Discussiones Mathematicae
Probability and Statistics 28 (2008) 229–246

GENERALIZED *F* TESTS AND SELECTIVE GENERALIZED *F* TESTS FOR ORTHOGONAL AND ASSOCIATED MIXED MODELS

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Abstract

The statistics of generalized F tests are quotients of linear combinations of independent chi-squares. Given a parameter, θ, for which we have a quadratic unbiased estimator, $\widetilde{\theta}$, the test statistic, for the hypothesis of nullity of that parameter, is the quotient of the positive part by the negative part of such estimator. Using generalized polar coordinates it is possible to obtain selective generalized F tests which are especially powerful for selected families of alternatives.

We build both classes of tests for the orthogonal and associated mixed models. The associated models are obtained adding terms to the orthogonal models.

Keywords: selective generalized F tests, generalized polar coordinates, associated models.

2000 Mathematics Subject Classification: 62J12, 62H15, 62H10.

1. Introduction

Generalized F tests were introduced by Michalski and Zmyślony (1996) and (1999), first for variance components and later for linear combinations of parameters in mixed linear models. The statistics of these tests are the quotients of the positive by the negative parts of quadratic unbiased estimators.

To obtain selective generalized F tests for the fixed effects part of mixed models generalized polar coordinates are used, (see Nunes and Mexia, 2004). The statistic of these tests is the statistic of the generalized F tests for the same hypothesis coupled with a vector of central angles. In this way it is possible to increase the test power for the selected family of alternatives. This possibility had already been considered for the usual F tests (see Dias, 1994). Moreover both F and selective F tests have been considered for balanced cross-nesting models (see Fonseca *et al.,* 2003, and Nunes *et al.,* 2006).

The distributions of the test statistics of generalized and selective generalized F tests have been studied (see Fonseca *et al.,* 2002, and Nunes and Mexia, 2006).

In what follows we consider generalized and selective generalized F tests for orthogonal mixed models. In this way we extend the results of Nunes *et al.,* (2006) for balanced cross-nesting models. We will obtain interesting monotonicity properties that enable us to consider the extension of our results to associated models. These models are obtained adding terms to the orthogonal mixed models. Actually such extension has already been considered (see Nunes and Mexia 2006), for balanced cross-nesting.

The next section is divided into two subsections, on distributions and algebraic model structure. The results presented in this section will be used in the study, first of generalized and then of selective generalized F tests, for orthogonal mixed models.

2. Preliminary results

2.1. Distributions

The vectors in this section will have k components, those of $\underline{1}$ [$\underline{p}_i$] being equal to 1 [0 except the i-th which is 1] and $\underline{q}_i = \underline{1} - \underline{p}_i$, $i = 1, ..., k$. Moreover $\underline{u} o \underline{v}$ will be the vector with components $u_i v_i$, $i = 1, ..., k$, and $\chi^2_{g,\delta}$ will be a chi-square with g degrees of freedom and non-centrality parameter δ. We will only consider independent chi-squares.

With $h < k$ let $F_h(.|\underline{a}, \underline{g}, \underline{\delta})$ be the distribution of

$$\Im_h(\underline{a}, \underline{g}, \underline{\delta}) = \frac{\sum_{i=1}^{h} a_i \chi^2_{g_i,\delta_i}}{\sum_{i=h+1}^{k} a_i \chi^2_{g_i,\delta_i}}. \tag{2.1}$$

In Nunes and Mexia (2006) it was shown that

$$F_h(z|\underline{a}, \underline{g}, \underline{\delta}) = e^{-\frac{1}{2}\sum_{i=1}^{k}\delta_i} \sum_{j_1=0}^{+\infty} ... \sum_{j_k=0}^{+\infty} \frac{\prod_{i=1}^{k}\left(\frac{\delta_i}{2}\right)^{j_i}}{\prod_{i=1}^{k} j_i!} F_h\Big(z|\underline{a}, \underline{g} + 2\underline{j}, \underline{0}\Big). \tag{2.2}$$

Consider the ℓ-th component of the non-centrality parameter vector $\underline{\delta}$, δ_ℓ, we can rewrite the previous expression as

$$F_h\Big(z|\underline{a}, \underline{g}, \underline{\delta}\Big) = e^{-\frac{\delta_\ell}{2}} \sum_{j=0}^{+\infty} \frac{\left(\frac{\delta_\ell}{2}\right)^j}{j!} F_h\Big(z|\underline{a}, \underline{g} + 2j\underline{p}_\ell, \underline{q}_\ell \underline{\delta}\Big). \tag{2.3}$$

