Discussiones Mathematicae
Probability and Statistics 36 (2016) 25–41
doi:10.7151/dmps.1183

K-TH RECORD VALUES FROM DAGUM DISTRIBUTION AND CHARACTERIZATION

Devendra Kumar

Department of Statistics
Amity Institute of Applied Sciences
Amity University Noida-201 303, India

e-mail: devendrastats@gmail.com

Abstract

In this study, we gave some new explicit expressions and recurrence relations satisfied by single and product moments of k-th lower record values from Dagum distribution. Next we show that the result for the record values from the Dagum distribution can be derived from our result as special case. Further, using a recurrence relation for single moments and conditional expectation of record values we obtain characterization of Dagum distribution. In addition, we use the established explicit expression of single moment to compute the mean, variance, coefficient of skewness and coefficient of kurtosis. Finally, we suggest two applications.

Keywords: sample, order statistics, lower record values, single moments, product moments, recurrence relations, Dagum distribution, characterization.

2010 Mathematics Subject Classification: 62G30, 62G99, 62E10.

1. Introduction

The Dagum distribution was introduced by Dagum [3] it is also called the inverse Burr XII distribution. The Burr XII distribution is widely known in various fields of science, the Dagum distribution is not much popular, perhaps, because of its difficult mathematical tractability. Dagum proposed his model as income distribution, its properties have been appreciated in economics and financial fields and its features have been extensively discussed in the studies of income and wealth. For more details and its applications on this distribution one may refer to Kleiber and Kotz [4] and Kleiber [5].

A random variable X is said to have Dagum distribution if its probability density function (*pdf*) is given by

$$f(x) = \alpha\beta\theta x^{-(\theta+1)}(1+\alpha x^{-\theta})^{-(\beta+1)}, \quad x > 0,\ \alpha, \beta, \theta > 0 \tag{1}$$

and the corresponding cumulative distribution function (*cdf*) is

$$F(x) = (1+\alpha x^{-\theta})^{-\beta}, \quad x > 0,\ \alpha, \beta, \theta > 0. \tag{2}$$

Therefore, in view of (1) and (2), we have

$$\alpha\beta\theta F(x) = x(\alpha + x^{\theta})f(x). \tag{3}$$

Here α is the scale parameter, while β and θ are shape parameters. For $\beta = 1$, the above distribution corresponds to the log-logistic distribution. The Dagum distribution has positive asymmetry, and it is unimodal for $\beta\theta > 1$ and zero-modal for $\beta\theta \leq 1$. The relation (3) will be used to derive some simple relations for the single and product moments of kth lower record values from the Dagum distribution. These recurrence relations will enable one to obtain all the single and product moments in a simple recursive manner.

The model of record statistics defined by Chandler [12] as a model for successive extremes in a sequence of independent and identically distributed (*iid*) random variables. This model takes a certain dependence structure into consideration. That is, the life-length distribution of the components in the system may change after each failure of the components. For this type of model, we consider the lower record statistics. If various voltages of equipment are considered, only the voltages less than the previous one can be recorded. These recorded voltages are the lower record value sequence. Record values are found in many situations of daily life as well as in many statistical applications. Often we are interested in observing new records and in recording them for, example, Olympic records or world records in sport. Record values are used in reliability theory. Moreover, these statistics are closely connected with the occurrences times of some corresponding nonhomogeneous Poisson process used in shock models. Feller [23] gave some examples of record values with respect to gambling problems. Resnick [21] discussed the asymptotic theory of records.

Theory of record values and its distributional properties have been extensively studied in the literature, Ahsanullah [13], Balakrishnan and Ahsanullah [15, 16, 17], Balakrishnan *et al*. [18], Grudzień and Szynal [24], Arnold *et al*. [1, 2], Kumar [7] and Kumar *et al*. [8]. Pawlas and Szynal [19, 20] and Saran and Singh [10] have established recurrence relations for single and product moments of kth record values from Weibull, Gumbel and linear exponential distribution. Kumar and Khan [6] are established recurrence relations for moments of kth record values from generalized beta II distribution. Kamps [22] investigated the importance of recurrence relations of order statistics in characterization.

Let $\{X_n, n \geq 1\}$ be a sequence of identically independently distributed random variables with *cdf* $F(x)$ and *pdf* $f(x)$. The jth order statistic of a sample $(X_1, X_2, \ldots, X_n)$ is denoted by $X_{j:n}$. For a fixed $k \geq 1$ we define the sequence $\{L^{(k)}(n), n \geq 1\}$ of $k-$th lower record times of $X_1, X_1 \ldots$ as follows:

$$L^{(k)}(1) = 1,$$
$$L^{(k)}(n+1) = \min\{j > L^{(k)}(n) : X_{k:L^{(k)}(n)+k-1} > X_{k:j+k-1}\}.$$

The sequences $\{Z_n^{(k)}, n \geq 1\}$ with $Z_n^{(k)} = X_{k:L^{(k)}(n)+k-1}$, $n = 1, 2, \ldots$, are called the sequences of kth lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Z_0^{(k)} = 0$. Note that $k = 1$ we have $Y_n^{(1)} = X_{L(n)}$, $n \geq 1$, i.e., record values of $\{X_n, n \geq 1\}$.

