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MLE FOR THE γ-ORDER GENERALIZED NORMAL DISTRIBUTION

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Abstract

The introduced three parameter (position μ, scale Σ and shape γ) multivariate generalized Normal distribution (γ-GND) is based on a strong theoretical background and emerged from Logarithmic Sobolev Inequalities. It includes a number of well known distributions such as the multivariate Uniform, Normal, Laplace and the degenerated Dirac distributions. In this paper, the cumulative distribution, the truncated distribution and the hazard rate of the γ-GND are presented. In addition, the Maximum Likelihood Estimation (MLE) method is discussed in both the univariate and multivariate cases and asymptotic results are presented.

Keywords: γ-order Normal distribution, cumulative distribution, truncated distribution, hazard rate, Maximum likelihood estimation.

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1. Introduction

Recall the definition of the γ-order generalized Normal distribution (γ-GND): The p-dimensional random variable X follows the γ-GND, $\mathcal{N}_\gamma^p(\mu,\Sigma)$ with mean vector $\mu \in \mathbb{R}^p$ and scale matrix $\Sigma \in \mathbb{R}^{p\times p}$, when the density function, f_X, is of the form

$$f_X(x;\ \mu,\Sigma,\gamma) = C_\gamma^p\,|\det\Sigma|^{-1/2}\exp\left\{-\tfrac{\gamma-1}{\gamma}Q_\theta(x)^{\frac{\gamma}{2(\gamma-1)}}\right\},\quad x\in\mathbb{R}^p, \tag{1}$$

with quadratic form $Q_\theta(x) = (x-\mu)\Sigma^{-1}(x-\mu)^T$, $x\in\mathbb{R}^p$, $\theta=(\mu,\Sigma)\in\mathbb{R}^{p\times(p\times p)}$, where the normality factor C_γ^p is defined as

$$C_\gamma^p = \pi^{-p/2}\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma(p\frac{\gamma-1}{\gamma}+1)}\left(\tfrac{\gamma-1}{\gamma}\right)^{p\frac{\gamma-1}{\gamma}}. \tag{2}$$

We denote $X\sim\mathcal{N}_\gamma^p(\mu,\Sigma)$. Notice that, for $\gamma=2$, $\mathcal{N}_2^p(\mu,\Sigma)$ is the well known multivariate Normal distribution.

Consider now the multivariate and elliptically contoured Uniform $\mathcal{U}^p(\mu,\Sigma)$, Normal $\mathcal{N}^p(\mu,\Sigma)$ and Laplace $\mathcal{L}^p(\mu,\Sigma)$ distributions, as well as the degenerate Dirac distribution $\mathcal{D}^p(\mu)$. Let U, N, L and D random variables following respectively $\mathcal{U}^p$, $\mathcal{N}^p$, $\mathcal{L}^p$ and $f_\mathcal{D}$ as above, adopting the following density functions:

$$f_U(x) = \frac{\Gamma(\frac{p}{2}+1)}{(\pi^p\det\Sigma)^{1/2}},\quad x\in\mathbb{R}^p \text{ with } Q_\theta(x)\le 1, \tag{3}$$

$$f_N(x) = \frac{1}{[(2\pi)^p\det\Sigma]^{1/2}}\exp\left\{-\tfrac{1}{2}Q_\theta(x)\right\},\quad x\in\mathbb{R}^p, \tag{4}$$

$$f_L(x) = \frac{\Gamma(\frac{p}{2}+1)}{p!(\pi^p\det\Sigma)^{1/2}}\exp\left\{-Q_\theta^{1/2}(x)\right\},\quad x\in\mathbb{R}^p, \tag{5}$$

$$f_D(x) = \begin{cases} +\infty, x=\mu, \\ 0, \quad x\in\mathbb{R}^p\setminus\mu. \end{cases} \tag{6}$$

All the above distributions are members of the γ-GND family for certain values of the shape parameter γ, see [12] for details. Thus, the order value γ, eventually, "bridges" distributions with complete different shape as well as "tailing" behavior. Indeed:

