

## TIGHTNESS OF CONTINUOUS STOCHASTIC PROCESSES

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### Abstract

Some sufficient conditions for tightness of continuous stochastic processes is given. It is verified that in the classical tightness sufficient conditions for continuous stochastic processes it is possible to take a continuous nondecreasing stochastic process instead of a deterministic function one.

### 1. INTRODUCTION

Given a separable metric space  $(X, \rho)$  denote by  $\mathcal{P}(X)$  the set of all probability measures on  $(X, \beta(X))$ , where as usual  $\beta(X)$  is a Borel  $\sigma$ -algebra on  $(X, \rho)$ . We call a subset  $\Lambda \subset \mathcal{P}(X)$  tight if for every  $\varepsilon > 0$  there is a compact set  $\mathcal{K} \subset X$  such that  $P(\mathcal{K}) \geq 1 - \varepsilon$  for every  $P \in \Lambda$ . Let  $\mathcal{P} = (\Omega, \mathcal{F}, P)$  be a complete probability space and let  $C_T = C([0, T], \mathbb{R}^m)$ . We shall consider  $C_T$  as a measurable space with its Borel  $\sigma$ -algebra  $\beta(C_T)$ . A continuous  $m$ -dimensional stochastic process  $x = (x_t)_{0 \leq t \leq T}$  on  $\mathcal{P}$  can be equivalently defined as  $(\mathcal{F}, \beta(C_T))$ -measurable random function  $x : \Omega \rightarrow C_T$ . For such continuous process we define its distribution on  $\beta(C_T)$  in the usual way by  $(Px^{-1})(A) = P(x^{-1}(A))$  for every  $A \in \beta(C_T)$ .

Having given a sequence  $(x^n)_{n=1}^\infty$  of continuous stochastic processes  $x^n : \Omega \rightarrow C_T$  we say that  $(x^n)_{n=1}^\infty$  is tight ([2, 3]) if and only if a sequence  $(P(x^n)^{-1})_{n=1}^\infty$  of its distributions is tight. It is known ([3], Theorem I.4.3) that a sequence  $(x^n)_{n=1}^\infty$  is tight if there are positive numbers  $\gamma, \alpha, \beta$  and  $M$  such that for every  $n = 1, 2, \dots$  one has  $E|x_0^n|^\gamma \leq M$  and  $E|x_t^n - x_s^n|^\alpha \leq M|t - s|^{1+\beta}$ . There are some weaker sufficient conditions for tightness of sequences of continuous stochastic processes. It is proved ([1], Theorem II.12.3) that a sequence  $(x^n)_{n=1}^\infty$  of continuous processes  $x^n : \Omega \rightarrow C_T$  is tight if a sequence  $(x_0^n)_{n=1}^\infty$  of  $m$ - dimensional random variables  $x_0^n : \Omega \rightarrow \mathbb{R}^m$  is tight and there are numbers  $\gamma \geq 0$  and  $\alpha > 1$  and a nondecreasing continuous function  $F : [0, T] \rightarrow \mathbb{R}$  such that for every  $n = 1, 2, \dots$  one has

$$P(\{|x_t^n - x_s^n| \geq \lambda\}) \leq \frac{1}{\lambda^\gamma} |F(t) - F(s)|^\alpha$$

for  $s, t \in [0, T]$  and a positive number  $\lambda$ . We shall show that the above result holds true if instead of a function  $F$  there is a real-valued continuous nondecreasing stochastic process  $(\Gamma(t))_{0 \leq t \leq T}$  such that  $E[\Gamma(T) - \Gamma(0)] < \infty$  and

$$P(\{|x_t^n - x_s^n| \geq \lambda\}) \leq \frac{1}{\lambda^\gamma} E|\Gamma(t) - \Gamma(s)|^\alpha$$

for  $s, t \in [0, T]$  and a positive number  $\lambda$ . The proof of such type result is obtained by modifications of the procedures given in ([1], Theorem II.12.1–Theorem II.12.3). To begin with let us introduce some adding notations. Having given a probability space  $\mathcal{P}$  and random variables  $\xi_i : \Omega \rightarrow \mathbb{R}^m$  for  $i = 1, \dots, n$  let us define  $S_k = \xi_1 + \dots + \xi_k$  for  $k = 1, \dots, n$  and  $S_0 = 0$ . Then let  $M_n = \max_{0 \leq k \leq n} |S_k|$  and  $M'_n = \max_{0 \leq k \leq n} (\min\{|S_k|, |S_n - S_k|\})$ . It is clear that  $M'_n \leq M_n$  and  $M_n \leq M'_n + |S_n|$  a.s. Therefore, for every  $\lambda > 0$  we have

$$(1) \quad P(\{M_n \geq \lambda\}) \leq P(\{M'_n \geq \lambda/2\}) + P(\{|S_n| \geq \lambda/2\}).$$

## 2. AUXILIARY RESULTS

We shall prove here some auxiliary results that are needed in the proof of the main result of the paper.

