, which is given by

$$M(x) := \{\xi \in Y \mid G(\xi, x, t) \leq 0,\, t \in B(x)\}.$$

The SVM $M(\cdot) : X \rightrightarrows Y$ is defined using a semi-infinite inequality system, where $G : Y \times X \times T \to \mathbb{R}$ and $B : X \rightrightarrows T$ is again an SVM. Thus, the problem possesses uncountably many constraints. The SVM $B(\cdot)$ is often called an index map of the semi-infinite (finite number of variables, infinite number of constraints) problem. In this section we are only interested in the robustness properties of the map $M(\cdot)$. Define the marginal function

$$m(\xi, x) := \begin{cases} \sup\limits_{t \in B(x)} G(\xi, x, t), & \text{if } B(x) \neq \emptyset; \\ -\infty, & \text{if } B(x) = \emptyset. \end{cases}$$

Obviously, we have that

$$M(x) := \{\xi \in Y \mid m(\xi, x) \leq 0\}.$$

Corollary 6.12. *Let $M(x) := \{\xi \in X \mid m(\xi, x) \leq 0\}$. If*

1. *for each fixed $x \in X, m(\cdot, x)$ is strictly quasi-convex and $M(x)$ is not a singleton, and*
2. *for each fixed $\xi \in Y, m(\xi, \cdot)$ is upper robust,*

then $M(\cdot)$ is lower robust.

Proof. See Corollary 6.5. ■

Next we will try to guarantee the assumptions on the marginal function $m(\cdot, \cdot)$ in Corollary 6.12 through the properties of G and $B(\cdot)$. We use the following definitions of convexity of functions.

Definition 6.13 (γ-strongly convex function)**.** Let Y be a linear space and $f : Y \to \mathbb{R}$. If for any $\xi_1, \xi_2 \in Y$ and $\alpha \in (0, 1)$ there are some fixed $c > 0$ and $0 < \gamma \leq 2$ such that

$$f(\alpha\xi_1 + (1-\alpha)\xi_2) \leq \alpha f(\xi_1) + (1-\alpha) f(\xi_2) - \frac{1}{2} c\alpha(1-\alpha)\|\xi_1 - \xi_2\|^\gamma.$$

In Definition 6.14, when $\gamma = 2$, f is called *strongly convex* (cf. [23]).

Definition 6.14 (γ-strongly quasi-convex function)**.** Let Y be a linear space and $f : Y \to \mathbb{R}$. If for any $\xi_1, \xi_2 \in Y$ and $\alpha \in (0,1)$ there are some fixed $c > 0$ and $0 < \gamma \leq 2$ such that

$$f(\alpha\xi_1 + (1-\alpha)\xi_2) \leq \max\Big\{f(\xi_1), f(\xi_2)\Big\} - \frac{1}{2}c\alpha(1-\alpha)\|\xi_1 - \xi_2\|^\gamma,$$

then f is called *γ-strongly quasi-convex.*

Example 6.15. *If $f : [a,b] \to \mathbb{R}, f'(x) \geq d > 0$, for all $x \in [a,b]$, then f is γ-strongly quasi-convex for $\gamma = 1$; and for $\gamma > 0$, whenever $[a,b]$ is a bounded interval. The verification of this statement is straightforward. Accordingly, the function $f(x) = x^3 + x$ is $\gamma-$strongly quasi-convex either for $x \in \mathbb{R}$ and $\gamma = 1$ or for $x \in [a,b]$ and $\gamma > 0$.*

Obviously, a γ-strongly quasi-convex function is strictly quasi-convex. Moreover, we have

Lemma 6.16. *γ-strongly convex implies γ-strongly quasi-convex.*

Proposition 6.17. *Let X and T be topological spaces, Y be a linear space and $B : X \rightrightarrows T$ be an SVM with compact values. If, for any fixed $x \in X$ and $t \in T$, $G(\cdot, x, t)$ is γ-strongly (quasi-) convex, then $m(\cdot, x)$ is strictly quasi-convex.*

Proof. For $\xi_1, \xi_2 \in Y$ and $\lambda_1, \lambda_2 \in [0,1]$ we have

$$\begin{aligned}
&m(\lambda_1\xi_1 + \lambda_2\xi_2, x) = \sup_{t\in B(x)} \Big[G(\lambda_1\xi_1 + \lambda_2\xi_2, x, t)\Big] \\
&\leq \sup_{t\in B(x)} \Big[\max\Big\{G(\xi_1, x, t), G(\xi_2, x, t)\Big\} - \frac{1}{2}c\alpha(1-\alpha)\|\xi_1 - \xi_2\|^\gamma\Big] \\
&= \max\Big\{\sup_{t\in B(x)} G(\xi_1, x, t), \sup_{t\in B(x)} G(\xi_2, x, t)\Big\} - \frac{1}{2}c\alpha(1-\alpha)\|\xi_1 - \xi_2\|^\gamma \\
&= \max\Big\{m(\xi_1, x), m(\xi_2, x)\Big\} - \frac{1}{2}c\alpha(1-\alpha)\|\xi_1 - \xi_2\|^\gamma.
\end{aligned}$$

This implies that $m(\cdot, x)$ is γ-strongly quasi-convex; hence, it is strictly quasi-convex. ∎

Observe that, in Proposition 6.17, to get the strict quasi-convexity of $m(\cdot, x)$, we required no robustness property of $B(\cdot)$. However, this is not the case for the robustness of $m(\xi, \cdot)$.

Proposition 6.18. *If for each fixed* $\xi \in Y$*,* $G(\xi, \cdot, \cdot)$ *is u.s.c. and* $B(\cdot)$ *is upper robust and compact-valued, then* $m(\xi, \cdot)$ *is upper robust.*

Proof. For a fixed $\overline{\xi} \in Y$, we could write

$$m(\overline{\xi}, x) := \sup_{t \in B(x)} G(\overline{\xi}, x, t).$$

Thus, if we let $\phi(x) := m(\overline{\xi}, x)$ and $\psi(x, t) := G(\overline{\xi}, x, t)$, then the claim follows from Theorem 5.7. ∎

Summing up, given Y a linear space and

$$M(x) = \{\xi \in Y \mid G(\xi, x, t) \leq 0,\, t \in B(x)\}$$

then $M(\cdot)$ will be a lower robust SVM

1. if
   (a) $G(\cdot, \cdot, \cdot)$ is u.s.c. and $M(x)$ is not a singleton;
   (b) $G(\cdot, x, t)$ is $(\gamma-)$ strongly $(quasi-)$ convex, with respect to $\xi \in Y$ for each $t \in B(x)$;
   (c) $B(\cdot)$ is upper robust and compact-valued;
2. or if
   (a) $G(\cdot, \cdot, \cdot)$ is u.s.c.;
   (b) for each $x \in X$, there exists $\hat{\xi}$ such that $G(\hat{\xi}, x, t) < 0$ for all $t \in B(x)$ (note that G u.s.c. implies there is $U(\hat{\xi}) : G(\xi, x, t) < 0$ for all $\xi \in U(\hat{\xi})$);
   (c) $B(\cdot)$ is upper robust and compact-valued.