Theorem 1. *The multivariate γ-GND r.v. X_γ, i.e., $X_\gamma\sim\mathcal{N}_\gamma^p(\mu,\Sigma)$ with p.d.f. f_{X_γ}, coincides for different values of the shape parameter γ with the Uniform, Normal, Laplace and Dirac distributions, as*

$$f_{X_\gamma} = \begin{cases} f_{\mathcal{D}}, & \textit{for} \quad \gamma = 0 \quad \textit{and} \quad p = 1, 2, \\ 0, & \textit{for} \quad \gamma = 0 \quad \textit{and} \quad p \geq 3, \\ f_{\mathcal{U}}, & \textit{for} \quad \gamma = 1, \\ f_{\mathcal{N}}, & \textit{for} \quad \gamma = 2, \\ f_{\mathcal{L}}, & \textit{for} \quad \gamma = \pm\infty. \end{cases} \tag{7}$$

2. C.D.F. FOR THE γ-GND

Recall that the cumulative distribution function (c.d.f.) of the standardized normally distributed $Z \sim \mathcal{N}(0,1)$ is given by

$$\Phi(z) = \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{erf}(\tfrac{z}{2}), \quad z \in \mathbb{R}, \tag{8}$$

with $\mathrm{erf}(\cdot)$ being the usual error function. For the γ-GND the generalized error function $\mathrm{Erf}_{\gamma/(\gamma-1)}$ is involved, defined as, [1],

$$\mathrm{Erf}_a(x) := \frac{\Gamma(a+1)}{\sqrt{\pi}} \int_0^x e^{-t^a}\, dt, \quad x \in \mathbb{R}. \tag{9}$$

The generalized error function, can be expressed, through the lower incomplete gamma function $q(a,x)$ or the upper (complementary) incomplete gamma function $\Gamma(a,x) = \Gamma(a) - q(a,x)$, in the form

$$\mathrm{Erf}_a(x) = \tfrac{\Gamma(a)}{\sqrt{\pi}}\gamma\left(\tfrac{1}{a}, x^a\right) = \tfrac{\Gamma(a)}{\sqrt{\pi}}\left[\Gamma\left(\tfrac{1}{a}\right) - \Gamma\left(\tfrac{1}{a}, x^a\right)\right], \quad x \in \mathbb{R}, \quad a \geq 0. \tag{10}$$

Theorem 2. *The c.d.f. F_{X_γ} of a γ-order normally distributed random variable $X_\gamma \sim \mathcal{N}_\gamma(\mu, \sigma^2)$ is given by*

$$F_{X_\gamma}(x) = \frac{1}{2} + \frac{\sqrt{\pi}}{2\,\Gamma(\frac{\gamma-1}{\gamma})\,\Gamma(\frac{\gamma}{\gamma-1})}\,\mathrm{Erf}_{\frac{\gamma}{\gamma-1}}\left\{(\tfrac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}}\tfrac{x-\mu}{\sigma}\right\} \tag{11}$$

$$= 1 - \frac{1}{2\,\Gamma(\frac{\gamma-1}{\gamma})}\,\Gamma\left(\tfrac{\gamma-1}{\gamma}, \tfrac{\gamma-1}{\gamma}(\tfrac{x-\mu}{\sigma})^{\frac{\gamma}{\gamma-1}}\right), \quad x \in \mathbb{R}. \tag{12}$$

Proof. From density function $f_{X\gamma}$, as in (1), we have

$$F_\gamma(x) = \int_0^x f_\gamma(t)dt = \frac{C_\gamma^1}{\sigma}\int_{-\infty}^x \exp\left\{-\tfrac{\gamma-1}{\gamma}\left|\tfrac{x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\} dt.$$

Applying the linear transformation $w = \frac{t-\mu}{\sigma}$, the above is reduced to

$$F_{X_\gamma}(x) = C_\gamma^1 \int\limits_{-\infty}^{\frac{x-\mu}{\sigma}} \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\} dw = \Phi_{Z_\gamma}(\tfrac{x-\mu}{\sigma}), \tag{13}$$

where Φ_{Z_γ} is the c.d.f. of the standardized γ-ordered Normal distribution with $Z_\gamma = \frac{1}{\sigma}(X_\gamma - \mu) \sim \mathcal{N}_\gamma(0,1)$. Moreover, Φ_{Z_γ} can be expressed in terms of the generalized error function. In particular

$$\Phi_{Z_\gamma}(z) = C_\gamma^1 \int\limits_{-\infty}^{z} \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\} dw = \Phi_{Z_\gamma}(0) + C_\gamma^1 \int\limits_{0}^{z} \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\} dw,$$