Besides this we have

$$(2.4)\quad \begin{cases} Pr\left(\dfrac{\sum_{i=1}^{h} a_i\chi^2_{g_i,\delta_i}}{\sum_{i=h+1}^{k} a_i\chi^2_{g_i,\delta_i}} < \dfrac{\sum_{i=1}^{h} a_i\chi^2_{g_i,\delta_i} + a_{i'}\chi^2_2}{\sum_{i=h+1}^{k} a_i\chi^2_{g_i,\delta_i}}\right) = 1,\ i' = 1,...,h \\ \\ Pr\left(\dfrac{\sum_{i=1}^{h} a_i\chi^2_{g_i,\delta_i}}{\sum_{i=h+1}^{k} a_i\chi^2_{g_i,\delta_i}} > \dfrac{\sum_{i=1}^{h} a_i\chi^2_{g_i,\delta_i}}{\sum_{i=h+1}^{k} a_i\chi^2_{g_i,\delta_i} + a_{i'}\chi^2_2}\right) = 1,\ i' = h+1,...,k \end{cases},$$

and since the second fractions will have distribution $F_h(.|\underline{a}, \underline{g} + 2\underline{p}_{i'}, \underline{\delta})$, $i' = 1, ..., k$, we have

$$(2.5)\quad \begin{cases} F_h(z|\underline{a}, \underline{g} + 2(j+1)\underline{p}_{i'}, \underline{\delta}) < F_h(z|\underline{a}, \underline{g} + 2j\underline{p}_{i'}, \underline{\delta}), \\ \qquad\qquad j = 0, ...,\ i' = 1, ..., h \\ F_h(z|\underline{a}, \underline{g} + 2j\underline{p}_{i'}, \underline{\delta}) < F_h(z|\underline{a}, \underline{g} + 2(j+1)\underline{p}_{i'}, \underline{\delta}) \\ \qquad\qquad j = 0, ...,\ i' = h+1, ..., k \end{cases}.$$

Now

$$(2.6)\quad \frac{\partial F_h(z|\underline{a}, \underline{g}, \underline{\delta})}{\partial \delta_{i'}} = \frac{1}{2} e^{-\frac{\delta_{i'}}{2}} \sum_{j=0}^{+\infty} \frac{\left(\frac{\delta_{i'}}{2}\right)^j}{j!}$$
$$\left(F_h\left(z|\underline{a}, \underline{g} + 2(j+1)\underline{p}_{i'}, \underline{q}_{i'}\underline{\delta}\right) - F_h\left(z|\underline{a}, \underline{g} + 2j\underline{p}_{i'}, \underline{q}_{i'}\underline{\delta}\right)\right),$$

so

$$
\left\{
\begin{array}{l}
\dfrac{\partial F_h(z|\underline{a},\underline{g},\underline{\delta})}{\partial \delta_{i'}} < 0,\ i' = 0,...,h \\[2ex]
\dfrac{\partial F_h(z|\underline{a},\underline{g},\underline{\delta})}{\partial \delta_{i'}} > 0,\ i' = h+1,...,k
\end{array}
\right. . \tag{2.7}
$$

Let us now assume that δ_i, $i = 1,...,k$, to be realizations of the non-negative random variables V_i, $i = 1,...,k$, components of $\underline{V}$, with distribution $G_{\underline{V}}$ and moment generation function $\lambda_{\underline{V}}$. We put

$$
\lambda_{\underline{V}}^{<\underline{j}>}(\underline{u}) = \frac{\partial^{j_1+...+j_k}\lambda_{\underline{V}}(\underline{u})}{\prod_{i=1}^{k}\partial u_i^{j_i}}, \tag{2.8}
$$

and point out that $\lambda_{\underline{V}}(\underline{u})$ is defined whenever $\underline{u} \leq \underline{0}$.