6.3. Piecewise semi-continuity of a SVM with a structure

In Section 4.3, we have considered piecewise semi-continuity properties of a general SVM. Correspondingly, we would like to characterize piecewise semi-continuity for set valued maps with given structures.

Recall that $M(x) = \{\xi \in Y \mid G(\xi, x, t) \leq 0,\ \forall\, t \in B(x)\}$. We give now a second characterization of lower robustness of $M(\cdot)$, besides the ones in Section 6.2, based on piecewise upper semi-continuity of $B(\cdot)$, joint upper semi-continuity of G and some weaker regularity condition of the system defining $M(x)$. Let X be a metric space, with metric ρ, let Y be a robust subset of some topological space, and T be a topological space.

Assumption (A): X has a robust partition $(X_i)_{i\in I}$, $I = \{0, 1, 2, \ldots, r+1\}$, where $X_0 := \{x \in X | B(x) = \emptyset\}$ and $X_{r+1} := \{x \in X \mid M(x) = \emptyset\}$ are among the robust partitions.

Assumption (B): $B : X \rightrightarrows T$ is compact-valued and $B|_{X_i}, i = 0, 1, 2, \ldots, r, r+1$ is u.s.c. with respect to the relative topology on X_i.

Definition 6.19 (local r-regularity)**.** The system

$$\begin{aligned} G(\xi, x, t) \;&\leq\; 0, \forall t \in B(x) \\ \xi \;&\in\; Y \end{aligned}$$

is r-regular at $(\xi^0, x^0) \in Y \times X_i$ if there is a semi-neighborhood $SNH(x^0) \subset X_i$ of x^0 with respect to the relative topology of X_i and a non-decreasing function $r_{\xi^0, x^0, SNH(x^0)} : \mathbb{R}_+ \to \mathbb{R}_+$ continuous at 0, with $r(0) = 0$ such that

$$\forall x \in SNH(x^0) : \operatorname{dist}(\xi^0, M(x)) \leq r\left(\max_{t\in B(x)} [G(\xi^0, x, t)]^+\right).$$

The r-regularity given in Definition 6.19 is quite weaker than the *metric regularity condition* given by Klatte and Henrion [15]. In fact, from the metric regularity follows the lower semi-continuity of $M(\cdot)$.

Theorem 6.20. *Let X and Y be normed spaces. If G is u.s.c. on $Y \times X \times T$, Assumptions* (A), (B) *are satisfied; and for all $i \in \{0, 1, \ldots, r, r+1\}$ and all $x^0 \in X_i$ the system defining $M(x)$ is r-regular at each (ξ^0, x^0), for each $\xi^0 \in M(x^0)$, then $M(\cdot)$ is l.r. on X.*

Proof. We show that $M(\cdot)$ is piecewise lower robust. That is we show that for each $i \in \{0, 1, \ldots, r+1\}$, $M(\cdot)$ is lower robust on X_i. If $x \in X_0$ we have that $M(x) = X$; hence, $M(\cdot)$ is continuous on X_0 in the relative topology. For all $x \in X_{r+1}$ we get, from $M(x) = \emptyset$ that $M(\cdot)$ is l.r. on X_{r+1}. Thus, it remains to discuss the case $1 \leq i \leq r$.

Thus, let $i \in \{1, \ldots, r\}$ and $x^0 \in X_i$. Let also $\xi^0 \in M(x^0)$. By definition of $M(\cdot)$ we have that

$$G(\xi^0, x^0, t) \leq 0, \forall t \in B(x^0).$$

If we let $g(\xi, x) := \max_{t\in B(x)}[G(\xi, x, t)]^+$, then $g(\xi^0, x^0) = 0$. Since $B(\cdot)$ is u.s.c. on X_i with respect to the relative topology of X_i, $g(\xi, \cdot)$ is u.s.c. at x^0 in the topology of X_i. Hence, given $\varepsilon > 0$, there is a neighborhood $V_\varepsilon(x^0)$

such that

$$g(\xi^0, x) \leq \varepsilon, \forall x \in V_\varepsilon(x^0) \cap X_i. \tag{2}$$

By the r-regularity at (ξ^0, x^0), we obtain that

$$\begin{aligned} &\forall x \in SNH(x^0) \cap \left[V_\varepsilon(x^0) \cap X_i\right] : \mathrm{dist}(\xi^0, M(x)) \\ &\leq r \left(\max_{t \in B(x)} \left[G(\xi^0, x, t)\right]^+ \right) = r(g(\xi^0, x)). \end{aligned}$$

Using (2) and the property of the function $r(\cdot)$, we obtain that

$$\forall x \in SNH(x^0) \cap \left[V_\varepsilon(x^0) \cap X_i\right] : \mathrm{dist}(\xi^0, M(x)) \leq r(\varepsilon).$$

Now, given an arbitrary neighborhood $U(\xi^0) \subset Y$ of ξ^0, there is $\varepsilon > 0$ (and a corresponding $V_\varepsilon(x^0)$) such that the open ball $\mathbf{B}_{2r(\varepsilon)}(\xi^0)$ is contained in $U(\xi^0)$. Accordingly, for each fixed $x \in SNH(x^0) \cap [V_\varepsilon(x^0) \cap X_i]$, we deduce that

$$\mathbf{B}_{2r(\varepsilon)}(\xi^0) \cap M(x) \neq \emptyset.$$

Accordingly, we find that

$$\forall x \in SNH(x^0) \cap \left[V_\varepsilon(x^0) \cap X_i\right] : M(x) \cap U(\xi^0) \neq \emptyset.$$

In other words

$$SNH(x^0) \cap \left[V_\varepsilon(x^0) \cap X_i\right] \subset M^{-1}(U(\xi^0)).$$