and as f_{Z_γ} is a symmetric density function around zero, we have

$$\Phi_{Z_\gamma}(z) = \exp\left\{-\tfrac{\gamma-1}{\gamma}|w|^{\frac{\gamma}{\gamma-1}}\right\} dw = \frac{1}{2} + C_\gamma^1 \int\limits_{0}^{z} \exp\left\{-\left|(\tfrac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}} w\right|^{\frac{\gamma}{\gamma-1}}\right\} dw,$$

and thus

$$\Phi_{Z_\gamma}(z) = \frac{1}{2} + C_\gamma^1 (\tfrac{\gamma}{\gamma-1})^{\frac{\gamma-1}{\gamma}} \int\limits_{0}^{(\frac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}} z} \exp\left\{-u^{\frac{\gamma}{\gamma-1}}\right\} du. \tag{14}$$

Substituting the normalizing factor, as in (2), and from the definition of the generalized error function, it is

$$\Phi_{Z_\gamma}(z) = \frac{1}{2} + \frac{\sqrt{\pi}}{2\,\Gamma(\frac{\gamma-1}{\gamma}+1)\,\Gamma(\frac{2\gamma-1}{\gamma-1})} \operatorname{Erf}_{\frac{\gamma}{\gamma-1}} \left\{(\tfrac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}} z\right\}, \quad z \in \mathbb{R}, \tag{15}$$

i.e., (11) holds, while (12) formed through (10). ■

It is essential for numeric calculations, to express (11) considering positive arguments for Erf. Indeed, through (14), we obtain

$$F_{X_\gamma}(x) = \frac{1}{2} + \frac{\operatorname{sgn}(x-\mu)\sqrt{\pi}}{2\,\Gamma(\frac{\gamma-1}{\gamma})\,\Gamma(\frac{\gamma}{\gamma-1})} \operatorname{Erf}_{\frac{\gamma}{\gamma-1}} \left\{(\tfrac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}} \left|\tfrac{x-\mu}{\sigma}\right|\right\}, \quad x \in \mathbb{R}, \tag{16}$$

while applying (10) into (16) we obtain

$$(17)\quad F_{X_\gamma}(x) = \frac{1+\operatorname{sgn}(x-\mu)}{2} - \frac{\operatorname{sgn}(x-\mu)}{2\,\Gamma(\frac{\gamma-1}{\gamma})}\,\Gamma\left(\tfrac{\gamma-1}{\gamma}, \tfrac{\gamma-1}{\gamma}\left|\tfrac{x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right), \quad x \in \mathbb{R},$$

where $\operatorname{sgn}(\cdot)$ is the usual sign function.

Proposition 3. *The c.d.f. of the positive-ordered $X_\gamma \sim \mathcal{N}_{\gamma>1}(\mu,\sigma^2)$ admits the following bounds, $B(x,\cdot)$, i.e.,*

$$(18)\qquad B(x;\ \tfrac{\gamma-1}{\gamma}) < F_{X_\gamma}(x) < B\left(x;\ \left[(\tfrac{\gamma-1}{\gamma})^{\frac{1}{\gamma}}\,\Gamma(\tfrac{\gamma-1}{\gamma})\right]^{\frac{\gamma-1}{\gamma}}\right), \quad x \in \mathbb{R},$$

where

$$(19)\qquad B(x;\ k) := \tfrac{1}{2} + \tfrac{1}{2}\operatorname{sgn}(x-\mu)\left(1 - \exp\left\{-k\left|\tfrac{x-\mu}{\sigma}\right|^{\frac{\gamma}{\gamma-1}}\right\}\right)^{\frac{\gamma-1}{\gamma}}, \quad k \in \mathbb{R}_+.$$

The inverted inequalities hold for the negative–ordered $X_\gamma \sim \mathcal{N}_{\gamma<0}(\mu,\sigma^2)$.

Proof. Applying the inequalities, [3],

$$(20)\qquad \Gamma(1+\tfrac{1}{a})\left[1-e^{-u(a)x^a}\right]^{1/a} < \int_0^x e^{-t^a}dt < \Gamma(1+\tfrac{1}{a})\left[1-e^{-v(a)x^a}\right]^{1/a},$$

where

$$u(a) = \begin{cases} \Gamma^{-a}(1+\frac{1}{a}), & 0<a<1,\\ 1, & a>1,\end{cases} \quad \text{and} \quad v(a) = \begin{cases} 1, & 0<a<1,\\ \Gamma^{-a}(1+\frac{1}{a}), & a>1,\end{cases}$$

into the definition of the generalized error function in (9) we obtain, through the additive identity of the gamma function, that

$$(21)\quad \tfrac{1}{\sqrt{\pi}}\,\Gamma(a)\,\Gamma(\tfrac{1}{a})\left[1-e^{-u(a)x^a}\right]^{1/a} < \operatorname{Erf}_a(x) < \tfrac{1}{\sqrt{\pi}}\,\Gamma(a)\,\Gamma(\tfrac{1}{a})\left[1-e^{-v(a)x^a}\right]^{1/a}.$$

