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ROBUSTNESS OF ESTIMATION OF FIRST-ORDER AUTOREGRESSIVE MODEL UNDER CONTAMINATED UNIFORM WHITE NOISE

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Abstract

The first-order autoregressive model with uniform innovations is considered. In this paper, we study the bias-robustness and MSE-robustness of modified maximum likelihood estimator of parameter of the model against departures from distribution of white noise. We used the generalized Beta distribution to describe these departures.

Keywords: autoregressive model, bias, MSE, robustness, generalized Beta distribution.

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1. Introduction

Consider the following autoregressive model

(1) $$X_t = \rho X_{t-1} + \varepsilon_t,\ t = \dots, -1, 0+1, \dots, \text{with } 0 \le \rho < 1,$$

where the ε_t's are *i.i.d* and distributed according to uniform distribution $U(0, \theta)$.

Bell and Smith (1986) studied the estimation and testing problem on the parameter ρ for model (1) with ε_t are i.i.d and non-negative. The study is established in three parametric cases: Gaussian, exponential and uniform as well as for the nonparametric case where it only assumed that ε's have a positive continuous distribution. In these different models, various types of point estimates and procedures of test for ρ are introduced by authors.

In the case of model (1) defined above, Nouali and Fellag (2002) obtained an approximation of bias of maximum likelihood estimator (MLE) for parameter ρ. A formula of approximate bias of MLE is given also when a single outlier occurs at specified time with a known amplitude. In the same model, Nouali and Fellag (2005) proposed a new testing procedures which perform a test on the parameter ρ in presence of single innovation outlier comparing to those proposed by Bell and Smith (1986).

Now, we assume X_0 distributed as $U(0, \theta/(1-\rho))$ and observe the segment of observations

$$X_1, X_2, \ldots, X_n, \qquad n \quad \text{fixed} \tag{2}$$

from a model (1). The maximum likelihood estimator (MLE) for ρ is (Bell and Smith, 1986) $\hat{\rho} = \min_{2 \le t \le n}(X_t/X_{t-1})$. Then, we can write

$$\hat{\rho} = \rho + \min_{2 \le t \le n} \left(\frac{\varepsilon_t}{X_{t-1}} \right). \tag{3}$$

Since the process is mean stationary with mean $m_\theta = \frac{\theta}{2(1-\rho)}$, we can use the method proposed by Anděl (1988) in exponential model which consist to substitute m_θ for X_{t-1} in (3). Then, the estimator becomes $\tilde{\rho} = \rho + \frac{1}{m_\theta} \min_{2 \le t \le n}(\varepsilon_t)$.

Let $b(\rho)$ and $MSE(\rho)$ be the bias and the mean square error of estimator $\tilde{\rho}$ in model (1) respectively. Hence,

$$b(\rho) = \frac{1}{m_\theta} E\left(\min_{2\leq t\leq n}(\varepsilon_t)\right) = \frac{2}{n}(1-\rho), \text{ (Nouali and Fellag, 2002)}$$

$$MSE(\rho) = \frac{1}{m_\theta^2} E\left(\min_{2\leq t\leq n}(\varepsilon_t)\right)^2 = \frac{8}{n(n+1)}(1-\rho)^2.$$

In fact, the pdf function $h(.)$ of random variable $Z = \min_{2\leq t\leq n}(\varepsilon_t)$ is obtained after simple computation

$$h(z) = \begin{cases} \dfrac{n-1}{\theta}\left(1-\dfrac{z}{\theta}\right)^{n-2} & \text{if} \quad 0 \leq z \leq \theta. \\ 0 & \text{else.} \end{cases}$$

with $E(Z) = \dfrac{\theta}{n}$ and $E(Z^2) = \dfrac{2\theta^2}{n(n+1)}$.

Our aim is to discuss a bias-robustness and MSE-robustness of estimator $\tilde{\rho}$ when the distribution of white noise is contaminated by generalized Beta distribution. Note that the bias-robustness of estimator $\tilde{\rho}$ is considered by Fellag and Ibazizen (2001) in the case of contaminated exponential white noise. The authors used a class of exponential power distribution to modeling the departures from original model.