Since, $SNH(x^0)$ is a robust set with respect to the topology of X_i, we have $[\mathrm{int}_{X_i} SNH(x^0)] \cap [V_\varepsilon(x^0) \cap X_i] \neq \emptyset$. Hence, $\mathrm{int}_{X_i} M^{-1}(U(\xi^0)) \neq \emptyset$. Moreover, x^0 is a robust point of $SNH(x^0) \cap [V_\varepsilon(x^0) \cap X_i]$; thereby, x^0 is robust point of $M^{-1}(U(\xi^0))$ in the relative topology of X_i. Since, $x^0 \in X_i$ is arbitrary, then we conclude that $M(\cdot)$ is lower robust on X_i in the relative topology. Therefore, $M(\cdot)$ is piecewise lower robust; and hence, it is lower robust (cf. Theorem 4.20). ■

Observe that the upper semi-continuity of $B(\cdot)$ is not assumed on the whole of X, whereas it is on each of the partitioning sets X_i of X.

Remark 6.21. Let X and Y be normed spaces. If the regularity condition given by Definition 6.19 holds at $(\xi^0, x^0) \in Y \times X$, where $\xi^0 \in M(x^0)$, for a neighborhood $V(x^0)$ of x^0 with respect to X and $B(\cdot)$ is u.s.c. on X, then $M(\cdot)$ will be lower semi-continuous at x^0. The verification of this follows from a slight modification of the proof of Theorem 6.20.

For a related result of upper robustness, we make the following assumption:

Assumption(C): $B|_{X_i}$, $i = 1, \dots, r$ is l.s.c. with respect to the relative topology of X_i.

Theorem 6.22. *Let Y be a compact set. If G is l.s.c. on $Y \times X \times T$ and Assumptions* (A) *and* (C) *are satisfied, then $M(\cdot)$ is u.r. on X.*

Proof. Considering $m(\xi, x) = \max_{t \in B(x)} G(\xi, x, t)$, we get $M(x) = \{\xi \in Y \mid m(\xi, x) \leq 0\}$. Then Assumption(C) and the lower semi-continuity of G yield, by Theorem 4, Aubin and Cellina [3] that m is l.s.c. on $Y \times X_i, 1 \leq i \leq r$, in relative topology. Furthermore, since Y is compact, we have that $M(\cdot)$ is u.s.c. on X_i in the relative topology of X_i, for each $i \in \{1, \dots, r\}$ (cf. Proposition 6.8).

From the above, we obtain that $M(\cdot)$ is piecewise-u.s.c. on X. Applying Theorem 4.15, we conclude that $M(\cdot)$ is upper robust. ■

Remark 6.23. In Theorems 6.20 and 6.22 it suffices to ensure the semi-continuity properties of G on $Y \times X_i \times T$ in the relative topology of X_i.

6.4. Characterization of robustness through constraint qualifications

In this section, we try to find out some results connecting certain Mangasarian-Fromovitz type constraint qualifications with the robustness of set-valued maps.

Lemma 6.24. *Suppose that X and O are robust and open sets, respectively. Then $X \cap O \neq \emptyset$ implies* $\operatorname{int} X \cap O \neq \emptyset$

Proof. Obvious. ■

Let W be a normed space. We put, for $\xi \in W$; $\lambda, \rho > 0$, the convex set

$$\mathcal{K}_\lambda(\xi, \rho) := cone(\xi + \rho \mathbf{B}) \cap \lambda \mathbf{B}$$

which has a nonempty interior.

Definition 6.25. In a normed linear space W we define the tangential cone $\mathcal{T}(X,\bar{x})$ of $X \subset W$ at $\bar{x} \in \operatorname{cl} X$ by

$$\mathcal{T}(X,\bar{x}) = \{\xi \in W \mid \forall \varepsilon > 0, \forall \rho > 0 : (\bar{x} + \mathcal{K}_\varepsilon(\xi,\rho)) \cap X \neq \emptyset\}$$

which is known to be closed and which has the well-known properties

$$\mathcal{T}(\operatorname{int} X,\bar{x}) \subset \mathcal{T}(X,\bar{x}) = \mathcal{T}(\operatorname{cl} X,\bar{x}).$$

Lemma 6.26. *If X is a robust set and $\bar{x} \in \operatorname{cl} X$ then $\mathcal{T}(X,\bar{x}) = \mathcal{T}(\operatorname{int} X,\bar{x})$.*

Proof. Observe that $\mathcal{T}(\operatorname{int} X,\bar{x}) \subset \mathcal{T}(X,\bar{x}) = \mathcal{T}(\operatorname{cl} X,\bar{x}) = \mathcal{T}(\operatorname{cl}(\operatorname{int} X),\bar{x})$ $= \mathcal{T}(\operatorname{int} X,\bar{x})$ ■

Lemma 6.27. *Suppose X is a robust subset of W, $\bar{x} \in X$ (or $\bar{x} \in \operatorname{cl} X$). If $\xi_0 \in \mathcal{T}(X,\bar{x})$ and $\lambda > 0$, then $\operatorname{int}([\bar{x} + \mathcal{K}_\lambda(\xi_0,\rho)] \cap X) \neq \emptyset$ and $\bar{x}$ is a robust point of [robust point to] the set $[\bar{x} + \mathcal{K}_\lambda(\xi_0,\rho)] \cap X$.*

Proof. Since $\mathcal{T}(X,\bar{x}) = \mathcal{T}(\operatorname{int} X,\bar{x})$ and $\operatorname{int} \mathcal{K}_\lambda(\xi_0,\rho) \neq \emptyset$ and convex (hence, robust) we get from Lemma 6.24 that $\forall \varepsilon > 0, \forall \rho > 0 : \emptyset \neq \operatorname{int}[\bar{x} + \mathcal{K}_\varepsilon(\xi_0,\rho)] \cap \operatorname{int} X \subset \operatorname{int}([\bar{x} + \mathcal{K}_\varepsilon(\xi_0,\rho)] \cap X)$. For each such $\varepsilon < \lambda$ we have $\emptyset \neq \operatorname{int} \mathcal{K}_\varepsilon(\xi_0,\rho) \subset \mathcal{K}_\varepsilon(\xi_0,\rho) \subset \mathcal{K}_\lambda(\xi_0,\rho)$ and $\operatorname{int} \mathcal{K}_\varepsilon(\xi_0,\rho) \subset \varepsilon\mathbf{B}$. Hence, in each ε-ball $\bar{x}+\varepsilon\mathbf{B}$ of $\bar{x}$ there are interior points of $[\bar{x}+\mathcal{K}_\lambda(\xi_0,\rho)] \cap X$; i.e., $\bar{x}$ is a robust point of $[\bar{x} + \mathcal{K}_\lambda(\xi_0,\rho)] \cap X$ if $\bar{x} \in X$; or a robust point to $[\bar{x} + \mathcal{K}_\lambda(\xi_0,\rho)] \cap X$ if $\bar{x} \in \operatorname{cl} X \setminus X$. ■