The distribution of $\Im_h(\underline{a},\underline{g},\underline{V})$ will be

$$
\begin{aligned}
& F_h(z|\underline{a},\underline{g},\lambda_{\underline{V}}) \\
&= \int_0^{+\infty} ... \int_0^{+\infty} e^{-\frac{1}{2}\sum_{i=1}^{k} v_i} \\
&\quad \sum_{j_1=0}^{+\infty} ... \sum_{j_k=0}^{+\infty} \frac{\prod_{i=1}^{k}\left(\frac{v_i}{2}\right)^{j_i}}{\prod_{i=1}^{k} j_i!} F_h(z|\underline{a},\underline{g}+2\underline{j},\underline{0})dG_{\underline{V}}(v) \qquad (2.9) \\
&= \sum_{j_1=0}^{+\infty} ... \sum_{j_k=0}^{+\infty} \frac{\lambda^{<\underline{j}>}\left(-\frac{1}{2}\underline{1}\right)}{\prod_{i=1}^{k}(2^{j_i} j_i!)} F_h(z|\underline{a},\underline{g}+2\underline{j},\underline{0}).
\end{aligned}
$$

It is also easy to see that, if $Pr(\underline{V} > \underline{0}) = 1$,

$$\text{(2.10)} \qquad \begin{cases} F_h(z|\underline{a}, \underline{g}, \lambda_{\underline{V}}) < F_h(z|\underline{a}, \underline{g}, \lambda_{\underline{q}_i \underline{V}}),\ i = 1, ..., h \\ F_h(z|\underline{a}, \underline{g}, \lambda_{\underline{V}}) > F_h(z|\underline{a}, \underline{g}, \lambda_{\underline{q}_i \underline{V}}),\ i = h+1, ..., k \end{cases},$$

since the i-th component of $\underline{q}_i \underline{V}$ will be null while the corresponding component of $\underline{V}$ will be positive with probability one.

2.2. Models structure

In this section we will use commutative Jordan algebras, CJA. These are linear spaces constituted by symmetric matrices that commute and containing the squares of their matrices. Seely (1971) showed that for any CJA $\mathcal{A}$ there exists one and only one basis, the principal basis $pb(\mathcal{A})$ of $\mathcal{A}$, constituted by pairwise orthogonal projection matrices.

If $\underline{Q} = pb(\mathcal{A}) = \{Q_1, ..., Q_\ell\}$, given an orthogonal projection matrix $Q \in \mathcal{A}$, we will have $Q = \sum_{j=1}^{\ell} a_j Q_j$ but, since Q is idempotent and the $Q_1, ..., Q_\ell$ are pairwise orthogonal, $a_j = 0$ or $a_j = 1$, $j = 1, ..., \ell$. Thus any orthogonal projection matrix belonging to a CJA will be the sum of all or part of the matrices in the principal basis.

Let us now consider symmetric matrices $M_1, ..., M_w$ belonging to a CJA $\mathcal{A}_1$ contained in another CJA $\mathcal{A}_2$. With $pb(\mathcal{A}_u) = \{Q_{u,1}, ..., Q_{u,\ell_u}\}$, $u = 1, 2$ we will have

$$\text{(2.11)} \qquad M_i = \sum_{j=1}^{\ell_u} b_{u,i,j} Q_{u,j},\ i = 1, ..., w,\ u = 1, 2,$$

as well as

$$\text{(2.12)} \qquad Q_{1,j} = \sum_{j' \in \varphi_j} Q_{2,j'},\ j = 1, ..., \ell_1,$$

where the $\varphi_1, ..., \varphi_{\ell_1}$ are pairwise disjunct sets. If we put $B_u = [b_{u,i,j}]$, $u = 1, 2$, we see that the columns of B_2 with indexers in a set φ_j, $j = 1, ..., \ell_1$, are equal. Thus $rank(B_1) = rank(B_2)$. Moreover, if

$$B_u = \begin{bmatrix} B_{u,1,1} & 0 \\ B_{u,2,1} & B_{u,2,2} \end{bmatrix},\ u = 1, 2, \tag{2.13}$$

where $B_{u,1,1}$ has m rows and t_u columns, so that $B_{u,2,1}$ will have $w-m$ rows and also t_u columns and $B_{u,2,2}$ $w-m$ rows and $\ell_u - t_u$ columns, $u = 1, 2$. We also will have $rank(B_{1,1,1}) = rank(B_{2,1,1})$ $rank(B_{1,2,1}) = rank(B_{2,2,1})$ and $rank(B_{1,2,2}) = rank(B_{2,2,2})$. Thus the row vectors of $B_{1,2,2,}$ are linearly independent if and only if the row vectors of $B_{2,2,2}$ are linearly independent. As we shall see this observation will be important.