2. Bias and MSE of $\tilde{\rho}$ under generalized beta white noise

The statistical model with uniform white noise is denoted by

$$M_{1,1} = (\mathbb{R}^+, \mathcal{B}^+, U(0,\theta), \theta > 0),$$

where $\mathbb{R}^+$ is the real-half, $\mathcal{B}^+$ is the family of Borel subsets of $\mathbb{R}^+$.

We can consider the following model as extension of original model $M_{1,1}$:

$$M_{\alpha,\beta} = (\mathbb{R}^+, \mathcal{B}^+, \mathrm{GBeta}_\theta(\alpha,\beta), \theta > 0, \alpha, \beta \in \mathbb{R}^+_\star),$$

where $GBeta_\theta(\alpha,\beta)$ is a general Beta distribution on three parameters defined on an interval $(0,\theta)$ with density

$$f_{\alpha,\beta}(x,\theta) = \frac{1}{\theta^{\alpha+\beta-1}B(\alpha,\beta)} x^{\alpha-1}(\theta - x)^{\beta-1}, 0 \le x \le \theta.$$

Notice that the uniform distribution over $(0,\theta)$ is a particular generalized Beta distribution of parameters θ, α and β with $\alpha = \beta = 1$.

Now, we suppose that the model $M_{1,1}$ is violated in such way that the innovations $\varepsilon_1, \ldots, \varepsilon_n$ are distributed according to $GBeta_\theta(\alpha,\beta)$ rather then uniform distribution over $(0,\theta)$ and we assume that θ is equal to 1 without loss of generality.

We have

$$E(\varepsilon_t) = \frac{\alpha}{\alpha+\beta}, \forall t = 1, \ldots, n.$$

We assume that X_0 is distributed according to $GBeta_{\tilde{\theta}}(1,1)$ with

$$\tilde{\theta} = \frac{2\alpha}{(\alpha+\beta)(1-\rho)},$$

then a process (X_t) is mean stationary with mean

$$m_{\alpha,\beta} = \frac{\alpha}{(\alpha+\beta)(1-\rho)}$$

and we can write

$$\tilde{\rho} = \rho + \frac{1}{m_{\alpha,\beta}} \min_{2\le t\le n}(\varepsilon_t) = \rho + \frac{(\alpha+\beta)(1-\rho)}{\alpha} \min_{2\le t\le n}(\varepsilon_t).$$

Let $H(z)$ and $h(z)$ be the distribution and the pdf function respectively of random variable $Z = \min_{2\le t\le n}(\varepsilon_t)$. After computation, we obtain

$$h(z) = (n-1).f_{\alpha,\beta}(z).[1 - F_{\alpha,\beta}(z)]^{n-2}.$$

The bias and MSE of $\tilde{\rho}$ in Model $M_{\alpha,\beta}$ are given by numerical formulas (4) and (5) respectively

$$b_{\alpha,\beta}(\rho) = \frac{(n-1)(\alpha+\beta)(1-\rho)}{\alpha B(\alpha,\beta)} . \int_0^1 z^{\alpha}.\,(1-z)^{\beta-1}.\,[1 - B_z(\alpha,\beta)]^{n-2}.\,dz, \forall n. \tag{4}$$

$$MSE_{\alpha,\beta}(\rho) = \frac{(n-1)(\alpha+\beta)^2(1-\rho)^2}{\alpha^2 B(\alpha,\beta)} . \int_0^1 z^{\alpha+2}.\,(1-z)^{\beta-1}.\,[1 - B_z(\alpha,\beta)]^{n-2}.\,dz, \forall n. \tag{5}$$

where

$$B_z(\alpha,\beta) = \int_0^z x^{\alpha-1}.\,(1-x)^{\beta-1}dx$$

denotes the incomplete Beta function.