Consider now the γ-order normally distributed $X_\gamma \sim \mathcal{N}_\gamma(\mu,\sigma^2)$ with $\gamma \in \mathbb{R}\setminus[0,1]$ and let $a = \frac{\gamma}{\gamma-1}$. Then, for the positive–ordered X_γ, i.e., for $\gamma > 1$, it is $a > 1$, while for the negative–ordered X_γ it is $0 < a < 1$. Therefore, defining $B(x;\ \cdot)$ as in (19), the bounds (18) for $\gamma > 1$ hold true, as (21) is applied to (16). For the negative–ordered case of $\gamma < 0$ the inverted bounds of (18) hold. ■

Example 4. *The c.d.f. of the normally distributed $X \sim \mathcal{N}(\mu,\sigma^2)$ admits the following bounds,*

$$\tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(x-\mu)\sqrt{1-e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}}<F_X(x)<\tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(x-\mu)\sqrt{1-e^{-\frac{2}{\pi}(\frac{x-\mu}{\sigma})^2}},$$

while for the (-1)*-ordered* $X_{-1}\sim\mathcal{N}_{-1}(\mu,\sigma^2)$*, it is*

$$\tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(x-\mu)\left(1-e^{-\sqrt{2|\frac{x-\mu}{\sigma}|}}\right)^2<F_{X_{-1}}(x)<\tfrac{1}{2}+\tfrac{1}{2}\operatorname{sgn}(x-\mu)\left(1-e^{-2\sqrt{|\frac{x-\mu}{\sigma}|}}\right)^2.$$

As the generalized error function Erf_a is defined in (10) through the upper incomplete gamma function $\Gamma(a^{-1},\cdot)$, series expansions can be used for a more "numerical–oriented" form of (10). Here we present some expansions of the usual c.d.f. of the $\mathcal{N}_\gamma$ family of distributions.

Corollary 5. *The usual c.d.f.* F_X *of* $X\sim\mathcal{N}_\gamma^1(\mu,\sigma^2)$ *can be expressed in the series expansion form*

$$F_X(x)=\tfrac{1}{2}+\frac{\left(\frac{\gamma-1}{\gamma}\right)^{\frac{\gamma-1}{\gamma}}}{\frac{2}{\gamma}\,\Gamma(\frac{\gamma-1}{\gamma})}(\tfrac{x-\mu}{\sigma})\sum_{k=0}^{\infty}\frac{\left(\frac{1-\gamma}{\gamma}|\frac{x-\mu}{\sigma}|^{\frac{\gamma}{\gamma-1}}\right)^k}{k![(k+1)\gamma-1]},\quad x\in\mathbb{R}. \tag{22}$$

Proof. Adopting the series expansion form of the lower incomplete gamma function,

$$q(a,x):=\int_0^x t^{a-1}e^{-t}dt=\sum_{k=0}^{\infty}\tfrac{(-1)^k}{k!(a+k)}x^{a+k},\quad x,a\in\mathbb{R}_+, \tag{23}$$

a series expansion form of the generalized error function is extracted through (10), i.e.

$$\operatorname{Erf}_a(x)=\tfrac{\Gamma(a+1)}{\sqrt{\pi}}\sum_{k=0}^{\infty}\tfrac{(-1)^k}{k!(ka+1)}x^{ka+1},\quad x,a\in\mathbb{R}_+. \tag{24}$$

Substituting now the series expansion form of (24) into (16) we get

$$F_{X_\gamma}(x)=\frac{1}{2}+(\gamma-1)C_\gamma^1\sum_{k=0}^{\infty}\tfrac{(-1)^k}{k!}\cdot\tfrac{(\frac{\gamma-1}{\gamma})^k}{\gamma(k+1)-1}\left|\tfrac{\log x-\mu}{\sigma}\right|^{\frac{k\gamma}{\gamma-1}+1},\quad x\in\mathbb{R}_+^*,$$

and expressing the infinite series using the integer powers k, and the fact that $\operatorname{sgn}(x)x=|x|$, $x\in\mathbb{R}$, we finally derive the series expansions as in (22). ∎