Definition 6.28. (MFCQ) Let X and T be nonempty subsets of normed spaces, $\overline{x} \in X$ and $\overline{t} \in B(\overline{x})$. Furthermore, let

$$I_0 := \{i \in I \mid h_i(\bar{x},\bar{t}) = 0\}$$

represent the active index set of $B(\overline{x})$. We say that the **(MFCQ)** is satisfied at $(\overline{x},\overline{t})$ iff

(i) h_i is Frechet-differentiable at $(\bar{x},\bar{t})$ for each $i \in I_0$ and h_i is continuous at $(\bar{x},\bar{t})$ for each $i \in I \setminus I_0$;

(ii) there are vectors $\xi_0 \in \mathcal{T}(X,\bar{x})$ and $\eta_0 \in \mathcal{T}(T,\bar{t})$ such that, for each $i \in I_0$,
$$D_x h_i(\bar{x},\bar{t})\,\xi_0 + D_t h_i(\bar{x},\bar{t})\,\eta_0 < 0.$$

Theorem 6.29. *Suppose, X and T are nonempty subsets of normed spaces, X is robust, $X \times T \subset W$, W is an open set, $h : W \to \mathbb{R}^p$ and $B : X \rightrightarrows T$ is a set-valued map defined by*

$$B(x) = \{t \in T \mid h_i(x,t) \le 0, \forall i \in I := \{1,2,\ldots,p\}\}.$$

If the **(MFCQ)** *is satisfied at all $(\bar{x},\bar{t}) \in \{\bar{x}\} \times B(\bar{x})$, then $B(\cdot)$ is lower robust at $\bar{x}$.*

Proof. We show that for an arbitrary $\varepsilon > 0$ the pre-image $B^{-1}(V_\varepsilon(\bar{t}))$ of the neighborhood $V_\varepsilon(\bar{t}) = (\bar{t} + \varepsilon\mathbf{B}) \cap T$ is a semi-neighborhood of $\bar{x}$, where

$$B^{-1}\left(V_\varepsilon\left(\bar{t}\right)\right) = \bigcup_{t \in V_\varepsilon(\bar{t})} \left\{x \in X \mid h_i\left(x,t\right) \le 0, \forall i \in I\right\}.$$

By the continuity and linearity of the derivative $(D_x h(\bar{x},\bar{t})\,, D_t h(\bar{x},\bar{t}))$, there are positive radii ρ_x and ρ_t such that for all $i \in I_0$

$$D_x h_i\left(\bar{x},\bar{t}\right)\xi + D_t h_i\left(\bar{x},\bar{t}\right)\eta < 0$$

for each $\lambda > 0$ and each $(\xi,\eta) \in \mathcal{K}_\lambda(\xi_0,\rho_x) \times \mathcal{K}_\lambda(\eta_0,\rho_t)$. The Taylor approximation of h_i at $(\bar{x},\bar{t})$ for $i \in I_0$

$$h_i(x,t) = h_i(\bar{x},\bar{t}) + D_x h_i(\bar{x},\bar{t})\,(x-\bar{x}) + D_t h_i(\bar{x},\bar{t})\,(t-\bar{t}) + o(x-\bar{x},t-\bar{t})$$

yields radii $\varepsilon > \gamma_x, \gamma_t > 0$ such that for all $(\xi,\eta) \in \mathcal{K}_{\gamma_x}(\xi_0,\rho_x) \times \mathcal{K}_{\gamma_t}(\eta_0,\rho_t)$

$$h_i(\bar{x}+\xi,\bar{t}+\eta) < 0$$

holds and the continuity of h_i, for $i \in I \setminus I_0$, yields radii $\varepsilon > \beta_x, \beta_t > 0$ such that for all $(\xi,\eta) \in \beta_x\mathbf{B} \times \beta_t\mathbf{B}$ again the inequality

$$h_i(\bar{x}+\xi,\bar{t}+\eta) < 0$$

is satisfied. It then follows that

$$\begin{aligned} B^{-1}(V_\varepsilon(\bar{t})) \supset B^{-1}((\bar{t} + \min(\gamma_t,\beta_t)\mathbf{B}) \cap T) &\supset (\bar{x} + [\mathcal{K}_{\gamma_x}(\xi_0,\rho_x) \cap \beta_x\mathbf{B}]) \cap X \\ &= (\bar{x} + \mathcal{K}_{\min(\gamma_x,\beta_x)}(\xi_0,\rho_x)) \cap X. \end{aligned}$$

Hence, by Lemma 6.27, $\bar{x}$ is a robust point of $(\bar{x} + \mathcal{K}_{\min(\gamma_x,\beta_x)}(\xi_0,\rho_x)) \cap X$ which implies that $B^{-1}(V_\varepsilon(\bar{t}))$ is a semi-neighborhood of $\bar{x}$. ∎

Remark 6.30. If the (MFCQ) is satisfied separately with respect to x at $\overline{x} \in X$, for all $\overline{t} \in B(\overline{x})$; i.e., there is $\xi \in \mathbb{R}^n$ such that

$$D_x h_i(\overline{x}, \overline{t})\xi < 0, \forall i \in I_0, \forall \overline{t} \in B(\overline{x}),$$

then this implies again only the robustness of $B(\cdot)$ at $\overline{x}$. However, if the (MFCQ) is satisfied separately with respect to t for all $t \in B(\bar{x})$, then, as it is well-known, $B(\cdot)$ turns out to be lower-semi-continuous at $\bar{x}$, since $\mathcal{K}_{\gamma_x}(\xi_0, \rho_x)$ can be replaced by the full neighborhood $\bar{x} + \gamma_x \mathbf{B}$.