In this paper, we study a behavior of bias and MSE of estimator $\tilde{\rho}$ in the following two particular extentions of orginal model:

$$M_\alpha = M_{\alpha,1} \text{ and } M_\beta = M_{1,\beta}.$$

We used a measure of robustness in sense of stability proposed by Zieliński (1977). In our case, if a property of estimator is a bias, this measure of robustness takes the following form

$$rb_{[\lambda_1,\lambda_2[}(\rho) = \sup_{\lambda_1\le\lambda<\lambda_2} b(\rho) - \inf_{\lambda_1\le\lambda<\lambda_2} b(\rho), \text{ in model } M_\lambda$$

with $\lambda = \alpha$ or β.

This function is called the "bias-robustness function". It represents a maximal oscillation of the bias of estimator $\tilde{\rho}$ in "supermodel" M_λ.

- We say that the estimator is "absolutely robust" according to bias criteria in model M_λ if

$$rb_\lambda(\rho) = 0, \forall\lambda > 0.$$

- When $rb_{\lambda\in I_1}(\rho) < rb_{\lambda\in I_2}(\rho)$ then the bias of the estimator is more stable for $\lambda \in I_1$ than for $\lambda \in I_2$, where I_1 and I_2 are two intervals of $\mathbb{R}^+_\star$.

In the same manner as bias-robustness function, we can define MSE-robustness function which denote by $rm(.)$. For more details on this robustness approach, we refer the reader to Zieliński (1977).

2.1. The behavior of bias and mse of $\tilde{\rho}$ in model M_α

In the model M_α, the density $h(z)$ is written

$$h(z) = \alpha(n-1).z^{\alpha-1}.[1-z^\alpha]^{n-2}.$$

The bias and MSE of estimator $\tilde{\rho}$ are given respectively by formula (6) and (7)

$$b_\alpha(\rho) = \frac{(1+\alpha)}{\alpha^2} B(1/\alpha, n)(1-\rho), \forall n. \tag{6}$$

$$MSE_\alpha(\rho) = \frac{(1+\alpha)^2}{\alpha^2}(n-1)B(1+2/\alpha, n-1)(1-\rho)^2, \forall n. \tag{7}$$

Let us adopt the following notations:

b_{min}, M_{min} : the minimal value of bias respectively of MSE.

b_{max}, M_{max} : the maximal value of bias respectively of MSE.

In order to illustrate a behavior of bias and MSE of estimator $\tilde{\rho}$ in model M_α, we give in Tables 1 and 2 the exact values of bias and MSE as function of α and ρ for different smallers values of size (n=5, 10, and 20). The values are computed using a formula (6) and (7).

Table 1. Values of the $b_\alpha(\rho)$ for $n = 5, 10$ and 20.

α	$n=5$			$n=10$			$n=20$		
	$\rho=0$	$\rho=0.4$	$\rho=0.9$	$\rho=0$	$\rho=0.4$	$\rho=0.9$	$\rho=0$	$\rho=0.4$	$\rho=0.9$
0.1	0.0110	0.0066	0.0011	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
0.2	0.0476	0.0286	0.0048	0.0030	0.0018	0.0003	0.0001	0.0001	0.0000
0.4	0.1492	0.0895	0.0149	0.0308	0.0185	0.0031	0.0059	0.0036	0.0006
0.6	0.2475	0.1485	0.0248	0.0819	0.0492	0.0082	0.0265	0.0159	0.0026
0.8	0.3310	0.1986	0.0331	0.1412	0.0847	0.0141	0.0598	0.0359	0.0060
1.0	0.4000	0.2400	0.0400	0.2000	0.1200	0.0200	0.1000	0.0600	0.0100
1.6	0.5453	0.3272	0.0545	0.3495	0.2097	0.0350	0.2253	0.1352	0.0225
2.0	0.6095	0.3657	0.0610	0.4257	0.2554	0.0426	0.2991	0.1795	0.0299
3.0	0.7121	0.4273	0.0712	0.5589	0.3353	0.0559	0.4411	0.2647	0.0441
4.0	0.7722	0.4633	0.0772	0.6432	0.3859	0.0643	0.5383	0.3230	0.0538
6.0	0.8395	0.5037	0.0840	0.7426	0.4456	0.0743	0.6592	0.3955	0.0659
8.0	0.8761	0.5257	0.0876	0.7989	0.4793	0.0799	0.7306	0.4383	0.0731
10	0.8992	0.5395	0.0899	0.8351	0.5010	0.0835	0.7773	0.4664	0.0777
b_{max}	1	0.6	0.1	1	0.6	0.1	1	0.6	0.1

Table 2. Values of the $MSE_{\alpha}(\rho)$ for $n = 5, 10$ and 20.