The joint *pdf* of kth lower record values $Z_1^{(k)}, Z_2^{(k)}, \ldots, Z_n^{(k)}$ can be given as the joint *pdf* of kth upper record values of $\{-X_n, n \geq 1\}$, Pawlas and Szynal [19]

$$f_{z_1^{(k)},\ldots,z_n^{(k)}}(z_1, \ldots, z_n) = k^n \left(\prod_{i=1}^{n-1} \frac{f(z_i)}{F(z_i)}\right) [F(z_n)]^{k-1} f(z_n), \quad z_1 > z_2 > \cdots > z_n.$$

In view of above equation, the marginal *pdf* of $X_{L(n)}^{(k)}$, $n \geq 1$ is given by

$$f_{X_{L(n)}^{(k)}}(x) = \frac{k^n}{(n-1)!}[-ln(F(x))]^{n-1}[F(x)]^{k-1}f(x), \tag{4}$$

and the joint *pdf* of $X_{L(m)}^{(k)}$ and $X_{L(n)}^{(k)}$, $1 \leq m < n$, $n > 2$ is given by

$$\begin{aligned} f_{X_{L(m)}^{(k)}, X_{L(n)}^{(k)}}(x, y) &= \frac{k^n}{(m-1)!(n-m-1)!}[-ln(F(x))]^{m-1} \\ &\times [-ln(F(y)) + ln(F(x))]^{n-m-1}[F(y)]^{k-1} \\ &\times \frac{f(x)}{F(x)} f(y), \ x > y, \quad 1 \leq m < n, \quad n \geq 2. \end{aligned} \tag{5}$$

Let $\{X_n, n \geq 1\}$ be a sequence of *iid* continuous random variables with *cdf* $F(x)$ and *pdf* $f(x)$. Let $X_{L(n)}$ be the nth lower record values, then the conditional *pdf* of $X_{L(n)}$ given $X_{L(m)} = x$, $1 \leq m < n$ in view of (4) and (5), is

$$f_{L(n)|L(m)}(y|x) = \frac{1}{(n-m-1)!}[-lnF(y) + lnF(x)]^{n-m-1} \frac{f(y)}{F(x)}, y < x. \tag{6}$$

We shall denote

$$\mu^{(r)}_{L(n):k} = E((X^{(k)}_{L(n)})^r) = \int_{-\infty}^{\infty} x^r f_{X^{(k)}_{L(n)}}(x)dx, \quad r, n = 1, 2, \ldots,$$

$$\mu^{(r,s)}_{L(m,n):k} = E((X^{(k)}_{L(m)})^r, (X^{(k)}_{L(n)})^s) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} x^r y^s f_{X^{(k)}_{L(m)}, X^{(k)}_{L(n)}}(x, y)dydx,$$

$$1 \le m \le n-1 \quad and \quad r, s = 1, 2, \ldots,$$

$$\mu^{(r,0)}_{L(m,n):k} = E((X^{(k)}_{L(m)})^r) = \mu^{(r)}_{L(m):k}, \quad 1 \le m \le n-1 \quad and \quad r = 1, 2, \ldots,$$

$$\mu^{(0,s)}_{L(m,n):k} = E((X^{(k)}_{L(n)})^s) = \mu^{(s)}_{L(n):k}, \quad 1 \le m \le n-1 \quad and \quad s = 1, 2, \ldots .$$

The paper is organized as follows. Section 2 gives some explicit expressions and recurrence relations for the single and product moments of kth lower record values from the Dagum distribution. In Section 3, some explicit expressions and recurrence relations for product moments of kth lower record values from Dagum distribution are derived. A characterization of the distribution based on recurrence relation for single moments and conditional expectation of record values is established in Section 4. Numerical results are presented in Section 5. Two applications is provided in Section 6. Section 7 contains a brief conclusion.

2. Relations for single moments

In this section we will derived the explicit expressions and recurrence relations for single moments of the kth lower record values from the Dagum distribution. We shall first establish the explicit expression for the single moment of kth lower record values by the following Theorem.

Theorem 1. *For the Dagum distribution given in* (2) *with fixed parameters* $\alpha, \beta, \theta > 0$, $k, n = 1, 2, \ldots,$ *and real* r *satisfying* $\frac{r}{\theta} + \beta k > 0$, *we have*

$$\mu^{(r)}_{L(n):k} = \varphi(n, \alpha, \beta, \theta, r) = (\beta k)^n \alpha^{r/\theta} \sum_{p=0}^{\infty} \frac{(r/\theta)_{(p)}}{p!\ [\beta k + p + (r/\theta)]^n}, \tag{7}$$

where

$$(t)_{(p)} = \begin{cases} t(t+1)\ldots(t+p-1), & p = 1, 2, \ldots \\ 1, & p = 0. \end{cases}$$

Proof. From (4), we have

$$\mu^{(r)}_{L(n):k} = \frac{k^n}{(n-1)!} \int_0^{\infty} x^r [F(x)]^{k-1} [-ln(F(x))]^{n-1} f(x)dx \tag{8}$$

$$
\begin{aligned}
&= \frac{\alpha^{r/\theta}(\beta k)^n}{(n-1)!}\int_0^1 z^{(r/\theta)+\beta k-1}(1-z)^{-r/\theta}[-lnz]^{n-1}dz \\
&= \frac{\alpha^{r/\theta}(\beta k)^n}{(n-1)!}\sum_{p=0}^{\infty}\frac{(r/\theta)_{(p)}}{p!}\int_0^1 z^{(r/\theta)+\beta k+p-1}[-lnz]^{n-1}dz \\
&= \frac{\alpha^{r/\theta}(\beta k)^n}{(n-1)!}\sum_{p=0}^{\infty}\frac{(r/\theta)_{(p)}}{p!}\int_0^\infty e^{-[(r/\theta)+\beta k+p]w}\ w^{n-1}dw,
\end{aligned}
$$

where $z = [F(x)]^{1/\beta}$ and $w = -lnz$. The result follows from the definition of the complete gamma function. ■

Specially, the first moment (mean) of the nth record values is

$$\mu_{L(n):k} = \mu^{(1)}_{L(n):k} = \varphi(n, \alpha, \beta, \theta, 1).$$

In addition, the variance of $L(n)$ is found to be

$$\sigma^2_{L(n):k} = \mu^{(2)}_{L(n):k} - [\mu^{(1)}_{L(n):k}]^2 = \varphi(n, \alpha, \beta, \theta, 2) - [\varphi(n, \alpha, \beta, \theta, 1)]^2.$$

Remark 2. Setting $k = 1$ in (7) we deduce the explicit expression for single moments of lower record values from the Dagum distribution.