Next we try to give a similar characterization for set-value maps defined with an infinite system. Thus, in the following, we suppose that X, Y, T are nonempty subsets of normed spaces, $B : X \rightrightarrows T$ is a set-valued map and the set-valued map $M : X \rightrightarrows Y$ is defined by

$$M(x) = \{y \in Y \mid G(y, x, B(x)) \leq 0\},$$

where $G(y, x, Q) \leq 0$ means that $G(y, x, t) \leq 0$ for all $t \in Q$ for a subset Q of T. We use further the active index set

$$E(x, y) = \{t \in T \mid G(y, x, t) = 0\} \subset T.$$

Definition 6.31. We say the (EMFCQ) is satisfied for the system

$$G(y, x, B(x)) \leq 0$$

with respect to $Y \times X$ at $(\bar{y}, \bar{x})$ iff

1. there is some $\tau > 0$ such that $G(\cdot, \cdot, \cdot)$ is F-differentiable at $(\bar{y}, \bar{x}, t)$ with respect to (y, x), the remainder property is satisfied uniformly in t on a compact subsets of T and $G(\bar{y}, \bar{x}, \cdot), D_y G(\bar{y}, \bar{x}, \cdot), D_x G(\bar{y}, \bar{x}, \cdot)$ are continuous at all $t \in (E(\bar{y}, \bar{x}) + \tau \mathbf{B}) \cap (B(\bar{x}) + \tau \mathbf{B}) \cap T$;
2. there are directions $\eta_0 \in \mathcal{T}(Y, \bar{y}), \xi_0 \in \mathcal{T}(X, \bar{x})$ such that for all $t \in E(\bar{y}, \bar{x}) \cap B(\bar{x})$

$$D_y G(\bar{y}, \bar{x}, t)\, \eta_0 + D_x G(\bar{y}, \bar{x}, t)\, \xi_0 < 0.$$

Theorem 6.32. *Suppose the robust subset X, the nonempty subset Y and the nonempty, compact subset T are supplied with topologies induced from normed spaces including X, Y and T respectively. Suppose $G : Y \times X \times T \to \mathbb{R}$ is continuous and $B : X \rightrightarrows T$ is upper semi-continuous on X and the defining system*

$$G(y, x, B(x)) \leq 0, x \in X, y \in Y$$

of the set-valued map $M : X \rightrightarrows Y$ satisfies the (EMFCQ) with respect to $Y \times X$ at $(y, \bar{x})$ for all $y \in M(\bar{x})$. Then $M(\cdot)$ is lower robust at $\bar{x}$.

Proof. Let first $B(\bar{x}) \neq \emptyset$. We show that for an arbitrary $\varepsilon > 0$ the pre-image $M^{-1}(V_\varepsilon(\bar{y}))$ of the neighborhood $V_\varepsilon(\bar{y}) = (\bar{y} + \varepsilon\mathbf{B}) \cap Y$, for an arbitrary $\bar{y} \in M(\bar{x})$, is a semi-neighborhood of $\bar{x}$. We have

$$M^{-1}(V_\varepsilon(\bar{y})) = \bigcup_{y \in V_\varepsilon(\bar{y})} \{x \in X \mid G(y, x, B(x)) \leq 0\}.$$

The (EMFCQ) implies the existence of directions $\eta_0 \in \mathcal{T}(Y, \bar{y}), \xi_0 \in \mathcal{T}(X, \bar{x})$ such that for all $t \in E(\bar{y}, \bar{x}) \cap B(\bar{x})$

$$D_y G(\bar{y}, \bar{x}, t)\eta_0 + D_x G(\bar{y}, \bar{x}, t)\xi_0 < 0$$

holds.

By the continuity and linearity of the derivative $(D_y G(\bar{y}, \bar{x}, t), D_x G(\bar{y}, \bar{x}, t))$, the continuity of $(D_y G(\bar{y}, \bar{x}, \cdot), D_x G(\bar{y}, \bar{x}, \cdot))$ and the compactness of $E(\bar{x}, \bar{y}) \cap B(\bar{x})$ there are positive radii ρ_y, ρ_x, $\delta < \tau$ such that

$$D_y G(\bar{y}, \bar{x}, t)\eta + D_x G(\bar{y}, \bar{x}, t)\xi < 0$$

for each $\lambda > 0$, each $(\eta, \xi) \in \mathcal{K}_\lambda(\xi_0, \rho_y) \times \mathcal{K}_\lambda(\eta_0, \rho_x)$ and each $t \in ((E(\bar{y}, \bar{x}) + \delta\mathbf{B}) \cap (B(\bar{x}) + \delta\mathbf{B})) \cap T$. Because of the upper semi-continuity of $B(\cdot)$ and the compactness of T and $B(x)$ there is a $\sigma(\delta) > 0$ such that

$$B(x) \subset (B(\bar{x}) + \delta\mathbf{B}) \cap T \tag{3}$$

for all $x \in X \cap (\bar{x} + \sigma\mathbf{B})$ (which is a relative open set in X). The Taylor approximation of G at $(\bar{y}, \bar{x}, t)$

$$G(y, x, t) = G(\bar{y}, \bar{x}, t) + D_y G(\bar{y}, \bar{x}, t)(y - \bar{y})$$

$$+ D_x G(\bar{y}, \bar{x}, t)(x - \bar{x}) + o(y - \bar{y}, x - \bar{x}, t)$$

and the continuity properties with respect to t and the uniform remainder property in t yields radii $\varepsilon > \gamma_y, \gamma_x > 0$ such that for all $(\eta, \xi) \in \mathcal{K}_{\gamma_y}(\eta_0, \rho_y) \times \mathcal{K}_{\gamma_x}(\xi_0, \rho_x)$ and for all

$$t \in ((E(\bar{y}, \bar{x}) + \delta\mathbf{B}) \cap (B(\bar{x}) + \delta\mathbf{B})) \cap T$$

the following inequality holds:

$$G(\bar{y} + \eta, \bar{x} + \xi, t) < 0.$$

The set-valued map $(y, x) \longmapsto E(y, x)$ is closed because of the continuity of G on $Y \times X \times T$ and the compactness of T implies the upper semi-continuity (cf. Hogan [11]). Hence, there is $\varepsilon > \mu(\delta) > 0$ such that for all $(y, x) \in ((\bar{y} + \mu\mathbf{B}) \cap Y) \times ((\bar{x} + \mu\mathbf{B}) \cap X)$

$$E(y, x) \subset E(\bar{y}, \bar{x}) + \delta\mathbf{B}.$$

Thus, using (3), we have

$$[(E(\bar{y}, \bar{x}) + \delta\mathbf{B}) \cap (B(\bar{x}) + \delta\mathbf{B})] \cap T \supset E(y, x) \cap B(x)$$

for all $(y, x) \in ((\bar{y} + \mu\mathbf{B}) \cap Y) \times ((\bar{x} + \mu\mathbf{B}) \cap X)$. So far we have proved that the inverse map of the active constraints contains the intersection of the semi-neighborhood $\mathcal{K}_{\gamma_x}(\xi_0, \rho_x)$ and the neighborhood $[(\bar{x} + \min\{\sigma, \mu\}\mathbf{B}) \cap X]$ of $\bar{x}$.