α	$n=5$			$n=10$			$n=20$		
	$\rho=0$	$\rho=0.4$	$\rho=0.9$	$\rho=0$	$\rho=0.4$	$\rho=0.9$	$\rho=0$	$\rho=0.4$	$\rho=0.9$
0.1	0.0114	0.0041	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2	0.0360	0.0129	0.0004	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
0.4	0.0972	0.0350	0.0010	0.0061	0.0022	0.0001	0.0003	0.0001	0.0000
0.6	0.1590	0.0572	0.0016	0.0214	0.0077	0.0002	0.0025	0.0009	0.0000
0.8	0.2158	0.0777	0.0022	0.0446	0.0161	0.0004	0.0086	0.0031	0.0001
1.0	0.2667	0.0960	0.0027	0.0727	0.0262	0.0007	0.0190	0.0069	0.0002
1.6	0.3884	0.1398	0.0039	0.1657	0.0597	0.0017	0.0702	0.0253	0.0007
3.0	0.5610	0.2020	0.0056	0.3496	0.1259	0.0035	0.2190	0.0788	0.0022
4.0	0.6349	0.2286	0.0063	0.4434	0.1596	0.0044	0.3116	0.1122	0.0031
6.0	0.7269	0.2617	0.0073	0.5705	0.2054	0.0057	0.4503	0.1621	0.0045
8.0	0.7819	0.2815	0.00787	0.6512	0.2344	0.0065	0.5450	0.1962	0.0055
10.0	0.8185	0.2946	0.0082	0.7067	0.2544	0.0071	0.6127	0.2206	0.0061
M_{max}	1	0.36	0.01	1	0.36	0.01	1	0.36	0.01

The results leads the following comments:

Comment 1. One can remak that the bias and MSE of estimator $\tilde{\rho}$ are an increasing functions of α. The variation of these two criterions is very small when $\alpha \in (0,1]$ than when $\alpha \in (1,+\infty[$.

Comment 2. When α is close to zero, the bias and MSE of estimator $\tilde{\rho}$ tend to value $b_{min} = M_{min} = 0$.

Comment 3. When α tend to infinity,

$$\begin{cases} b_\alpha(\rho) & \to b_{max} = (1-\rho). \\ MSE_\alpha(\rho) & \to M_{max} = (1-\rho)^2. \end{cases}$$

After computations, we obtain the following expressions of bias and MSE robustness functions:

$$rb_{(0,1]}(\rho) = b_1(\rho) - b_{min} = \frac{2(1-\rho)}{n}.$$

$$rb_{(1,+\infty[}(\rho) = b_{max} - b_1(\rho) = \frac{(n-2)(1-\rho)}{n}.$$

$$rm_{(0,1]}(\rho) = MSE_1(\rho) - M_{min} = \frac{8(1-\rho)^2}{n(n+1)}.$$

$$rm_{(1,+\infty[}(\rho) = M_{max} - MSE_1(\rho) = \frac{n(n+1)-8}{n(n+1)}.(1-\rho)^2.$$

and we can establish the following inequalities

$$rb_{(0,1]}(\rho) \ < rb_{(1,+\infty[}(\rho), \forall n.$$

$$rm_{(0,1]}(\rho) \ < rm_{(1,+\infty[}(\rho), \forall n \geq 4.$$

The two inequalities show that the estimator $\tilde{\rho}$ is more stable for $\alpha \in (0,1]$ than for $\alpha \in (1,+\infty[$ with respect to bias and MSE.