Recurrence relations for single moments of kth lower record values from *pdf* (2) can be derived in the following theorem.

Theorem 3. *Under assumptions of Theorem* 1, *and convention* $\mu^{(0)}_{L(n):k} = 1$, $\mu^{(r)}_{L(0):k} = 0$, *we have*

$$\frac{1}{\alpha}\mu^{(r+\theta)}_{L(n):k} = \frac{\beta\theta k}{r}\mu^{(r)}_{L(n-1):k} - \left(1 + \frac{\beta\theta k}{r}\right)\mu^{(r)}_{L(n):k}. \tag{9}$$

Proof. Integrating (8) by parts taking $[F(x)]^{k-1}f(x)$ as the part to be integrated and the rest of the integrand for differentiation, we get

$$
\begin{aligned}
\mu^{(r)}_{L(n):k} &= \mu^{(r)}_{L(n-1):k} - \frac{rk^n}{k(n-1)!}\int_0^\infty x^{r-1}[F(x)]^k[-ln(F(x))]^{n-1}dx \\
&= \mu^{(r)}_{L(n-1):k} - \frac{rk^n}{k(n-1)!}\int_0^\infty x^{r-1}[F(x)]^{k-1}\left(\frac{\alpha x + x^{\theta+1}}{\alpha\beta\theta}\right) \\
&\times\ f(x)[-ln(F(x))]^{n-1}dx
\end{aligned}
$$

$$= \mu^{(r)}_{L(n-1):k} - \frac{rk^n}{\beta\theta k(n-1)!}\left\{ \int_0^\infty x^r [F(x)]^{k-1} f(x)[-ln(F(x))]^{n-1} dx \right.$$

$$\left. + \frac{1}{\alpha}\int_0^\infty x^{r+\theta}[F(x)]^{k-1} f(x)[-ln(F(x))]^{n-1}dx \right\}.$$

The result follows. ■

Remark 4. Setting $k = 1$ in (9) we deduce the recurrence relation for single moments of lower record values from the Dagum distribution.

The recurrence relations and identities have great significance because they are useful in reducing the number of operations necessary to obtain a general form for the function under consideration and they reduce the amount of direct computation, time and labour. This concept is well-established in the statistical literature, see Arnold *et al.* [1]. Furthermore, they are used in characterizing distributions, which is an important area, permitting the identification of population distribution from the properties of the sample.

3. Relations for product moments

In this section we will derived the explicit expression and recurrence relations for product moments of the kth lower record values from the Dagum distribution. We shall first establish the explicit expression for the product moment of kth lower record values by the following Theorem.

Theorem 5. *For the Dagum distribution given in* (2) *with fixed parameters* $\alpha, \beta, \theta > 0$, $k, n = 1, 2, \ldots,$ *and real* r *and* s *satisfying* $\frac{s}{\theta} + \beta k > 0$, $\frac{s+r}{\theta} + \beta k > 0$, *we have*

$$\mu^{(r,s)}_{L(m,n):k} = (\beta k)^n \alpha^{(r+s)/\theta} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(s/\theta)_{(p)}}{p!\ q![\beta k + p + (s/\theta)]^{n-m}}$$

$$\times \frac{(r/\theta)_{(q)}}{[\beta k + p + q + (r+s)/\theta]^m}. \tag{10}$$

Proof. From (1.5), we have

$$\mu^{(r,s)}_{L(m,n):k} = \frac{k^n}{(m-1)!(n-m-1)!}\int_0^\infty x^r[-ln(F(x))]^{m-1}\frac{f(x)}{F(x)} I(x)dx, \tag{11}$$

where

$$I(x) = \int_0^x y^s[-ln(F(y)) + ln(F(x))]^{n-m-1}[F(y)]^{k-1} f(y)dy. \tag{12}$$

By setting $w = [-ln(F(y)) + ln(F(x))]$ in (12), we obtain

$$I(x) = \beta^{n-m}\alpha^{s/\theta}\sum_{p=0}^{\infty}\frac{(s/\theta)_{(p)}\ [F(x)]^{k+\{(s/\theta)+p\}/\beta}\ (n-m-1)!}{p![\beta k+(s/\theta)+p]^{n-m}}.$$

On substituting the above expression of $I(x)$ in (11) and simplifying the resulting equation, we obtain the result given in (10). ■

For simplicity, we denote the (1, 1)th moment of $\mu_{L(m):k}$ and $\mu_{L(n):k}$, which is also called the simple product moment, by $\mu_{L(m,n):k}$. The simple product moments are used for evaluating the covariances, in other words

$$\sigma_{L(m,n):k} = cov(\mu_{L(m):k}, \mu_{L(n):k}) = \mu_{L(m,n):k} - \mu_{L(m):k}\ \mu_{L(n):k}.$$

Remark 6. Setting $k = 1$ in (10) we deduce the explicit expression for product moments of lower record values from the Dagum distribution.