The complement C of $((E(\bar{y}, \bar{x}) + \delta\mathbf{B}) \cap (B(\bar{x}) + \delta\mathbf{B})) \cap T$ with respect to $\mathrm{cl}(B(\bar{x}) + \delta\mathbf{B}) \cap T$ is a compact set in T (note that $\mathrm{cl}(B(\bar{x}) + \delta\mathbf{B}) \cap T \supset B(x)$). Here is $G(\bar{y}, \bar{x}, t) < 0$; i.e., G is non-active at $(\bar{x}, \bar{y}, t)$ for an arbitrary $t \in C$. Hence, for all $t \in C$ and some $\beta_y > 0, \sigma > \beta_x > 0$ we have, by the continuity of G, that

$$G(y, x, t) < 0$$

for all $(y, x) \in ((\bar{y} + \beta_y\mathbf{B}) \cap X) \times ((\bar{x} + \beta_x\mathbf{B}) \cap Y)$. It follows

$$\begin{aligned} M^{-1}(V_\varepsilon(\bar{y})) &\supset M^{-1}((\bar{y} + \min(\gamma_y, \beta_y, \mu)\mathbf{B}) \cap Y) \\ &\supset (\bar{x} + [\mathcal{K}_{\gamma_x}(\xi_0, \rho_x) \cap \beta_x\mathbf{B} \cap \mu\mathbf{B}]) \cap X \\ &= (\bar{x} + \mathcal{K}_{\min(\gamma_x, \beta_x, \mu)}(\xi_0, \rho_x)) \cap X. \end{aligned}$$

Hence, by Lemma 6.27, $\bar{x}$ is a robust point of $(\bar{x} + \mathcal{K}_{\min(\gamma_x, \beta_x, \mu)}(\xi_0, \rho_x)) \cap X$ which implies that $M^{-1}(V_\varepsilon(\bar{y}))$ is a semi-neighborhood of $\bar{x}$.

Furthermore, if $B(\bar{x}) = \emptyset$, then there is a neighborhood U of $\bar{x}$ such that $B(x) = \emptyset$ for all $x \in U$. It follows immediately that $M(\bar{x}) \equiv Y$ on U. This even implies the continuity of $M(\cdot)$ at $\bar{x}$. ∎

Remark 6.33. In the same manner as for the finite case, we get again lower robustness if we have the (EMFCQ) being satisfied separately with respect to $\overline{x} \in X$, for all $y \in M(\bar{x})$; and the lower semi-continuity if we have the (EMFCQ) being satisfied separately with respect to y for all $y \in M(\bar{x})$. In both proofs, the compactness of $B(x)$ plays an important role.

Remark 6.34. Both Theorems 6.29 and 6.32 can be extended to piecewise upper semi-continuous set-valued maps $B(\cdot)$. Taking that $\{X_k\}_{k \in J}$ is a robust partition of the robust set X, we demand the assumptions of the theorems to hold true for each component X_k with respect to its relative topology. Naturally, we have to take the tangential cones with respect to X_k and not with respect to X. For instance, in Theorem 6.32, the piecewise upper semi-continuity of $B(\cdot)$ with the validity of the regularity condition (EMFCQ) on each X_k imply that $M(\cdot)$ is piecewise lower robust. Hence, $M(\cdot)$ will be lower robust (see Theorem 4.20). Note that in this case, the regularity separately in t (Theorem 6.29) or in y (Theorem 6.32) does not yield lower semi-continuity, but at least lower robustness (see Theorem 4.20).

Using a right hand-side perturbation of the defining system of $M(\cdot)$, Klatte and Henrion [15] have shown the equivalence of (EMFCQ) and metric-regularity, which in turn implies the r-regularity. This equivalence requires the lower semi-continuity of $B(\cdot)$. Under such instances, $M(\cdot)$ will be lower semi-continuous. However, for us the upper semi-continuity of $B(\cdot)$ along with a weaker form (EMFCQ) (Definition 6.31) is enough to derive the lower robustness of $M(\cdot)$. Indeed, it would have been very interesting to find out the relation between r-regularity (of Definition 6.19) and the (EMFCQ) (Definition 6.31). But, this has been left out for a future research activity.

Furthermore, for set-valued maps with a given structure we have explicitly considered inequality constraints. However, when equality constraints are assumed to be present one may need certain stronger regularity conditions to guarantee the corresponding robustness properties.

In many instances, the upper semi-continuity (upper robustness) of a set-valued map of the form

$$B(x) = \{t \in T \mid h_i(x,t) \leq 0\}$$

follows easily, if we demand that T is a compact set and $h_i, i = 1, \ldots, p$, are continuous. In this respect, the uniform lower robustness (Definition 6.9) and the r-regularity (Definition 6.10) assumptions of Proposition 6.11 will become superfluous. In any case, one may need to note that the validity of continuity properties on partitioning sets imply robustness on the whole.

7. Conclusion

Before we wind up we need to note that:

Given a function $f : X \to \mathbb{R}$ and a set valued map $M : X \rightrightarrows Y$ with X being a second countable and Y a separable spaces, then *upper robust functions have a dense set of upper semi-continuity*; *a lower robust set-valued map has a dense set of lower semi-continuity*, etc. These yield the most vital property for numerical computations; namely, *approximatability.*

With respect to both functions and set-valued maps we have shown that *continuity (robustness) properties on partitioning sets imply robustness on the whole.*

Marginal functions are useful, for instance, in the stability analysis of optimization problems, in the study of multilevel optimization problems, in the characterization of the feasible set of a generalized semi-infinite optimization problems, etc. In particular, they could be used to define penalty functions for certain optimization problems (cf. [1] and [2]). However, under general assumptions, they are usually discontinuous. Hence, robust analysis of marginal functions is a new and a general approach to these functions.