**Proposition 1.** *Let  $\gamma \geq 0$  and  $\alpha > 1/2$  be given and suppose there are positive random variables  $u_1, \dots, u_n$  such that  $E(\sum_{l=1}^n u_l)^{2\alpha} < \infty$  and*

$$(2) \quad P(\{|S_j - S_i| \geq \lambda, |S_k - S_j| \geq \lambda\}) \leq \frac{1}{\lambda^{2\gamma}} E(u_{i+1} + \dots + u_k)^{2\alpha}$$

*is satisfied for  $0 \leq i \leq j \leq k \leq n$  and every  $\lambda > 0$ . Then there is a number  $K_{\gamma, \alpha}$  such that for every positive  $\lambda$  one has*

$$(3) \quad P\left(\left\{M'_n \geq \lambda\right\}\right) \leq \frac{K_{\gamma, \alpha}}{\lambda^{2\gamma}} E(u_1 + \dots + u_n)^{2\alpha}.$$

**Proof.** Let  $\delta = 1/(2\gamma + 1)$ . We have  $2^\delta [1/2^{2\alpha\delta} + 1/K^\delta] \leq 1$  for sufficiently large  $K > 0$ . We shall show that (3) is satisfied if  $K$  satisfies the above inequality and  $K \geq 1$ . It can be verified ([1], Theorem II.12.1) that the minimal number  $K$  satisfying the above inequality is given by

$$K_{\gamma, \alpha} = \left[ \frac{1}{2^{1/(2\gamma+1)}} - \left( \frac{1}{2^{1/(2\gamma+1)}} \right)^{2\alpha} \right]^{-(2\gamma+1)}.$$

The proof of (3) we get by the induction procedure with respect to  $n$ . For  $n = 1$  the inequality (3) is trivial. Let  $n = 2$ . Immediately from (2) for  $K \geq 1$  it follows

$$\begin{aligned} P\left(\left\{M'_2 \geq \lambda\right\}\right) &= P(\{\min[|S_1|, |S_2 - S_1|] \geq \lambda\}) \\ &\leq \frac{1}{\lambda^{2\gamma}} E(u_1 + u_2)^{2\alpha} \leq \frac{K}{2^{2\gamma}} E(u_1 + u_2)^{2\alpha} \end{aligned}$$

for  $\lambda > 0$ . Assume now that (3) is satisfied for any positive integer  $k < n$ . We shall show that it is also satisfied for  $k = n$ . Let  $v = E(u_1 + \dots + u_n)^{2\alpha}$ ,  $v_0 = 0$  and  $v_h = E(u_1 + \dots + u_h)^{2\alpha}$  with  $1 \leq h \leq n$ . We can assume that  $v > 0$ . We have  $v_{h-1} \leq v_h$ . Then  $0 \leq v_1/v \leq v_2/v \leq \dots \leq v_{n-1}/v \leq 1$ . Therefore  $[0, 1] = \bigcup_{h=1}^n [v_{h-1}/v, v_h/v]$ . By virtue of the assumption  $\alpha > 1/2$  we have  $1/2^{2\alpha} \in [0, 1]$ . Therefore, there is  $1 \leq h \leq n$  such that  $v_{h-1}/v \leq 1/2^{2\alpha} \leq v_h/v$ . Similarly as in [1] we define  $U_1, U_2, D_1$  and  $D_2$  by setting

$$U_1 = \max_{0 \leq j \leq h-1} \min \{|S_j|, |S_{h-1} - S_j|\},$$

$$U_2 = \max_{h \leq j \leq n} \min \{|S_j - S_h|, |S_n - S_j|\},$$

$$D_1 = \min \{|S_{h-1}|, |S_n - S_{h-1}|\}, \quad D_2 = \min \{|S_h|, |S_n - S_h|\}.$$