Making use of (2), we can drive recurrence relations for product moments of kth lower record values.

Theorem 7. *Under assumptions of Theorem* 5*, we have*

$$\frac{1}{\alpha}\mu^{(r,s+\theta)}_{L(m,n):k} = \frac{\beta\theta k}{s}\mu^{(r,s)}_{L(m,n-1):k} - \left(1+\frac{\beta\theta k}{s}\right)\mu^{(r,s)}_{L(m,n):k}. \tag{13}$$

Proof. Integrating (12) by parts treating $[F(y)]^{k-1}f(y)$ for integration and the rest of the integrand for differentiation, and substituting the resulting expression in (11), we get

$$\begin{aligned}
\mu^{(r,s)}_{L(m,n):k} &= \mu^{(r,s)}_{L(m,n-1):k} - \frac{sk^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_0^x x^r y^{s-1}[-ln(F(x))]^{m-1} \\
&\times [-ln(F(y)) + ln(F(x))]^{n-m-1}[F(y)]^k\frac{f(x)}{F(x)}dydx \\
&= \mu^{(r,s)}_{L(m,n-1):k} - \frac{sk^n}{(m-1)!(n-m-1)!}\int_0^\infty\int_0^x x^r y^{s-1}[-ln(F(x))]^{m-1} \\
&\times [-ln(F(y)) + ln(F(x))]^{n-m-1}[F(y)]^{k-1}\left(\frac{\alpha y + y^{\theta+1}}{\alpha\beta\theta}\right)f(y)\frac{f(x)}{F(x)}dydx \\
&= \mu^{(r,s)}_{L(m,n-1):k} - \frac{s\ k^n}{\beta\theta k(m-1)!(n-m-1)!}\left\{\int_0^\infty\int_0^x x^r y^s\right. \\
&\times [-ln(F(x))]^{m-1}[-ln(F(y)) + ln(F(x))]^{n-m-1}[F(y)]^{k-1}
\end{aligned}$$

$$\times\ f(y)\frac{f(x)}{F(x)}dydx+\frac{1}{\alpha}\int_0^\infty\int_0^x x^r y^{s+\theta}[-ln(F(x))]^{m-1}$$
$$\times\ [-ln(F(y))+ln(F(x))]^{n-m-1}[F(y)]^{k-1}f(y)\frac{f(x)}{F(x)}dydx\Bigg\}.$$

The result follows. ∎

As a check, put $s=0$ in (10), (13) and use (9), we have $\mu^{(r,0)}_{L(m,n):k}=\mu^{(r)}_{L(n):k}$.

Remark 8. Setting $k=1$ in (13), we deduce the recurrence relation for product moments of lower record values from the Dagum distribution.

4. Characterization

This section contains two characterization Theorems of Dagum distribution based on recurrence relation for kth lower record values and conditional expectation of lower record values.

Let $L(a,\ b)$ stand for the space of all integrable functions on $(a,\ b)$. A sequence $(h_n)\subset L(a,\ b)$ is called complete on $L(a,\ b)$ if for all functions $g\in L(a,\ b)$ the condition

$$\int_a^b g(x)h_n(x)dx=0,\qquad n\in N,$$

implies $g(x)=0$ a.e. on $(a,\ b)$. We start with the following result of Lin [9].

Proposition 9. *Let n_0 be any fixed non-negative integer, $-\infty\le a<b\le\infty$ and $g(x)\ge 0$ an absolutely continuous function with $g'(x)\ne 0$ a.e. on (a,b). Then the sequence of functions $\{(g(x))^n e^{-g(x)},\ n\ge n_0\}$ is complete in $L(a,b)$ iff $g(x)$ is strictly monotone on $(a,\ b)$.*

Using the above Proposition 9 we get a stronger version of Theorem 3.

Theorem 10. *Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $F(x)>0$ for all $x>0$. Then*

$$\frac{1}{\alpha}\mu^{(r+\theta)}_{L(n):k}=\frac{\beta\theta k}{r}\mu^{(r)}_{L(n-1):k}-\left(1+\frac{\beta\theta k}{r}\right)\mu^{(r)}_{L(n):k}\tag{14}$$

holds for fixed $\alpha,\beta,\theta>0$, positive integer k and all positive integer n if and only if

$$F(x)=(1+\alpha x^{-\theta})^{-\beta},\qquad x>0.$$

Proof. The necessary part follows immediately from equation (9). On the other hand if the recurrence relation in equation (14) is satisfied, then using equations (4), we have

$$
\begin{aligned}
&\frac{k^n}{(n-1)!}\int_0^\infty x^r[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx \\
&= \frac{(n-1)k^n}{k(n-1)!}\int_0^\infty x^r[F(x)]^{k-1}[-ln(F(x))]^{n-2}f(x)dx \\
&-\frac{r\ k^n}{\beta\theta k(n-1)!}\int_0^\infty x^r[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx \\
&-\frac{r\ k^n}{\alpha\beta\theta k(n-1)!}\int_0^\infty x^{r+\theta}[F(x)]^{k-1}[-ln(F(x))]^{n-1}f(x)dx.
\end{aligned} \tag{15}
$$

Integrating the first integral on the right hand side of equation (15), by parts, we get

$$\frac{r\ k^n}{k(n-1)!}\int_0^\infty x^r[F(x)]^{k-1}[-ln(F(x))]^{n-1}\left\{F(x)-\frac{x}{\beta\theta}f(x)-\frac{x^{\theta+1}}{\alpha\beta\theta}f(x)\right\}dx=0.$$