We considered set-valued maps with given structures defined through only inequality systems; i.e., excluding equality constraints. The presence of equality constraints is believed to create theoretical difficulties, since robust sets are required to have non-empty interiors.

Klatte and Henrion [15] have shown that metric regularity is equivalent to a strong form of (MFCQ), from which follows that $M(\cdot)$ is lower semi-continuous. However, we used here the upper semi-continuity of $B(\cdot)$ along with the weaker (MFCQ) (Definition 6.28) to guarantee the lower robustness of $M(\cdot)$. The relationship between the (MFCQ) (Definition 6.28) and the r-regularity (Definition 6.19) still remains to be found and it.

8. Summary of definitions and results

Based on the suggestion of our Referee we have compiled a review of the main issues and definitions into a tabular form.

X, Y, T topological spaces

Set:	$A \subset X$		
	A open:	$A = \operatorname{int} A$	
	A robust:	$\operatorname{cl} A = \operatorname{cl} \operatorname{int} A$	Def. 2.1
	U neighborhood of $\bar{x}$	$\bar{x} \in \operatorname{int} U$	
	S semi-neighborhood (SNH) of $\bar{x}$	$\bar{x} \in S \cap \operatorname{cl} \operatorname{int} S$	
Function:	$f : X \to Y$		
	continuous:	$f^{-1}(\text{open}) = \text{open}$	
	robust:	$f^{-1}(\text{open}) = \text{robust}$	Definition 3.1
Functional:	$f : X \to \mathbb{R}$		
	lower semi continuous **(l.s.c.)**:	$[f > c] = \text{open}$	p. 563 in [6]
	lower robust (**l.r.**):	$[f > c] = \text{robust}$	Def. 2.14
	upper semi continuous (**u.s.c.**):	$[f < c] = \text{open}$	p. 563 in [6]
	upper robustness **(u.r.)**:	$[f < c] = \text{robust}$	Def. 2.10
SVM:	$M : X \rightrightarrows Y$		
(Set-Valued Map)	lower semi continuous (**l.s.c.**):	$M^{-1}(\text{open}) = \text{open}$	Def. 1.4.1/Prop. 1.4.4 in [4]
	lower robust (**l.r.**):	$M^{-1}(\text{open}) = \text{robust}$	Def. 4.1
	upper semi continuous (**u.s.c.**):	$M^{+}(\text{open}) = \text{open}$	Def. 1.4.2/Prop. 1.4.4 in [4]
	upper robust (**u.r.**):	$M^{+}(\text{open}) = \text{robust}$	Def. 4.5
	ε-**l.s.c.** at $\bar{x}$ if for all $x \in U_\varepsilon(\bar{x})$	$M(\bar{x}) \subset M(x) + \varepsilon \mathrm{B}$	Def. 6 in [3]
	ε-**l.r.** at $\bar{x}$ if for all $x \in SNH_\varepsilon(\bar{x})$	$M(\bar{x}) \subset M(x) + \varepsilon \mathrm{B}$	Def. 4.11
	ε-**u.s.c.** at $\bar{x}$ if for all $x \in U_\varepsilon(\bar{x})$	$M(x) \subset M(\bar{x}) + \varepsilon \mathrm{B}$	Def. 5 in [3]
	ε-**u.r.** at $\bar{x}$ if for all $x \in SNH_\varepsilon(\bar{x})$	$M(x) \subset M(\bar{x}) + \varepsilon \mathrm{B}$	Def. 4.8

SVM:	$M : X \rightrightarrows Y$			$M(x)$		$M : X \rightrightarrows Y$			$M(x)$	
	ε-l.s.c.	$\Rightarrow$	l.s.c.		p. 45 in [3]	ε-u.s.c.	$\Leftarrow$	u.s.c.		p. 45 in [3]
	ε-l.s.c.	$\Leftarrow$	l.s.c.	compact	p. 45 in [3]	ε-u.s.c.	$\Rightarrow$	u.s.c.	compact	p. 45 in [3]
	ε-l.r.	$\Rightarrow$	l.r.		Prop. 4.12	ε-u.r.	$\Leftarrow$	u.r.		Prop. 4.9
	ε-l.r.	$\not\Leftarrow$	l.r.	compact	Ex. 4.13	ε-u.r.	$\Rightarrow$	u.r.	compact	Prop. 4.10

$\varphi(x) := \inf_{y \in M(x)} \psi(x, y)$

Marginal functional $\varphi : X \to \mathbb{R}$					
$M(x)$	M	ψ		φ	
	l.s.c.	u.s.c.	$\Longrightarrow$	u.s.c.	Theorem 4 in [3]
	l.r.	u.s.c.	$\Longrightarrow$	u.r.	Theorem 5.1
	l.s.c.	u.r.	$\Longrightarrow$	u.r.	left to the reader (see Remark 5.9)
compact	u.s.c.	l.s.c.	$\Longrightarrow$	l.s.c	Theorem 5 in [3]
compact	u.r.	l.s.c.	$\Longrightarrow$	l.r.	Corollary 5.8
compact	u.s.c.	l.r.	$\Longrightarrow$	l.r.	left to the reader

$B(x) := \{t \in T \mid h_i(x,t) \leq 0\ , i = 1, 2, \dots, r\}$

SVM B	**by finite inequalities**	$B : X \rightrightarrows T$	$h(x,\cdot)$	$h(\cdot,t)$	$h(\cdot,\cdot)$	
	lower semi continuity:		Slater or sqc	u.s.c.		Theorem 12 in [11]
	lower robustness:		Slater or sqc	one u.r./ others u.s.c.		Cor. 6.5, Prop. 6.6
	upper semi continuity:	T compact			u.s.c.	Theorem 3.2.1 in [5]
	upper robustness:		uniform l.r., r-regular			Prop. 6.11
		$B : X \rightrightarrows T$		**MFCQ** for $h \leq 0$	$h(\cdot,\cdot)$	
	lower semi continuity:			in t	cont.	Remark 6.30
	lower robustness:			in (x,t) or in x	cont.	Theorem 6.29