Let us observe that for  $1 \leq h \leq n$ , taken above, we have  $v_{h-1} \leq (2^{2\alpha} - 1)v/2^{2\alpha}$  and  $z_{h+1} \leq (2^{2\alpha} - 1)v/2^{2\alpha}$ , where  $z_{h+1} = E(u_{h+1} + \dots + u_n)^{2\alpha}$ . Indeed, we have  $v_{h-1} \leq v/2^{2\alpha} \leq (2^{2\alpha} - 1)v/2^{2\alpha}$ . Furthermore

$$\begin{aligned} \frac{v_h}{v} + \frac{z_{h+1}}{v} &= \frac{E \left[ (u_1 + \dots + u_h)^{2\alpha} + (u_{h+1} + \dots + u_n)^{2\alpha} \right]}{v} \\ &\leq \frac{E [(u_1 + \dots + u_h) + (u_{h+1} + \dots + u_n)]^{2\alpha}}{v} = 1 \end{aligned}$$

and  $1 - v_h/v \leq 1 - 1/2^{2\alpha} = (2^{2\alpha} - 1)/2^{2\alpha}$ . Therefore,  $z_{h+1} \leq (2^{2\alpha} - 1)v/2^{2\alpha}$ . Let us observe that (2) will be satisfied if we take  $h - 1$  instead of  $n$ . Since  $h - 1 < n$  then we can assume that (3) is satisfied for random variables  $\xi_1, \dots, \xi_{h-1}$  and  $u_1, \dots, u_{h-1}$ . Hence and the above inequalities we obtain

$$P(\{U_1 \geq \lambda\}) \leq \frac{K}{\lambda^{2\gamma}} E(u_1 + \dots + u_{h-1})^{2\alpha} \leq \frac{K(2^{2\alpha} - 1)}{2^{2\alpha} \lambda^{2\gamma}} v.$$

Similarly, taking in (2) indexes  $h \leq i \leq j \leq n$ , we shall only consider random variables  $\xi_{h+1}, \dots, \xi_n$  and  $u_{h+1}, \dots, u_n$  and we can assume that (3) is satisfied for these random variables because  $n - h < n$ . Hence and the above inequalities we obtain

$$P(\{U_2 \geq \lambda\}) \leq \frac{K}{\lambda^{2\gamma}} E(u_{h+1} + \dots + u_n)^{2\alpha} \leq \frac{K(2^{2\alpha} - 1)}{2^{2\alpha} \lambda^{2\gamma}} v.$$

Next, by (2) we have

$$P(\{D_1 \geq \lambda\}) \leq \frac{1}{\lambda^{2\gamma}} E(u_1 + \dots + u_n)^{2\alpha} = \frac{v}{\lambda^{2\gamma}}$$

and

$$P(\{D_2 \geq \lambda\}) \leq \frac{v}{\lambda^{2\gamma}}.$$

Let us observe that in particular cases:  $h = 1$  and  $h = n$  the above inequalities are trivial, respectively. Similarly as in ([1], Theorem II.12.1) we can verify that  $M'_n \leq \max[U_1 + D_1, U_2 + D_2]$  and therefore

$$(4) \quad P(\{M'_n \geq \lambda\}) \leq P(\{U_1 + D_1 \geq \lambda\}) + P(\{U_2 + D_2 \geq \lambda\}).$$

On the other hand we have

$$(5) \quad P(\{U_1 + D_1 \geq \lambda\}) \leq P(\{U_1 \geq \lambda_0\}) + P(\{D_1 \geq \lambda_1\})$$

$$\leq \left[ \frac{1}{\lambda_0^{2\alpha}} \frac{K(2^{2\alpha} - 1)}{2^{2\alpha}} + \frac{1}{\lambda_1^{2\alpha}} \right] v$$

for positive numbers  $\lambda_0$  and  $\lambda_1$  such that  $\lambda = \lambda_0 + \lambda_1$ . It can be verified ([1], Theorem II.12.1) that for positive numbers  $C_0, C_1, \lambda, \delta$  and  $\gamma$  such that  $\delta = 1/(2\gamma + 1)$  we have

$$\min_{\lambda_0 + \lambda_1 = \lambda} \left[ \frac{C_0}{\lambda_0^{2\gamma}} + \frac{C_1}{\lambda_1^{2\gamma}} \right] = \frac{1}{\lambda_1^{2\gamma}} \left[ C_0^\delta + C_1^\delta \right]^{1/\delta},$$

where minimum is taken over all positive numbers  $\lambda_0$  and  $\lambda_1$  such that  $\lambda_0 + \lambda_1 = \lambda$ . Therefore (5) implies

$$P(\{U_1 + D_1 \geq \lambda\}) \leq \frac{v}{\lambda^{2\gamma}} \left[ \left( \frac{K(2^{2\alpha} - 1)}{2^{2\alpha}} \right)^\delta + 1 \right]^{1/\delta}.$$