It now follows from Proposition with $g(x) = [-ln(F(x))]$ that

$$x(\alpha + x^\theta)f(x) = \alpha\beta\theta F(x)$$

or

$$\frac{f(x)}{F(x)} = \frac{\alpha\beta\theta}{x(\alpha+x^\theta)} = \frac{\alpha\beta\theta x^{-(\theta+1)}}{(1+\alpha x^{-\theta})}. \tag{16}$$

Integrating both sides of (16), we get

$$\int\frac{f(x)}{F(x)}dx = \int\frac{\alpha\beta\theta x^{-(\theta+1)}}{(1+\alpha x^{-\theta})}dx.$$

or

$$logF(x) = -\beta log(1+\alpha x^{-\theta}),$$

which proves that

$$F(x) = (1+\alpha x^{-\theta})^{-\beta}, \qquad x > 0.$$ ■

Theorem 11. *Let X be an absolutely continuous random variable with cdf $F(x)$ and pdf $f(x)$ on the support $(0,\infty)$, then for $m < n$,*

$$E[X_{L(n)}|X_{L(m)} = x] = \alpha^{1/\theta}\sum_{p=0}^{\infty}\frac{(1/\theta)_{(p)}}{p!(1+\alpha x^{-\theta})^{p+1}}\left(\frac{p}{\beta}+\frac{1}{\beta\theta}+1\right)^{m-n}, \tag{17}$$

if and only if

$$F(x) = (1+\alpha x^{-\theta})^{-\beta}, \qquad x > 0.$$

Proof. From (6), we have

$$\begin{aligned}
E[X_{L(n)}|X_{L(m)} = x] &= \frac{1}{(n-m-1)!}\int_0^x y\Big[ln\Big(\frac{F(x)}{F(y)}\Big)\Big]^{n-m-1}\frac{f(y)}{F(x)}dy \\
&= \frac{\alpha^{1/\theta}}{(n-m-1)!}\int_0^\infty \Big[(1+\alpha x^{-\theta})e^{\frac{t}{\beta}} - 1\Big]^{-1/\theta} \times t^{n-m-1}e^{-t}dt \\
&= \frac{\alpha^{1/\theta}}{(n-m-1)!}\int_0^\infty (1+\alpha x^{-\theta})^{-1/\theta}\ e^{\frac{-t}{\beta\theta}} \\
&\times \left(1 - \frac{e^{-t/\beta}}{1+\alpha x^{-\theta}}\right)^{-1/\theta} t^{n-m-1}\ e^{-t}dt \\
&= \frac{1}{(n-m-1)!}\left(\frac{\alpha}{1+\alpha x^{-\theta}}\right)^{1/\theta}\sum_{p=0}^{\infty}\frac{(1/\theta)_{(p)}}{p!(1+\alpha x^{-\theta})^p} \\
&\times \int_0^\infty t^{n-m-1}\ exp\left[-\left(\frac{p}{\beta}+\frac{1}{\beta\theta}+1\right)t\right]dt, \quad p \geq \beta,
\end{aligned}$$

where $t = ln\Big(\frac{F(x)}{F(y)}\Big)$. The result follows from the definition of the complete gamma function.

To prove sufficient part, we have from (6) and (17)

$$\frac{1}{(n-m-1)!}\int_0^x y[-ln(F(y)) + ln(F(x))]^{n-m-1}f(y)dy = F(x)H_{n-m}(x), \tag{18}$$

where

$$H_{n-m}(x) = \alpha^{1/\theta}\sum_{p=0}^{\infty}\frac{(1/\theta)_{(p)}}{p!(1+\alpha x^{-\theta})^{p+1}}\left(\frac{p}{\beta}+\frac{1}{\beta\theta}+1\right)^{m-n}.$$

Differentiating (18) both sides with respect to x, we get

$$\frac{1}{(n-m-2)!}\int_0^x y[-ln(F(y)) + ln(F(x))]^{n-m-2}\frac{f(x)}{F(x)}f(y)dy$$

$$= f(x)H_{n-m}(x) + F(x)H'_{n-m}(x),$$

$$\frac{f(x)}{F(x)} = \frac{H'_{n-m}(x)}{[H_{n-m-1}(x) - H_{n-m}(x)]} = \frac{\alpha\beta\theta}{x(\alpha + x^{\theta})},$$

which proves that

$$F(x) = (1+\alpha x^{-\theta})^{-\beta}, \qquad x > 0.$$

■

5. Numerical results

Here, we investigate how the moments of lower record values from the Dagum distribution vary with respect to α, β and θ. The recurrence relations obtained in the preceding sections allow us to evaluate the means, variances and covariances of all order statistics for all sample sizes in a simple recursive manner and can be used for various inferential purposes; for example, they are useful in determining BLUEs of location/scale parameters and best linear unbiased predictors (BLUPs) of censored failure times. More details on BLUEs and BLUPs based on order statistics can be seen in (Balakrishnan and Cohen [14], Arnold *et al.* [1]).

In Table 1 we have computed the values of first four moments for $r = 1(1)4$, $\theta = 2(1)5$ and $\alpha,\ \beta = 1, 2$. From Table 1, one can see that the moments are decreasing with respect to n. In Table 2 we have computed the variances for $\theta = 3(1)5$, $\alpha, \beta = 1, 2$ and different values of n. We can see that variances are decreasing with respect to n and θ but increasing with respect to α and β. Table 3 shows the coefficients of skewness and coefficient of kurtosis.