2.2 The behavior of bias and mse of $\tilde{\rho}$ in model M_β

As in above section, we give the exact formulas of bias and MSE of estimator $\tilde{\rho}$. Also, we derive the expressions of bias-robustness function and MSE-robustness function.

So, in model M_β, the density $h(z)$ is written

$$h(z) = \beta.\,(n-1).\,[1-z]^{\beta(n-1)-1}.$$

The expressions of bias and MSE of estimator $\tilde{\rho}$ take the following forms

(8) $$b_\beta(\rho) = \frac{(1+\beta)}{1+\beta(n-1)}.\,(1-\rho), \forall n \geq 1.$$

(9) $$MSE_\beta(\rho) = \frac{2(1+\beta)^2}{(1+\beta(n-1))(2+\beta(n-1))}.\,(1-\rho)^2, \forall n \geq 1.$$

We present in Tables 3 and 4 the exact values of bias and MSE respectively of estimator $\tilde{\rho}$ as function of β and ρ for different smallers values of size (n=5, 10, and 20). Here, the values are computed using a formula (8) and (9).

Table 3. Values of the $b_\beta(\rho)$ for $n = 5, 10$ and 20.

β	$n=5$			$n=10$			$n=20$		
	$\rho=0$	$\rho=0.4$	$\rho=0.9$	$\rho=0$	$\rho=0.4$	$\rho=0.9$	$\rho=0$	$\rho=0.4$	$\rho=0.9$
b_{max}	1	0.6	0.1	1	0.6	0.1	1	0.6	0.1
0.01	0.9711	0.5826	0.0971	0.9118	0.5559	0.0918	0.8487	0.50096	0.0848
0.1	0.7857	0.4714	0.0786	0.5789	0.3474	0.0579	0.3793	0.2276	0.0379
0.2	0.6667	0.4000	0.0667	0.4286	0.2571	0.0429	0.2500	0.1500	0.0250
0.4	0.5385	0.3231	0.0538	0.3043	0.1826	0.0304	0.1628	0.0977	0.0163
0.6	0.4706	0.2824	0.0471	0.2500	0.1500	0.0250	0.1290	0.0774	0.0129
0.8	0.4286	0.2571	0.0429	0.2195	0.1317	0.0220	0.1111	0.0667	0.0111
1.0	0.4000	0.2400	0.0400	0.2000	0.1200	0.0200	0.1000	0.0600	0.0100
1.6	0.3514	0.2108	0.0351	0.1688	0.1013	0.0169	0.0828	0.0497	0.0083
2.0	0.3333	0.2000	0.0333	0.1579	0.0947	0.0158	0.0769	0.0462	0.0077
3.0	0.3077	0.1846	0.0308	0.1429	0.0857	0.0143	0.0690	0.0414	0.0069
4.0	0.2941	0.1765	0.0294	0.1351	0.0811	0.0135	0.0649	0.0390	0.0065
6.0	0.2800	0.1680	0.0280	0.1273	0.0764	0.0127	0.0609	0.0365	0.0061
8.0	0.2727	0.1636	0.0273	0.1233	0.0740	0.0123	0.0588	0.0353	0.0059
10	0.2683	0.1610	0.0268	0.1209	0.0725	0.0121	0.0576	0.0346	0.0058
b_{min}	0.25	0.15	0.025	0.1111	0.0666	0.0111	0.0526	0.0315	0.0052

Table 4. Values of the $MSE_{\beta}(\rho)$ for $n = 5, 10$ and 20.