In a similar way we obtain

$$P(\{U_2 + D_2 \geq \lambda\}) \leq \frac{v}{\lambda^{2\gamma}} \left[ \left( \frac{K(2^{2\alpha} - 1)}{2^{2\alpha}} \right)^\delta + 1 \right]^{1/\delta}.$$

Therefore (4) implies

$$P(\{M'_n \geq \lambda\}) \leq \frac{2v}{\lambda^{2\gamma}} \left[ \left( \frac{K(2^{2\alpha} - 1)}{2^{2\alpha}} \right)^\delta + 1 \right]^{1/\delta}.$$

For  $\alpha > 1/2$  and sufficiently large  $K \geq 1$  satisfying  $2^\delta [1/2^{2\alpha\delta} + 1/K^\delta] \leq 1$  we have

$$\left[ \left( \frac{K(2^{2\alpha} - 1)}{2^{2\alpha}} \right)^\delta + 1 \right]^{1/\delta} \leq K.$$

Indeed, we have

$$\left[ \left( \frac{2^{2\alpha} - 1}{2^{2\alpha}} \right)^\delta + \frac{1}{K^\delta} \right] \rightarrow \left( \frac{2^{2\alpha} - 1}{2^{2\alpha}} \right)^\delta$$

as  $K \rightarrow \infty$ . Therefore for sufficiently large  $K \geq 1$  we get

$$\left[ \left( \frac{K(2^{2\alpha} - 1)}{2^{2\alpha}} \right)^\delta + 1 \right]^{1/\delta} = K \left[ \left( \frac{2^{2\alpha} - 1}{2^{2\alpha}} \right)^\delta + \frac{1}{K^\delta} \right]^{1/\delta} \leq K.$$

Then for such sufficiently large  $K \geq 1$  we finally obtain

$$P(\{M'_n \geq \lambda\}) \leq \frac{K_{\gamma,\alpha}}{\lambda^{2\gamma}} E(u_1 + \dots + u_n)^{2\alpha}$$

with  $K_{\gamma,\alpha} = 2K$ . ■

**Proposition 2.** Let  $\gamma \geq 1$  and an integer  $\alpha > 1$  be given and suppose there are random variables  $\xi_i : \Omega \rightarrow \mathbb{R}^m$  and  $u_i : \Omega \rightarrow \mathbb{R}^+$  for  $i = 1, \dots, n$  such that  $E(u_1 + \dots + u_n)^\alpha < \infty$  and

$$(6) \quad P(\{|S_j - S_i| \geq \lambda\}) \leq \frac{1}{\lambda^\gamma} E(u_{i+1} + \dots + u_j)^\alpha$$

for every  $\lambda > 0$  and  $0 \leq i < j \leq n$ . Then there is a positive number  $K'_{\gamma, \alpha}$  such that

$$(7) \quad P(\{M_n \geq \lambda\}) \leq \frac{K'_{\gamma, \alpha}}{\lambda^\gamma} E(u_1 + \dots + u_n)^\alpha$$

**Proof.** Taking into account inequalities  $P(E_1 \cap E_2) \leq [P(E_1)]^{1/2}[P(E_2)]^{1/2}$  and  $xy \leq (x+y)^2$  for  $E_1, E_2 \in \mathcal{F}$  and  $x, y \in \mathbb{R}$ , we can easily see that (6) implies

$$\begin{aligned} & P(\{|S_j - S_i| \geq \lambda, |S_k - S_j| \geq \lambda\}) \\ & \leq [P(\{|S_j - S_i| \geq \lambda\})]^{1/2} [P(\{|S_k - S_j| \geq \lambda\})]^{1/2} \\ & \leq \frac{1}{\lambda^{\gamma/2}} \left[ E \left( \sum_{i < l \leq j} u_l \right)^\alpha \right]^{1/2} \cdot \frac{1}{\lambda^{\gamma/2}} \left[ E \left( \sum_{j < l \leq k} u_l \right)^\alpha \right]^{1/2} \\ & \leq \frac{1}{\lambda^\gamma} E \left[ \left( \sum_{i < l \leq j} u_l \right)^\alpha + \left( \sum_{j < l \leq k} u_l \right)^\alpha \right] \leq \frac{2}{\lambda^\gamma} E \left[ \sum_{i < l \leq j} u_l + \sum_{j < l \leq k} u_l \right]^\alpha \\ & = \frac{1}{\lambda^\gamma} (u_{i+1} + \dots + u_k)^\alpha. \end{aligned}$$