Table 1. First four moments of lower records.

	$r=1,\ \theta=2$				$r=1,\ \theta=3$			
	$\alpha=1$		$\alpha=2$			$\alpha=1$	$\alpha=2$	
n	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$
1	1.48440	2.18374	2.09926	3.08828	1.19098	1.57591	1.50054	1.98552
2	0.60662	1.03430	0.85789	1.46271	0.67084	0.98272	0.84521	1.23816
3	0.34486	0.67698	0.48771	0.95739	0.45858	0.74511	0.57778	0.93879
4	0.21422	0.48870	0.30295	0.69112	0.33029	0.59960	0.41614	0.75545
5	0.13779	0.36811	0.19486	0.52058	0.24283	0.49536	0.30595	0.62412
6	0.09010	0.28356	0.12742	0.40102	0.18024	0.41499	0.22709	0.52285
7	0.05942	0.22132	0.08403	0.31300	0.13443	0.35049	0.16937	0.44159
8	0.03937	0.17415	0.05567	0.24629	0.10051	0.29750	0.12663	0.37483
9	0.02615	0.13776	0.03698	0.19483	0.07525	0.25334	0.09481	0.31918
10	0.01740	0.10936	0.02460	0.15466	0.05638	0.21618	0.07104	0.27237
	$r=1,\ \theta=4$				$r=1,\ \theta=5$			
	$\alpha=1$		$\alpha=2$			$\alpha=1$	$\alpha=2$	
n	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$
1	1.10292	1.37284	1.31160	1.63259	1.06449	1.27384	1.22278	1.46325
2	0.72221	0.97246	0.85886	1.15646	0.76090	0.97097	0.87404	1.11535
3	0.54159	0.79156	0.64406	0.94133	0.60350	0.82418	0.69324	0.94673
4	0.42136	0.67239	0.50108	0.79961	0.49240	0.72321	0.56562	0.83075
5	0.33257	0.58212	0.39550	0.69226	0.40621	0.64411	0.46662	0.73989
6	0.26423	0.50907	0.31422	0.60539	0.33680	0.57821	0.38688	0.66418
7	0.21062	0.44780	0.25047	0.53253	0.27992	0.52142	0.32155	0.59896
8	0.16817	0.39533	0.19998	0.47013	0.23294	0.47153	0.26758	0.54165
9	0.13439	0.34980	0.15982	0.41599	0.19398	0.42718	0.22282	0.49070
10	0.10745	0.30999	0.12778	0.36864	0.16158	0.38744	0.18561	0.44505

			$r=2,\ \theta=3$			$r=2,\ \theta=4$		
	$\alpha=1$		$\alpha=2$			$\alpha=1$	$\alpha=2$	
n	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$
1	2.01912	3.23328	3.20516	5.13250	1.48440	2.18374	2.09926	3.08828
2	0.58229	1.13446	0.92433	1.80085	0.60662	1.03430	0.85789	1.46271
3	0.27452	0.63544	0.43578	1.00870	0.34486	0.67698	0.48771	0.95739
4	0.14788	0.41093	0.23474	0.65231	0.21422	0.48870	0.30295	0.69112
5	0.08391	0.28250	0.13320	0.44843	0.13779	0.36811	0.19486	0.52058
6	0.04881	0.20053	0.07748	0.31832	0.09010	0.28356	0.12742	0.40102
7	0.02877	0.14503	0.04566	0.23023	0.05942	0.22132	0.08403	0.31300
8	0.01708	0.10613	0.02711	0.16847	0.03937	0.17415	0.05567	0.24629
9	0.01018	0.07826	0.01616	0.12423	0.02615	0.13776	0.03698	0.19483
10	0.00608	0.05800	0.00966	0.09207	0.01740	0.10936	0.02460	0.15466
			$r=2,\ \theta=5$			$r=3,\ \theta=4$		
	$\alpha=1$		$\alpha=2$			$\alpha=1$	$\alpha=2$	
n	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$
1	1.28688	1.78113	1.69805	2.35021	2.42946	4.02799	4.08585	6.77424
2	0.63966	0.99817	0.84404	1.31710	0.58310	1.20732	0.98066	2.03047
3	0.40620	0.71424	0.53598	0.94244	0.24975	0.62309	0.42002	1.04791
4	0.27551	0.55037	0.36354	0.72622	0.12542	0.38105	0.21094	0.64085
5	0.19178	0.43812	0.25306	0.57810	0.06700	0.25030	0.11268	0.42095
6	0.13513	0.35482	0.17830	0.46818	0.03686	0.17066	0.06200	0.28701
7	0.09580	0.29025	0.12641	0.38299	0.02060	0.11892	0.03465	0.20000
8	0.06814	0.23892	0.08991	0.31525	0.01162	0.08400	0.01954	0.141287
9	0.04856	0.19745	0.06407	0.26053	0.00658	0.05987	0.01107	0.10069
10	0.03464	0.16361	0.04570	0.21588	0.00374	0.04293	0.00630	0.07220
			$r=3,\ \theta=5$			$r=4,\ \theta=5$		
	$\alpha=1$		$\alpha=2$			$\alpha=1$	$\alpha=2$	
n	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$
1	1.76714	2.74222	2.67849	4.15643	2.74017	4.62814	4.77091	8.05807
2	0.58776	1.08776	0.89088	1.64874	0.58767	1.25990	1.02320	2.19361
3	0.29883	0.64923	0.45294	0.98404	0.23736	0.61816	0.41327	1.07628
4	0.17033	0.43883	0.25817	0.66514	0.11432	0.36545	0.19904	0.63629
5	0.10154	0.31289	0.15390	0.47425	0.05893	0.23359	0.10261	0.40670
6	0.06183	0.22943	0.09372	0.34776	0.03138	0.15548	0.05464	0.27071
7	0.03807	0.17103	0.05771	0.25923	0.01701	0.10598	0.02961	0.18452
8	0.02359	0.12880	0.03576	0.19522	0.00931	0.07331	0.01620	0.12765
9	0.01467	0.09764	0.02223	0.14800	0.00512	0.05121	0.00892	0.08917
10	0.00914	0.07435	0.01385	0.11270	0.00283	0.03601	0.00493	0.06270

Table 2. Variances of lower records.