β	$n=5$			$n=10$			$n=20$		
	$\rho=0$	$\rho=0.4$	$\rho=0.9$	$\rho=0$	$\rho=0.4$	$\rho=0.9$	$\rho=0$	$\rho=0.4$	$\rho=0.9$
M_{max}	1	0.6	0.1	1	0.6	0.1	1	0.6	0.1
0.01	0.9616	0.342	0.0096	0.8956	0.3224	0.0090	0.7829	0.2818	0.0078
0.1	0.7202	0.2593	0.0072	0.4392	0.1581	0.0044	0.2140	0.0770	0.0021
0.2	0.5714	0.2057	0.0057	0.2707	0.0974	0.0027	0.1034	0.0372	0.0010
0.6	0.3422	0.1232	0.0034	0.1081	0.0389	0.0011	0.0308	0.0111	0.0003
0.4	0.4188	0.1508	0.0042	0.1522	0.0548	0.0015	0.0475	0.0171	0.0005
0.8	0.2967	0.1068	0.0030	0.0859	0.0309	0.0009	0.0233	0.0084	0.0002
1	0.2667	0.0960	0.0027	0.0727	0.0262	0.0007	0.0190	0.0069	0.0002
1.6	0.2175	0.0783	0.0022	0.0535	0.0193	0.0005	0.0133	0.0048	0.0001
2	0.2000	0.0720	0.0020	0.0474	0.0171	0.0005	0.0115	0.0042	0.0001
3	0.1758	0.0633	0.0018	0.0394	0.0142	0.0004	0.0094	0.0034	0.0001
4	0.1634	0.0588	0.0016	0.0356	0.0128	0.0004	0.0083	0.0030	0.0001
6	0.1508	0.0543	0.0015	0.0318	0.0115	0.0003	0.0073	0.0026	0.0001
8	0.1444	0.0520	0.0014	0.0300	0.0108	0.0003	0.0069	0.0025	0.0001
10	0.1405	0.0506	0.0014	0.0289	0.0104	0.0003	0.0066	0.0024	0.0001
M_{min}	0.1250	0.0450	0.0012	0.0246	0.0088	0.0002	0.0055	0.0019	0.0000

Some properties obtained in the model M_β are given in the following comments:

Comment 1. The inverse situation is presented in model M_β to that observed in model M_α. The bias and MSE of estimator $\tilde{\rho}$ are a decreasing functions of β. The two criterions increase strongly when $\beta \in (0,1]$ but decrease slowly when $\beta \in (1,+\infty[$.

Comment 2. When β is close to zero, the bias and MSE of estimator $\tilde{\rho}$ tend to value $b_{max} = M_{max} = (1-\rho)$.

Comment 3. When α tend to infinity,

$$\begin{cases} b_\alpha(\rho) \to b_{min} = \dfrac{(1-\rho)}{n-1}. \\ \\ MSE_\alpha(\rho) \to M_{min} = \dfrac{2(1-\rho)^2}{(n-1)^2}. \end{cases}$$

After computations, we obtain the following expressions of bias and MSE robustness functions:

$$rb_{(0,1]}(\rho) = b_{max} - b_1(\rho) = \tfrac{(n-2)}{n}.(1-\rho).$$

$$rb_{(1,+\infty[}(\rho) = b_1(\rho) - b_{min} = \frac{(n-2)}{n(n-1)}.(1-\rho).$$

$$rm_{(0,1]}(\rho) = M_{max} - MSE_1(\rho) = \frac{n(n+1)-8}{n(n+1)}.(1-\rho)^2.$$

$$rb_{(1,+\infty[}(\rho) = MSE_1(\rho) - M_{min} = \frac{6n^2-9n-4}{(n-1)(n^2+n)}.(1-\rho)^2.$$

This leads to

$$rb_{(1,+\infty[}(\rho) \ < rb_{(0,1]}(\rho), \forall n.$$

$$rm_{(1,+\infty[}(\rho) \ < rm_{(0,1]}(\rho), \forall n \geq 4.$$

In this situation, we can say that the estimator $\tilde{\rho}$ is more stable for $\beta \in (1, +\infty[$ than for $\beta \in (0, 1]$ with respect to bias and MSE.

3. Conclusions

The bias and the MSE of estimator $\tilde{\rho}$ present a very small oscillation under model M_α when $\alpha \in (0, 1]$ than for $\alpha \in (1, +\infty[$.

In model M_β, we get completely opposite behavior to that obtained in model M_α. We can conclude that the estimator $\tilde{\rho}$ has a good properties (more robust) in model M_α for $\alpha \in (0, 1]$ and in model M_β for $\beta \in (1, +\infty[$ with respect to bias and mean square error.

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