Let us consider a normal mixed model

$$\underline{Y} = \sum_{i=1}^{m} X_i \underline{\beta}_i + \sum_{i=m+1}^{w} X_i \underline{\widetilde{\beta}}_i, \tag{2.14}$$

where $\underline{\beta}_1, ..., \underline{\beta}_m$ are fixed and the $\underline{\widetilde{\beta}}_{m+1}, ..., \underline{\widetilde{\beta}}_w$ are normal, independent with null mean vectors and variance-covariance matrices $\sigma_i^2 I_{c_i}$, $i = m+1, ..., w$. Many times $X_w = I_n$ and $\underline{\widetilde{\beta}}_w = \underline{e}$, an error vector. Then $\underline{Y}$ will be normal with mean vector

$$\underline{\mu} = \sum_{i=1}^{m} X_i \underline{\beta}_i \tag{2.15}$$

and variance-covariance matrix

$$\Sigma\!\!\!/(\underline{Y}) = \sum_{i=m+1}^{w} \sigma_i^2 M_i, \tag{2.16}$$

where $M_i = X_i X_i^\top$, $i = 1, ..., w$. This model is orthogonal when the matrices M_i commute.

Now, see Schott (1997, pg 157), the matrices M_i, $i = 1, ..., w$, commute if and only if they are diagonalized by an orthogonal matrix P.

Thus if the model is orthogonal, $M_1, ..., M_w \in \mathcal{V}(P)$ with $\mathcal{V}(P)$ the family of matrices diagonalized by P which is a CJA. So, the model is orthogonal if and only if the matrices $M_1, ..., M_w$ belong to a CJA. Since intersecting CJA gives CJA there will be a minimal CJA $\dot{\mathcal{A}} = \mathcal{A}(\underline{M})$ containing $\underline{M} = \{M_1, ..., M_w\}$, the CJA generated by $\underline{M}$. With $\underline{\dot{Q}} = \{\dot{Q}_1, ..., \dot{Q}_{\dot{\ell}}\} = pb(\dot{\mathcal{A}})$ we have

$$M_i = \sum_{j=1}^{\dot{\ell}} \dot{b}_{i,j} \dot{Q}_j, \; i = 1, ..., w. \tag{2.17}$$

Now the space Ω spanned by $\underline{\mu}$ is the range space of

$$\sum_{i=1}^{m} M_i = \sum_{j \in \mathcal{D}} \left(\sum_{i=1}^{m} \dot{b}_{i,j} \right) \dot{Q}_j, \tag{2.18}$$

with $\mathcal{D}= \{j : \sum_{i=1}^{m} \dot{b}_{i,j} \neq 0\}$. Thus the orthogonal projection matrix T on Ω will be

$$T = \sum_{j \in \mathcal{D}} \dot{Q}_j. \tag{2.19}$$

We can always reorder the matrices in $pb(\dot{\mathcal{A}})$ to get $\mathcal{D}= \{1, ..., \dot{d}\}$. Then, since the matrices M_i, $i = 1, ..., w$, are positive semi-definite,

$$\dot{b}_{i,j} = 0, \; j = \dot{d} + 1, ..., \dot{\ell}, \; i = 1, ..., m,$$

and so

$$\dot{B} = \begin{bmatrix} \dot{B}_{1,1} & 0 \\ \dot{B}_{2,1} & \dot{B}_{2,2} \end{bmatrix}. \tag{2.20}$$

As we saw, if for another CJA containing $\underline{M}$, we have

$$M_i = \sum_{j=1}^{k} b_{i,j} Q_j, \ i = 1, ..., w, \tag{2.21}$$

we will have

$$B = \begin{bmatrix} B_{1,1} & 0 \\ B_{2,1} & B_{2,2} \end{bmatrix} \tag{2.22}$$

and the row vectors of $B_{2,2}$ are linearly independent if and only if the row vectors of $\dot{B}_{2,2}$ are linearly independent.