	$\theta=3$			
	$\alpha=1$		$\alpha=2$	
n	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$
1	0.600687	0.749788	0.953540	1.190210
2	0.132264	0.168721	0.209950	0.267810
3	0.064224	0.080251	0.101950	0.127373
4	0.038789	0.051410	0.061568	0.081605
5	0.024944	0.037118	0.039595	0.058904
6	0.016324	0.028313	0.025910	0.044948
7	0.010699	0.022187	0.016974	0.035228
8	0.006978	0.017624	0.011075	0.027972
9	0.004517	0.014079	0.007171	0.022354
10	0.002901	0.011266	0.004613	0.017885
	$\theta=4$			
	$\alpha=1$		$\alpha=2$	
n	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$
1	0.267967	0.299050	0.378965	0.422930
2	0.085033	0.088622	0.120250	0.125310
3	0.051540	0.050413	0.072897	0.071288
4	0.036676	0.036592	0.051869	0.051744
5	0.027187	0.029246	0.038440	0.041356
6	0.020283	0.024408	0.028686	0.034523
7	0.015059	0.020795	0.021295	0.029412
8	0.011089	0.017864	0.015678	0.025268
9	0.008089	0.015400	0.011438	0.021782
10	0.005854	0.013266	0.008272	0.018765
	$\theta=5$			
	$\alpha=1$		$\alpha=2$	
n	$\beta=1$	$\beta=2$	$\beta=1$	$\beta=2$
1	0.153741	0.158462	0.202859	0.209109
2	0.060691	0.055387	0.080094	0.073094
3	0.041988	0.034967	0.055398	0.046142
4	0.033052	0.027337	0.043614	0.036074
5	0.026773	0.023242	0.035326	0.030663
6	0.021696	0.020493	0.028624	0.027045
7	0.017445	0.018371	0.023016	0.024237
8	0.013879	0.016579	0.018311	0.021865
9	0.010932	0.014967	0.014421	0.019744
10	0.008532	0.013500	0.011249	0.01781

Table 3. Coefficients of skewness and kurtosis based on lower record values.

	Coefficient of skewness			
	$\alpha = 1$		$\alpha = 2$	
n	$\beta = 1$	$\beta = 2$	$\beta = 1$	$\beta = 2$
1	1.160521	1.103942	1.160632	1.103571
2	0.580742	0.844494	0.579866	0.843099
3	0.349731	0.447952	0.350413	0.447455
4	0.352642	0.277884	0.352095	0.276578
5	0.430523	0.211948	0.429243	0.213247
6	0.533212	0.194638	0.535022	0.197633
7	0.645364	0.212851	0.647810	0.212017
8	0.765715	0.237504	0.768017	0.240633
9	0.882758	0.276343	0.881751	0.281873
10	0.997347	0.318764	0.999092	0.323542
	Coefficient of kurtosis			
	$\alpha = 1$		$\alpha = 2$	
n	$\beta = 1$	$\beta = 2$	$\beta = 1$	$\beta = 2$
1	4.782052	3.882246	4.781325	3.882445
2	4.128797	4.888647	4.131847	4.894499
3	3.228035	3.745018	3.226523	3.753257
4	3.000637	3.297152	3.003395	3.313524
5	2.974357	3.098906	2.991363	3.091831
6	3.082535	2.368889	3.076892	2.996861
7	3.299072	2.337784	3.271439	2.941616
8	3.535337	1.492047	3.492471	2.928391
9	3.790521	1.295355	3.837180	2.908994
10	4.176110	1.341425	4.212107	2.938245

6. Application

In this Section we suggest some application based on moments discussed in Section 2.

(i) **Estimation:** The moments of record values given in Section 2 can be used to obtain the best linear unbiased estimates (BLUEs) of the scale parameters of the Digum distribution

$$f(x) = \alpha\beta\theta x^{-(\theta+1)}(1+\alpha x^{-\theta})^{-(\beta+1)}, \quad x > 0, \ \alpha, \beta, \theta > 0. \tag{19}$$

This type of work have been done by Sultan and Moshref [11].

(ii) **Characterization:** The Dagum distribution given in (2) can be characterization by using recurrence relation and conditional expectation of record value as Theorem 10 and Theorem 11, respectively.