Corollary 6. *For the negative-ordered* $X \sim \mathcal{N}_\gamma(\mu, \sigma^2)$ *with* $\gamma = \frac{1}{1-n} \in \mathbb{R}_-$, $n \in \mathbb{N}$, $n \geq 2$, *we obtain the c.d.f.'s* F_X *as finite series expansion, i.e.,*

$$F_X(x) = \frac{1 + \operatorname{sgn}(x-\mu)}{2} - \frac{\operatorname{sgn}(x-\mu)}{2\exp\left\{n\left|\frac{x-\mu}{\sigma}\right|^{1/n}\right\}} \sum_{k=0}^{n-1} \frac{n^k}{k!} \left|\frac{x-\mu}{\sigma}\right|^{k/n}, \quad x \in \mathbb{R}. \tag{25}$$

Proof. Applying the following known finite expansion form of the upper incomplete gamma function,

$$\Gamma(n,x) = (n-1)!e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}^* = \mathbb{N} \setminus 0,$$

into (17) we readily get (25). ∎

Example 7. *For the* (-1)*-ordered normally distributed* $X_{-1} \sim \mathcal{N}_{-1}(\mu, \sigma^2)$ *we have*

$$F_{X_{-1}}(x) = \frac{1 + \operatorname{sgn}(x-\mu)}{2} - \operatorname{sgn}(x-\mu) \frac{1 + 2\sqrt{\left|\frac{x-\mu}{\sigma}\right|}}{2\exp\left\{2\sqrt{\left|\frac{x-\mu}{\sigma}\right|}\right\}},$$

while for the $(-1/2)$*-ordered normally distributed* $X_{-1/2} \sim \mathcal{N}_{-1/2}(\mu, \sigma^2)$, *it is*

$$F_{X_{-1/2}}(x) = \frac{1 + \operatorname{sgn}(x-\mu)}{2} - \operatorname{sgn}(x-\mu) \frac{1 + 3\sqrt[3]{\left|\frac{x-\mu}{\sigma}\right|} + 9\sqrt[3]{\left(\frac{x-\mu}{\sigma}\right)^2}}{2\exp\left\{3\sqrt[3]{\left|\frac{x-\mu}{\sigma}\right|}\right\}}.$$

Table 1 provides the probability values $F_{X_\gamma}(x) = \Pr\{X_\gamma \leq x\}$, for $x = -3, -2, \ldots, 3$ for various $X_\gamma \sim \mathcal{N}_\gamma(0,1)$. The column for $x = 0$ is omitted as $F_{X_\gamma}(0) = 1/2$ for every γ value ($\mathcal{N}_\gamma(0,1)$ is a symmetric distribution around the mean 0). Moreover, the last column provide also the 1st quartile points $Q_{X_\gamma}(1/4)$ of X_γ, i.e., $\Pr\{X_\gamma \leq Q_{X_\gamma}(1/4)\} = 1/4$ for various γ values. For the 3rd quartile points $Q_{X_\gamma}(3/4)$, it is $Q_{X_\gamma}(3/4) = -Q_{X_\gamma}(1/4)$ due to the symmetric form of the γ-GND around the mean 0. These quartiles evaluated using the quantile function of X_γ,

$$\begin{aligned} Q_{X_\gamma}(P) &:= \inf\left\{x \in \mathbb{R} \mid F_{X_\gamma}(x) \geq P\right\} \\ &= \operatorname{sgn}(2P-1)\sigma \left[\tfrac{\gamma}{\gamma-1}\Gamma^{-1}\left(\tfrac{\gamma-1}{\gamma}, |2P-1|\right)\right]^{\frac{\gamma-1}{\gamma}}, \quad P \in (0,1), \end{aligned} \tag{26}$$

for $P = 1/4, 3/4$, that derived through (17). The values of the inverse upper incomplete gamma function $\Gamma^{-1}(\frac{\gamma-1}{\gamma}, \cdot)$ were numerically calculated.

Table 1. Probability mass values $F_{X_\gamma}(x)$ for various $x \in \mathbb{R}$ as well as the 1st quartile points $Q_{X_\gamma}(1/4)$, for certain r.v. $X_\gamma \sim \mathcal{N}_\gamma(0,1)$.