We then have $b_{i,j} = 0$, $j = d+1, ..., k$; $i = 1, ..., m$, and

$$\not\Sigma(\underline{Y}) = \sum_{i=m+1}^{w} \sigma_i^2 \sum_{j=1}^{k} b_{i,j} Q_j = \sum_{j=1}^{k} \gamma_j Q_j, \tag{2.23}$$

with

$$\gamma_j = \sum_{i=m+1}^{w} b_{i,j} \sigma_i^2. \tag{2.24}$$

Putting $\underline{\gamma}(1) = (\gamma_1, ..., \gamma_d)$, $\underline{\gamma}(2) = (\gamma_{d+1}, ..., \gamma_k)$ and $\underline{\sigma}^2 = (\sigma_{m+1}^2, ..., \sigma_w^2)$ we have

$$\begin{cases} \underline{\gamma}(1) = B_{2,1}^{\top} \underline{\sigma}^2 \\ \underline{\gamma}(2) = B_{2,2}^{\top} \underline{\sigma}^2 \end{cases}, \tag{2.25}$$

and, since the column vectors of $B_{2,1}^{\top}$ are linearly independent, we will have

$$
(2.26) \qquad \begin{cases} \underline{\sigma}^2 = \left(B_{2,2}^{\top}\right)^+ \underline{\gamma}(2) \\ \\ \underline{\gamma}(1) = B_{2,1}^{\top}\left(B_{2,2}^{\top}\right)^+ \underline{\gamma}(2) \end{cases},
$$

where $(B_{2,2}^{\top})^+$ is the Moore-Penrose inverse of $B_{2,2}^{\top}$.

Let now the row vectors of A_j constitute an orthonormal basis for the range space of Q_j, then $Q_j = A_j^{\top} A_j$ and $A_j A_j^{\top} = I_{g_j}$, with $g_j = rank(Q_j)$, $j = 1, ..., k$. We may assume that the observations vector spans R^n so that $\sum_{j=1}^{k} A_j^{\top} A_j = \sum_{j=1}^{k} Q_j = I_n$. Then with

$$
(2.27) \qquad \begin{cases} \underline{\eta}_j = A_j \underline{\mu}, \ j = 1, ..., k \\ \widetilde{\underline{\eta}}_j = A_j \underline{Y}, \ j = 1, ..., k \end{cases},
$$

we will have $\underline{\eta}_j = \underline{0}$, $j = d+1, ..., k$, and

$$
(2.28) \qquad \begin{cases} \underline{\mu} = \sum_{j=1}^{d} A_j^{\top} \underline{\eta}_j \\ \\ \underline{Y} = \sum_{j=1}^{k} A_j^{\top} \widetilde{\underline{\eta}}_j \end{cases}.
$$

These expressions show the central part that vectors $\underline{\eta}_1, ..., \underline{\eta}_d$ $[\widetilde{\underline{\eta}}_1, ..., \widetilde{\underline{\eta}}_k]$ play in our model.

3. Generalized F tests

We start by obtaining sufficient and complete statistics. Since the Q_j, $j = 1, ..., k$, are pairwise orthogonal projection matrices we will have

$$\not{\Sigma}(\underline{Y})^{-1} = \sum_{j=1}^{k} \frac{1}{\gamma_j} Q_j = \sum_{j=1}^{k} \frac{1}{\gamma_j} A_j^{\top} A_j, \tag{3.1}$$

so that, with $S_j = \|\underline{\widetilde{\eta}}_j\|^2$, $j = d+1, ..., k$,

$$(\underline{Y} - \underline{\mu})^{\top} \not{\Sigma}(\underline{Y})^{-1} (\underline{Y} - \underline{\mu}) = \sum_{j=1}^{k} \frac{\|\underline{\widetilde{\eta}}_j - \underline{\eta}_j\|^2}{\gamma_j}$$

$$= \sum_{j=1}^{d} \frac{\|\underline{\widetilde{\eta}}_j - \underline{\eta}_j\|^2}{\gamma_j} + \frac{S_j}{\gamma_j}. \tag{3.2}$$

Using the factorization theorem and the fact the normal distribution belongs to the exponential family with, for these models, a parametric space that contain open sets, we establish the first part of the thesis of