7. Conculding remarks

Some new explicit expressions and recurrence relations for single and product moments of kth lower record values from the Dagum distribution are established. In addition the single moments are calculated for some choices of the parameters. Further, characterization of this distribution has also been obtained on using a recurrence relation for single moments and conditional expectation of the record value. The relation between our results and some other results in literature are listed below:

1. When $\beta = 1$ we deduce

$$\mu_{L(n):k}^{(r)} = k^n \alpha^{r/\theta} \sum_{p=0}^{\infty} \frac{(r/\theta)_{(p)}}{p!\ [k+p+(r/\theta)]^n},$$

$$\frac{1}{\alpha}\mu_{L(n):k}^{(r+\theta)} = \frac{\theta k}{r}\mu_{L(n-1):k}^{(r)} - \left(1+\frac{\theta k}{r}\right)\mu_{L(n):k}^{(r)},$$

$$\begin{aligned} \mu_{L(m,n):k}^{(r,s)} &= k^n \alpha^{(r+s)/\theta} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(s/\theta)_{(p)}}{p!\ q![k+p+(s/\theta)]^{n-m}} \\ &\times \frac{(r/\theta)_{(q)}}{[k+p+q+(r+s)/\theta]^m}, \end{aligned}$$

$$\frac{1}{\alpha}\mu_{L(m,n):k}^{(r,s+\theta)} = \frac{\theta k}{s}\mu_{L(m,n-1):k}^{(r,s)} - \left(1+\frac{\theta k}{s}\right)\mu_{L(m,n):k}^{(r,s)}.$$

Which are the explicit expression and recurrence relations of the single and product moments of lower record values from log-logistic distribution.

2. The recurrence relations for the single and product moments of record values can be used to calculate the different moments for any order and sample size in a simple regressive manner. The recurrence relations reduce the round off error for calculating the moments compare with the numerical integration techniques.

Acknowledgements

The author expresses their sincere thanks to the associate editor and anonymous reviewers for their useful comments on an earlier version of this manuscript which led to this improved version.

References

[1] B.C. Arnold, N. Balakrishnan and H.N. Nagaraja, A First Course in Order Statistics (John Wiley, New York, 1992).

[2] B.C. Arnold, N. Balakrishnan and H.N. Nagaraja, Records (John Wiley, New York, 1998).

[3] C. Dagum, *A new model of personal income distribution: Specification and estimation*, Econ. Appl. **XXX** (1977) 413–436.

[4] C. Kleiber and S. Kotz, Statistical Size Distribution in Economics and Actuarial Sciences (John Wiley & Sons, Inc., Hoboken, NJ, 2003).

[5] C. Kleiber, A guide to the Dagum distribution, in Modeling Income Distributions and Lorenz Curves Series: Economic Studies in Inequality, Social Exclusion and Well-Being, 5, C. Duangkamon (Springer, New York, NY, 2008).

[6] D. Kumar and M.I. Khan, *Recurrence relations for moments of kth record values from generalized beta II distribution and a characterization*, Seluk Journal of Applied Mathematics **13** (2012) 75–82.

[7] D. Kumar, *Explicit Expressions and statistical inference of generalized Rayleigh distribution based on lower record values*, Math. Meth. Stat. **24** (2015) 225–241.

[8] D. Kumar, N. Jain and S. Gupta, *The type I generalized half logistic distribution based on upper record values*, J. Probab. Stat. **2015** (2015) 01–11.

[9] G.D. Lin, *On a moment problem*, Tohoku Math. J. **38** (1986) 595–598.

[10] J. Saran and S.K. Singh, *Recurrence relations for single and product moments of kth record values from linear exponential distribution and a characterization*, Asian J. Math. Stat. **1** (2008) 159–164.

[11] K.S. Sultan and M.E. Moshref, *Record values from generalized Pareto distribution and associated inference*, Metrika **51** (2000) 105–116.

[12] K.N. Chandler, *The distribution and frequency of record values*, J. Roy. Statist. Soc., Ser. B **14** (1952) 220–228.

[13] M. Ahsanullah, Record Statistics (Nova Science Publishers, New York, 1995).

[14] N. Balakrishnan and A.C. Cohan, Order statistics and inference: estimation methods (Boston, MA, Academic Press, 1991).

[15] N. Balakrishnan and M. Ahsanullah, *Recurrence relations for single and product moments of record values from generalized Pareto distribution*, Comm. Statist. Theory and Methods **23** (1994) 2841–28526.

[16] N. Balakrishnan and M. Ahsanullah, *Relations for single and product moments of record values from Lomax distribution*, Sankhyā Ser. B **56** (1994) 140–146.

[17] N. Balakrishnan and M. Ahsanullah, *Relations for single and product moments of record values from exponential distribution*, J. Appl. Statist. Sci. **2** (1995) 73–87.

[18] N. Balakrishnan, P.S. Chan and M. Ahsanullah, *Recurrence relations for moments of record values from generalized extreme value distribution*, Comm. Statist. Theory and Methods **22** (1993) 1471–1482.

[19] P. Pawlas and D. Szynal, *Relations for single and product moments of kth record values from exponential and Gumbel distributions*, J. Appl. Statist. Sci. **7** (1998) 53–61.

[20] P. Pawlas and D. Szynal, *Recurrence relations for single and product moments of kth record values from weibull distribution and a characterization*, J. Appl. Statist. Sci. **10** (2000) 17–25.

[21] S.I. Resnick, *Record values and related statistics*, Ann. Probab. **2** (1973) 650–662.

[22] U. Kamps, *Characterizations of distributions by recurrence relations and identities for moments of order statistics*, in: N. Balakrishnan, and C.R. Rao, Handbook of Statistics 16 Order Statistics: Theory & Methods, Alavi, Lick and Schwenk (Ed(s)), (North-Holland, Amsterdam, 1998) 291–311.

[23] W. Feller, An introduction to probability theory and its applications (John Wiley & New York, 1966).

[24] Z. Grudzień and D. Szynal, *Characterization of uniform and exponential distributions via moments of kth record values with random indices*, J. Appl. Statist. Sci. **5** (1997) 259–266.

Received 16 October 2015
Revised 12 February 2016