Then the assumption (2) of Proposition 1, with  $\gamma/2$  and  $\alpha/2$  instead of  $\gamma$

and  $\alpha$ , respectively is satisfied. Therefore, by virtue of Proposition 1, we obtain

$$P\left(\{M'_n \geq \lambda\}\right) \leq \frac{\tilde{K}}{\lambda^\gamma} E(u_1 + \dots + u_n)^\alpha$$

with  $\tilde{K} = K_{\gamma/2, \alpha/2}$ . On the other hand (6) implies

$$P(\{|S_n| \geq \lambda\}) \leq \frac{1}{\lambda^\gamma} E(u_1 + \dots + u_n)^\alpha.$$

Hence and (1) we obtain

$$P(\{M_n \geq \lambda\}) \leq \frac{K'_{\gamma, \alpha}}{\lambda^\gamma} E(u_1 + \dots + u_n)^\alpha$$

with  $K'_{\gamma, \alpha} = 2^\gamma (\tilde{K} + 1)$ . ■

### 3. TIGHTNESS OF CONTINUOUS PROCESSES

We shall prove now the main result of the paper

**Theorem 3.** *A sequence  $(x^n)_{n=1}^\infty$  of continuous  $m$ - dimensional stochastic processes  $x^n = (x^n(t))_{0 \leq t \leq T}$  on a probability space  $\mathcal{P} = (\Omega, \mathcal{F}, P)$  is tight if for every  $\epsilon > 0$  there is a number  $a > 0$  such that  $P(|x^n(0)| > a) \leq \epsilon$  for  $n \geq 1$  and there are  $\gamma \geq 0$ , an integer  $\alpha > 1$  and a continuous nondecreasing stochastic process  $(\Gamma(t))_{0 \leq t \leq T}$  on  $\mathcal{P}$  such that  $E[\Gamma(T) - \Gamma(0)] < \infty$  and*

$$(8) \quad P(\{|x^n(t) - x^n(s)| \geq \lambda\}) \leq \frac{1}{\lambda^\gamma} E|\Gamma(t) - \Gamma(s)|^\alpha$$

for every  $n \geq 1$ ,  $\lambda > 0$  and  $s, t \in [0, T]$ .

**Proof.** For simplicity we consider the case  $T = 1$ . By virtue of ([1], Theorem II.8.3) it is enough only to verify that for every  $\epsilon > 0$  and  $\eta > 0$  there is a  $\delta \in (0, 1)$  such that  $\delta^{-1}$  is an integer and

$$(9) \quad \sum_{j < \delta^{-1}} P \left( \left\{ \sup_{j\delta \leq s \leq (j+1)\delta} |x^n(s) - x^n(j\delta)| \geq \varepsilon \right\} \right) \leq \eta$$

for every  $n \geq 1$ . Fix for  $n \geq 1$  and  $j \geq 1$ . For a positive integer  $k$  consider  $m$ - dimensional random variables  $\xi_1^j, \dots, \xi_k^j$  defined by

$$\xi_i^j = x^n \left( j\delta + \frac{i}{k}\delta \right) - x^n \left( j\delta + \frac{i-1}{k}\delta \right)$$

for  $i = 1, \dots, k$ . Immediately from (8) it follows that (6) is satisfied with

$$u_l^j = \Gamma \left( j\delta + \frac{l}{k}\delta \right) - \Gamma \left( j\delta + \frac{l-1}{k}\delta \right)$$

for  $l = 1, 2, \dots, k$ , because we have

$$\begin{aligned} & P(\{|S_j - S_i| \geq \lambda\}) \\ &= P \left( \left\{ \left| x^n \left( j\delta + \frac{i}{k}\delta \right) - x^n \left( j\delta + \frac{j}{k}\delta \right) \right| \geq \lambda \right\} \right) \\ &\leq \frac{1}{\lambda^\gamma} E \left| \Gamma \left( j\delta + \frac{i}{k}\delta \right) - \Gamma \left( j\delta + \frac{j}{k}\delta \right) \right|^\alpha \\ &= \frac{1}{\lambda^\gamma} E \left( \sum_{i < l \leq j} \left[ \Gamma \left( j\delta + \frac{l}{k}\delta \right) - \Gamma \left( j\delta + \frac{l-1}{k}\delta \right) \right] \right)^\alpha \\ &= \frac{1}{\lambda^\gamma} E \left( u_{i+1}^j + \dots + u_j^j \right)^\alpha. \end{aligned}$$