γ	$F_{X_\gamma}(-3)$	$F_{X_\gamma}(-2)$	$F_{X_\gamma}(-1)$	$F_{X_\gamma}(1)$	$F_{X_\gamma}(2)$	$F_{X_\gamma}(3)$	$Q_{X_\gamma}(\frac{1}{4})$
-50	0.0260	0.0690	0.1846	0.8154	0.9310	0.9740	-0.6936
-10	0.0304	0.0742	0.1869	0.8131	0.9258	0.9696	-0.6951
-5	0.0357	0.0802	0.1895	0.8105	0.9198	0.9643	-0.6967
-2	0.0502	0.0950	0.1958	0.8042	0.9050	0.9498	-0.7004
-1	0.0699	0.1131	0.2030	0.7970	0.8869	0.9301	-0.7042
$-1/2$	0.0970	0.1361	0.2116	0.7884	0.8639	0.9030	-0.7082
$-1/10$	0.1656	0.1889	0.2299	0.7701	0.8111	0.8344	-0.7142
1	0.	0.	0.	1.	1.	1.	-0.5
2	0.0013	0.0228	0.1587	0.8413	0.9772	0.9987	-0.6745
3	0.0071	0.0402	0.1699	0.8301	0.9598	0.9929	-0.6833
4	0.0112	0.0480	0.1742	0.8258	0.9520	0.9888	-0.6865
5	0.0138	0.0523	0.1765	0.8235	0.9477	0.9862	-0.6881
10	0.0193	0.0604	0.1805	0.8195	0.9396	0.9807	-0.6909
50	0.0238	0.0663	0.1833	0.8167	0.9337	0.9762	-0.6927
$\pm\infty$	0.0249	0.0677	0.1839	0.8161	0.9323	0.9751	-0.6931

Figure 1 illustrates Theorem 2 with $X_\gamma \sim \mathcal{N}_\gamma(0,1)$ in a compact form, including all the c.d.f. $F_{X_\gamma}(x)$ for every $\gamma \in [-10,0) \cup [1,10]$ and $x \in [-3,3]$. The known c.d.f. of the Uniform ($\gamma = 1$) and Normal ($\gamma = 2$) distributions are also depicted. The c.d.f. of $\mathcal{N}_{\gamma=\pm 10}(0,1)$, which approximates the c.d.f. of the Laplace distribution $\mathcal{L}(0,1) = \mathcal{N}_{\pm\infty}(0,1)$, as well as the c.d.f. of $\mathcal{N}_{-0.005}(0,1)$, which approximates the degenerate Dirac distribution $\mathcal{D}(0)$, are clearly presented. Notice the smooth–bringing of $F_{X_\gamma}(x)$ between these significant distributions which are included into the γ-GND family of distributions for $\gamma \in \mathbb{R} \cup \{\pm\infty\} \setminus (0,1)$. Moreover, upon the formed surface, the quantile functions $Q_{X_\gamma}(P)$ are depicted as curves with $P = 0.05, 0.1, \ldots 0.95$ with the 1st and 3rd quartile $Q_{X_\gamma}(1/4)$ and $Q_{X_\gamma}(3/4)$ distinguished.

From (1) and (12) or (17) the following holds.

Corollary 8. *The hazard rate $h_{X_\gamma} = f_{X_\gamma}/(1 - F_{X_\gamma})$ of a γ-order normally distributed random variable $X_\gamma \sim \mathcal{N}_\gamma(\mu,\sigma^2)$ is given by*

$$h_{X_\gamma}(x) = \frac{(\frac{\gamma}{\gamma-1})^{1/\gamma} \exp\left\{\frac{\gamma-1}{\gamma}|\frac{x-\mu}{\sigma}|^{\frac{\gamma}{\gamma-1}}\right\}}{\Gamma\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}(\frac{x-\mu}{\sigma})^{\frac{\gamma}{\gamma-1}}\right)}, \quad x \in \mathbb{R}, \tag{27}$$

or

$$h_{X_\gamma}(x) = \frac{(\frac{\gamma}{\gamma-1})^{1/\gamma} \exp\left\{\frac{\gamma-1}{\gamma}|\frac{x-\mu}{\sigma}|^{\frac{\gamma}{\gamma-1}}\right\}}{1 - \operatorname{sgn}(x-\mu) q\left(\frac{\gamma-1}{\gamma}, \frac{\gamma-1}{\gamma}|\frac{x-\mu}{\sigma}|^{\frac{\gamma}{\gamma-1}}\right)}, \quad x \in \mathbb{R}, \tag{28}$$