Theorem 1. *The* $\underline{\widetilde{\eta}}_1, ..., \underline{\widetilde{\eta}}_d$ *and* $S_{d+1}, ..., S_k$ *constitute a sufficient complete statistic. Moreover the* $\underline{\widetilde{\eta}}_1, ..., \underline{\widetilde{\eta}}_d$, $\underline{\widetilde{\gamma}}(2)$ *with components* $\widetilde{\gamma}_j = \frac{S_j}{g_j}$, $j = d+1, ..., k$, $\underline{\widetilde{\sigma}}^2 = (B_{2,2}^{\top})^{+} \underline{\widetilde{\gamma}}(2)$ *and* $\underline{\widetilde{\gamma}}(1) = B_{2,1}^{\top} (B_{2,2}^{\top})^{+} \underline{\widetilde{\gamma}}(2)$ *will be UMVUE.*

Proof. The second part of the thesis follows from the first part and from the theorem of Blackwell-Lehman-Scheffé. ■

Now we can put

$$\sigma_i^2 = \sum_{j \in \varphi_i^{+}} b_{i,j} \gamma_j - \sum_{j \in \varphi_i^{-}} b_{i,j} \gamma_j, \ i = m+1, ..., w, \tag{3.3}$$

with $\varphi_i^{+} \cup \varphi_i^{-} \subseteq \{d+1, ..., k\}$. Thus the positive and the negative parts of an unbiased estimator for σ_i^2 will be $\sum_{j \in \varphi_i^{+}} b_{i,j} \frac{S_j}{g_j}$ and $\sum_{j \in \varphi_i^{-}} b_{i,j} \frac{S_j}{g_j}$ and the statistic for testing

$$H_0 : \sigma_i^2 = 0,\ i = m+1, ..., w, \tag{3.4}$$

will be

$$\Im = \frac{\displaystyle\sum_{j\in\varphi_i^+} b_{i,j}\frac{S_j}{g_j}}{\displaystyle\sum_{j\in\varphi_i^-} b_{i,j}\frac{S_j}{g_j}} = \frac{\displaystyle\sum_{j\in\varphi_i^+} \frac{b_{i,j}\gamma_j}{g_j}\chi^2_{g_j}}{\displaystyle\sum_{j\in\varphi_i^-} \frac{b_{i,j}\gamma_j}{g_j}\chi^2_{g_j}}. \tag{3.5}$$

The orthogonal model

$$\underline{Y} = \sum_{j=1}^{k} A_j^\top \underline{\widetilde{\eta}}_j \tag{3.6}$$

has associated models given by

$$\underline{Y}_a = \underline{Y} + \underline{Y}_p, \tag{3.7}$$

where

$$\underline{Y}_p = \sum_{j=1}^{k} A_j^\top \underline{Z}_j. \tag{3.8}$$

The $\underline{Z}_1, ..., \underline{Z}_k$ being independent of the $\underline{\widetilde{\eta}}_1, ..., \underline{\widetilde{\eta}}_k$. We take

$$V_j = \frac{1}{\gamma_j}\|\underline{Z}_j\|^2,\ j = 1, ..., k, \tag{3.9}$$

and represent by G_i the joint distribution of the V_j with $j \in \varphi_i^+ \cup \varphi_i^-$, $i = m+1, ..., w$. With $h_i = \sharp(\varphi_i^+)$, $k_i - h_i = \sharp(\varphi_i^-)$, $\underline{a}_i$ the vector of the coefficients for the positive and the negative parts of the estimator and $\underline{g}_i$ the vector of number of degrees of freedom, the distribution of $\Im$ for the orthogonal model be $F_{h_i}(.|\underline{a}_i, \underline{g}_i)$, $i = m+1, ..., w$.

When we go over to the associated models we get the distribution $F_{h_i}(.|\underline{a}_i, \underline{g}_i, G_i)$, $i = m+1, ..., w$. Our results of Section 2.1 show that the effects of the $\underline{Z}_j$, $j = 1, ..., h_i$, is to "increase" the test statistic possibly leading to pseudo-significant results. Moreover the effect of $\underline{Z}_j$, $j = h_i + 1, ..., k_i$, will be to "decrease" the statistic leading to loss of power.

Likewise if we go to the fixed effects part, and given

$$\underline{\psi} = W\underline{\eta}_j, \; j = 1, ..., d, \tag{3.10}$$

we have the UMVUE $\underline{\widetilde{\psi}} = W\underline{\widetilde{\eta}}_j$, $j = 1, ..., d$.