$M(x) := \{y \in Y \mid G(y,x,t) \geq 0\ \forall t \in B(x)\}$

SVM M	**by infinite inequalities**	$M : X \rightrightarrows Y$	G	$G(\cdot,x,t)$	$B(\cdot)$	
	lower robustness:	$B(x)$ compact	u.s.c.	Slater or γ-sqc	u.r.	Prop. 6.18
		$M : X \rightrightarrows Y$	G	**EMFCQ** for $G \geq 0$	$B(\cdot)$	
	lower semi continuity:	T compact	cont.	in y	u.s.c.	Prop. 1.2.8 in [1]
	lower robustness:	T compact	cont.	in (y,x) or in x	u.s.c.	Theorem 6.32

Robust partition: $X = \bigcup_{k=1}^{m} X_k$, X_k robust and pairwise disjoint

SVM:	$M : X \rightrightarrows Y$ or **Functional** $f : X \to \mathbb{R}$			
	l.s.c. (u.s.c.) on X_k in relative topology	$\not\Rightarrow$	l.s.c. (u.s.c.) on X	
	l.r. (u.r.) on X_k in relative topology	$\Rightarrow$	l.r. (u.r.) on X	Theorem 4.20
	piecewise l.s.c. (u.s.c.) in relative topology	$\Rightarrow$	l.r. (u.r.) on X	Theorem 4.14 or Theorem 4.15

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References

[1] A. Geletu, *A Coarse Solution of Generalized Semi-infinite Optimization via Robust Analysis of Marginal Functions and Global Optimization*, Phd. Dissertation, Techncal University of Ilmanu, Institute of Mathematics, Department of Operations Research and Stochastics, December 17, 2004.

[2] A. Geletu and A. Hoffmann, *A conceptual method for solving generalized semi-infinite programming problems via global optimization by exact discontinuous penalization*, European J. of OR, V. **157** (2004), 3–15.

[3] J.-P. Aubin and A. Cellina, Differential Inclusions, Springer Verlag, Berlin 1984.

[4] J.-P. Aubin and H. Frankowska, Set-Valued Analysis, Birkhäuser, Basel 1990.

[5] B. Bank, J. Guddat, D. Klatte, B. Kummer and K. Tammer, Non-Linear Parametric Optimization, Akademie-Verlag, Berlin 1982.

[6] M.S. Bazaraa, H.D. Sherali and C.M. Shetty, Nonlinear Programming, 2nd. ed., John Wiley & Sons, Inc. 1993.

[7] S. Chew and Q. Zheng, Integral Global Optimization, Springer-Verlag, Berlin 1988.

[8] J. Hichert, *Methoden zur Bestimmung des wesentlichen Supremums mit Anwendung in der globalen Optimierung*, Phd. Dissertation, TU-Ilmenau, 1999, Berichte aus der Mathematik, Shaker-Verlag, Aachen 2001.

[9] J. Hichert, A. Hoffmann and H.X. Phú, *Convergence speed of an integral method for computing essential supremum*, in Developments in Global Optimization, I.M. Bomze, T. Csendes, R. Horst, P.M. Pardalos, Kluwer Academic Publishers, Dodrecht, Boston, London 1997, 153–170.

[10] J. Hichert, A. Hoffmann, H.X. Phú and R. Reinhardt, *A primal-dual integral method in global optimzation*, Discuss. Math. Differential Inclusions, Control and Optimization **20** (2) (2000), 257–278.

[11] W.W. Hogan, *Point-to-set maps in mathematical programming*, SIAM Review **15** (3) (1973) 591–603.

[12] S. Hu and N.S. Papageorgiou, Handbook of Multivalued Analysis, Volume I, Kluwer Academic Publishers 1997.

[13] H.T. Jongen, J.-J. Rückmann and O. Stein, *Generalized semi-infinite optimization: a first order optimality condition and examples*, Math. Prog. **83** (1998), 145–158.

[14] L.W. Kantorowitsch and G.P. Akilow, Funktionalanalysis in normierten Räumen, Akademie-Verlag, Berlin 1978.

[15] D. Klatte and R. Henrion, *Regularity and stability in non-linear semi-infinite optimization*, in Semi-infinite Programming, R. Reemtsen and J.-J. Rückmann (eds.), pp. 69–102, Kluwer Academic Pres, 1998.

[16] M.M. Kostreva and Q. Zheng, *Integral global optimization method for solution of nonlinear complementarity problems*, J. Global Opt. **5** (1994), 181–193.

[17] H.X. Phú and A. Hoffmann, *Essential supremum and supremum of summable functions*, Numerical Functional Analysis and Optimization **17** (1 & 2) (1996), 167–180.

[18] S. Shi, Q. Zheng and D. Zhuang, *On existence of robust minimizers*, in The State of the Art in Global Optimization, C.A. Fouldas and P.M. Pardalos (eds.), pp. 47–56, Kluwer Academic Publishers, 1996.

[19] S. Shi, Q. Zheng and D. Zhuang, *Discontinuous robust mappings are approximatable*, American Math. Soc. Trans. **347** (12) (1995), 4943–4957.

[20] S. Shi, Q. Zheng and D. Zhuang, *Set valued robust mappings and approximatable mappings*, J. Math. Anal. Appl. **183** (1994), 706–726.

[21] O. Stein, *On level sets of marginal functions, Optimization*, **48** (2000) 43–67.

[22] O. Stein, Bi-level Strategies in Semi-infinite Programming, Kluwer Academic Publishers 2003.

[23] J.-B.H. Urruty and C. Lemaréchal, Convex Analysis and Minimization Algorithms I, Springer Verlag 1993.

[24] G.-W. Weber, Generalized Semi-infinite Optimization and Related Topics, Postdoctoral Thesis, Dept. of Mathematics, Darmstadt University of Technology 1999.

[25] K. Yosida, Functional Analysis, 6th edition, Springer-Verlag, Berlin-Heidelberg-New York 1980.

[26] Q. Zheng, Integral Global Optimization of Robust Discontinuous Functions, Ph. D. Dissertation, Clemson University, December 1992.

[27] Q. Zheng and L. Zhang, *Global minimization of constrained problems with discontinuous penality functions*, Compt. Math. Appl. **37** (1999), 41–58.

[28] Q. Zheng and D. Zhuang, *Integral global minimization: algorithms, implementations and numerical tests*, J. Global Optim. **7** (1995), 421–454.

[29] Q. Zheng and D. Zhuang, *The approximation of fixed points of robust mappings*, in Advances in Optimization and Approximation, D.-Z. Du & J. Sun (eds.), pp. 376–389, Kluwer Acadmic publishers 1994.

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