Therefore, by virtue of Proposition 2 we get

$$\begin{aligned} & P \left( \left\{ \max_{0 \leq i \leq k} \left| x^n \left( j\delta + \frac{i}{k}\delta \right) - x^n(j\delta) \right| \geq \varepsilon \right\} \right) \\ & \leq \frac{K'_{\gamma, \alpha}}{\varepsilon^\gamma} E(u_1 + \dots + u_k)^\alpha \\ & = \frac{K'_{\gamma, \alpha}}{\varepsilon^\gamma} E[\Gamma((j+1)\delta) - \Gamma(j\delta)]^\alpha. \end{aligned}$$

Similarly as in ([1], Theorem II.12.3), by continuity of  $x^n$ , hence it follows

$$\begin{aligned} & P \left( \left\{ \sup_{j\delta \leq s \leq (j+1)\delta} |x^n(s) - x^n(j\delta)| \geq \varepsilon \right\} \right) \\ & \leq \frac{K'_{\gamma, \alpha}}{\varepsilon^\gamma} E[\Gamma((j+1)\delta) - \Gamma(j\delta)]^\alpha. \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{j < \delta^{-1}} P \left( \left\{ \sup_{j\delta \leq s \leq (j+1)\delta} |x^n(s) - x^n(j\delta)| \geq \varepsilon \right\} \right) \\ & \leq \frac{K'_{\gamma, \alpha}}{\varepsilon^\gamma} E \left\{ \Lambda_\alpha \sum_{j < \delta^{-1}} [\Gamma((j+1)\delta) - \Gamma(j\delta)] \right\}, \end{aligned}$$

where

$$\Lambda_\alpha = \left[ \max_{j < \delta^{-1}} |\Gamma((j+1)\delta) - \Gamma(j\delta)| \right]^{\alpha-1}.$$

Hence it follows

$$\begin{aligned} & \sum_{j < \delta^{-1}} P \left( \left\{ \sup_{j\delta \leq s \leq (j+1)\delta} |x^n(s) - x^n(j\delta)| \geq \varepsilon \right\} \right) \\ & \leq \frac{K'_{\gamma, \alpha}}{\varepsilon^\gamma} E [\Lambda_\alpha (\Gamma(1) - \Gamma(0))], \end{aligned}$$

because  $\sum_{j < \delta^{-1}} [\Gamma((j+1)\delta) - \Gamma(j\delta)] \leq \Gamma(1) - \Gamma(0)$  a.s. By the continuity of a stochastic process  $\Gamma = (\Gamma(t))_{0 \leq t \leq 1}$  and the assumption  $\alpha > 1$  we get  $\lim_{\delta \rightarrow 0} H_\alpha(\omega) = 0$  for a.e.  $\omega \in \Omega$ , where  $H_\alpha(\omega) = \sup_{0 \leq t \leq 1} [\Gamma(t+\delta)(\omega) - \Gamma(t)(\omega)]^{\alpha-1}$  for  $\omega \in \Omega$ . Hence, by the properties of  $\Gamma$ , it follows that  $\lim_{\delta \rightarrow 0} E [H_\alpha (\Gamma(1) - \Gamma(0))] = 0$ . But  $\Lambda_\alpha \leq H_\alpha$  a.s. Then

$$\begin{aligned} & \sum_{j < \delta^{-1}} P \left( \left\{ \sup_{j\delta \leq s \leq (j+1)\delta} |x^n(s) - x^n(j\delta)| \geq \varepsilon \right\} \right) \\ & \leq \frac{K'_{\gamma, \alpha}}{\varepsilon^\gamma} E [H_\alpha (\Gamma(1) - \Gamma(0))]. \end{aligned}$$

Therefore for every  $\eta > 0$  there is  $\delta > 0$  such that  $\delta^{-1}$  is a positive integer and for every  $n \geq 1$  one has

$$\sum_{j < \delta^{-1}} P \left( \left\{ \sup_{j\delta \leq s \leq (j+1)\delta} |x^n(s) - x^n(j\delta)| \geq \varepsilon \right\} \right) \leq \eta.$$

Then (9) is satisfied for every  $n \geq 1$ , which together with the tightness of a sequence  $(x^n(0))_{n=1}^\infty$ , implies the tightness of  $(x^n)_{n=1}^\infty$ . ■

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