Moreover we want to test

$$H_0 : \underline{\psi} = \underline{\psi}_0. \tag{3.11}$$

Since $\underline{\widetilde{\psi}}$ will be normal with mean vector $\underline{\psi}$ and variance-covariance matrix

$$W A_j \Sigma(\underline{Y}) A_j^\top W^\top = \gamma_j W W^\top,$$

the quadratic form

$$U = \left(\underline{\widetilde{\psi}} - \underline{\psi}_0\right)^\top \left(WW^\top\right)^+ \left(\underline{\widetilde{\psi}} - \underline{\psi}_0\right)$$

will be (see Mexia 1990), the product by γ_j of χ^2_{g,δ_0} with $g = rank(W)$

and

$$\delta_0 = \frac{1}{\gamma_j}\left(\underline{\psi} - \underline{\psi}_0\right)^\top \left(WW^\top\right)^+ \left(\underline{\psi} - \underline{\psi}_0\right). \tag{3.12}$$

When H_0 holds, $\delta_0 = 0$, and if the row vectors of W are linearly independent, $WW^\top$ will be positive definite so $(WW^\top)^+ = (WW^\top)^{-1}$ and the hypothesis may be rewritten as

$$H_0 : \lambda = 0, \tag{3.13}$$

with $\lambda = \delta_0\gamma_j$. In what follows we will restrict ourselves to this case. Now

$$E(U) = g\gamma_j + \lambda \tag{3.14}$$

and for γ_j we have the UMVUE

$$\widetilde{\gamma}_j = \sum_{v\in\varphi_j^+} c_{j,v}\widetilde{\gamma}_v - \sum_{v\in\varphi_j^-} c_{j,v}\widetilde{\gamma}_v, \tag{3.15}$$

where $\varphi_j^+ \cup \varphi_j^- \subseteq \{d+1, ..., k\}$ and the $c_{j,v}$ an element of $B_{2,1}^\top (B_{2,2}^\top)^+$.

Thus for λ we have the quadratic unbiased estimator

$$\widetilde{\lambda} = \left(U + g\sum_{v\in\varphi_j^-} c_{j,v}\frac{S_v}{g_v}\right) - \left(g\sum_{v\in\varphi_j^+} c_{j,v}\frac{S_v}{g_v}\right). \tag{3.16}$$

So we will have the test statistic with distribution $F_{h_j}(.|\underline{a}, \underline{g}, \delta_0\underline{p}_1)$, where $\underline{a}$ has components γ_j, $gc_{j,v}\frac{\gamma_v}{g_v}$, $v \in \varphi_j^-$ and $gc_{j,v}\frac{\gamma_v}{g_v}$, $v \in \varphi_j^+$ while the components of $\underline{g}$ will be g, g_v, $v \in \varphi_j^-$ and g_v, $v \in \varphi_j^+$ and $h_j = \sharp(\varphi_j^-) + 1$.

Since in the test statistic

$$\mathcal{F} = \frac{U + g \sum_{v \in \varphi_j^-} c_{j,v} \dfrac{S_v}{g_v}}{g \sum_{v \in \varphi_j^+} c_{j,v} \dfrac{S_v}{g_v}} \tag{3.17}$$

only the term U may have non null centrality parameters, from our results in Section 2.1, it follows that this test will be strictly unbiased.

If we go over to associated models we can reason as above to show that:

- the $\underline{Z}_{j'}$, with $j' \in \varphi_j^-$, "increase" the statistics leading, possibility, to situations of pseudo-significance;
- the $\underline{Z}_{j'}$, with $j' \in \varphi_j^+$, "degrease" the statistics leading to a loss of test power.

Moreover, if we replace $\underline{\widetilde{\eta}}_j$ by $\underline{\widetilde{\eta}}_j + \underline{Z}_j$ we will have, with $\underline{\psi}_0 = W\underline{\eta}_0$,

$$U = \left(\underline{\widetilde{\eta}}_j + \underline{Z}_j - \underline{\eta}_0\right)^\top W^\top \left(WW^\top\right)^{-1} W \left(\underline{\widetilde{\eta}}_j + \underline{Z}_j - \underline{\eta}_0\right)$$

so when H_0 holds and $\delta_0 = 0$, the perturbations $\underline{Z}_j$ may lead to pseudo-significant results.

4. Selective generalized F tests

To obtain selective F tests we use generalized polar coordinates. Let $\underline{\psi}$ have s components. Given a point in R^s with cartesian coordinates $(x_1, ..., x_s)$, and generalized polar coordinates $(r, \theta_1, ..., \theta_{s-1})$, we will have $r = \|\underline{x}\|$ and

$$x_j = r\ell_j(\underline{\theta}),$$

where $\underline{\theta} = (\theta_1, ..., \theta_{s-1})$ and

$$
(4.1) \qquad \begin{cases} \ell_1(\underline{\theta}) & = & \cos\theta_1 \cdots \cos\theta_{s-1} \\ \vdots & & \\ \ell_j(\underline{\theta}) & = & \cos\theta_1 \cdots \cos\theta_{s-j} \sin\theta_{s-j+1}, \; j = 2, ..., s-1 \\ \vdots & & \\ \ell_s(\underline{\theta}) & = & \sin\theta_1 \end{cases} .
$$

For the central angles we have the bounds

$$
(4.2) \qquad \begin{cases} -\frac{\pi}{2} \leq \theta_j \leq \frac{\pi}{2} \quad ; \quad j = 1, ..., s-2 \\ \\ 0 \leq \theta_{s-1} < 2\pi \end{cases} ,
$$

which define the domain $\mathcal{D}$ of variation of the central angles.

Given $\underline{x}$ the corresponding vector of the central angles will be $\underline{\theta}(\underline{x})$.

The use of generalized polar coordinates enables us to obtain tests for alternatives

$$
(4.3) \qquad H_1 : \underline{\psi} = \underline{\psi}_1
$$

to

$$
(4.4) \qquad H_0 : \underline{\psi} = \underline{\psi}_0
$$

such that, $\underline{\theta}(\underline{\psi}_1 - \underline{\psi}_0) \in \mathcal{D}_1 \subset \mathcal{D}$.

In the previous section we presented a statistic $\mathcal{F}$ for the (non-selective) generalized F test for H_0. Now , when H_0 holds $\mathcal{F}$ is independent of

$$
\underline{\Theta} = \underline{\theta}\Big(\widetilde{\underline{\psi}} - \underline{\psi}_0\Big)
$$

(see Nunes and Mexia, 2004) thus we now use as test statistic the pair $(\mathcal{F}, \underline{\Theta})$, rejecting H_0 when $\mathcal{F} > f$ and $\underline{\Theta} \in \mathcal{D}_1$. The test level will be the product of $F_{h_j}(f|\underline{a}, \underline{g})$, (see Nunes and Mexia, 2004),

$$Pr\left(\underline{\Theta} \in \mathcal{D}_1 | H_0\right)$$

$$= \frac{2^{-1}\Gamma\left(\frac{s}{2}\right)}{\pi^{s/2}} \int_{\mathcal{D}_1} ... \int \cos\theta_1^{s-2}, ..., \ \cos\theta_{s-2} \prod_{j=1}^{s-1} d\theta_j.$$

Many times, when $\underline{\psi} = \underline{\eta}_j$, $\theta(x) \in \mathcal{D}_1$ if and only if the g_j components satisfy ℓ order relations. Since $\underline{\widetilde{\eta}}_j - \underline{\eta}_j$ will be normal with null mean vector, when H_0 holds, and variance-covariance matrix $\gamma_j I_{g_j}$ we will have

$$Pr\left(\underline{\Theta} \in \mathcal{D}_1 | H_0\right) = \frac{\ell!}{g_j!}.$$

Going over to the associated models, we will assume only perturbations $\underline{Z}_j$, $j = d+1, ..., k$, so that $\underline{\widetilde{\psi}}$ will have the same distribution as before and $\mathcal{F}$ will continue to be independent from $\underline{\Theta}$ when H_0 holds. We can now reasons as before to see that:

- the $\underline{Z}_{j'}$, with $j' \in \varphi_j^-$, "increase" the test statistic leading, possibility, to pseudo-significance;
- the $\underline{Z}_{j'}$, with $j' \in \varphi_j^+$, "degrease" the test statistic originating loss of test power.

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Received 18 March 2008
Revised 4 November 2008