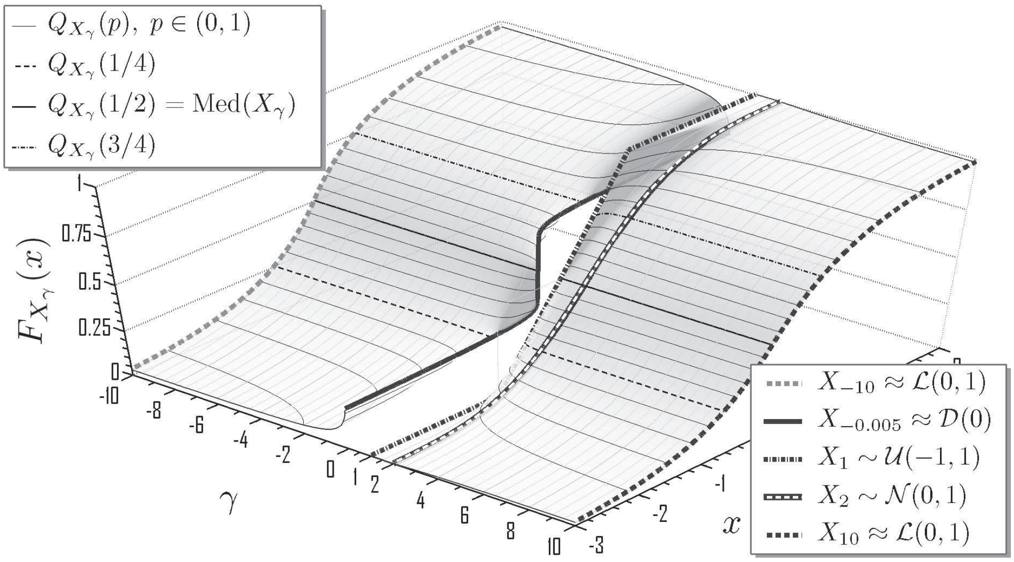

Figure 1. Surface graph of all the c.d.f. $F_{X_\gamma}(x)$ along x-axis and γ-axis, where $X_\gamma \sim \mathcal{N}_\gamma(0,1)$ as well as the quantile functions $Q_{X_\gamma}(P)$, $P \in [0,1]$ (surface curves).

where $q(a,x) = \Gamma(a) - \Gamma(a,x)$, $x \in \mathbb{R}$, $a \in \mathbb{R}_+$, *being the lower incomplete gamma function.*

Example 9. *For the Laplace distributed random variable* $Y := X_{\pm\infty} \sim \mathcal{N}_{\pm\infty}(\mu,\sigma^2) = \mathcal{L}(\mu,\sigma)$, *using the fact that* $q(1,x) = 1 - e^x$, $x \in \mathbb{R}$, (28) *for* $\gamma \to \pm\infty$, *can be written as*

$$h_Y(y) = \begin{cases} 2\exp\{|\frac{y-\mu}{\sigma}|\} - 1, & \textit{for} \quad y \leq \mu, \\ 1, & \textit{for} \quad y > \mu, \end{cases} \tag{29}$$

which is the hazard rate of the Laplace distribution, as expected.

Figure 2 illustrates also in a compact form, the hazard rates of $X_\gamma \sim \mathcal{N}_\gamma(0,1)$ for every $\gamma \in [-10,0) \cup [1,10]$ and $x \in [-3,3]$. The hazard rate of the Uniform ($\gamma = 1$) and Normal ($\gamma = 2$) distributions are clearly depicted. The hazard rate of $\mathcal{N}_{\gamma=\pm 10}(0,1)$, which approximate the hazard rate of the Laplace distribution $\mathcal{L}(0,1) = \mathcal{N}_{\pm\infty}(0,1)$, as well as the one of $\mathcal{N}_{-0.005}(0,1)$, which approximates the degenerate Dirac distribution $\mathcal{D}(0)$, are also distinguished.

The truncated γ-GND can be derived through the p.d.f. and c.d.f. of a univariate r.v. from $\mathcal{N}_\gamma(\mu,\sigma^2)$. Recall the p.d.f. f_X as in (1) and c.d.f. F_X as in (11). We shall say that X follows the *right-truncated* γ*-GND at* $x = \rho$ with

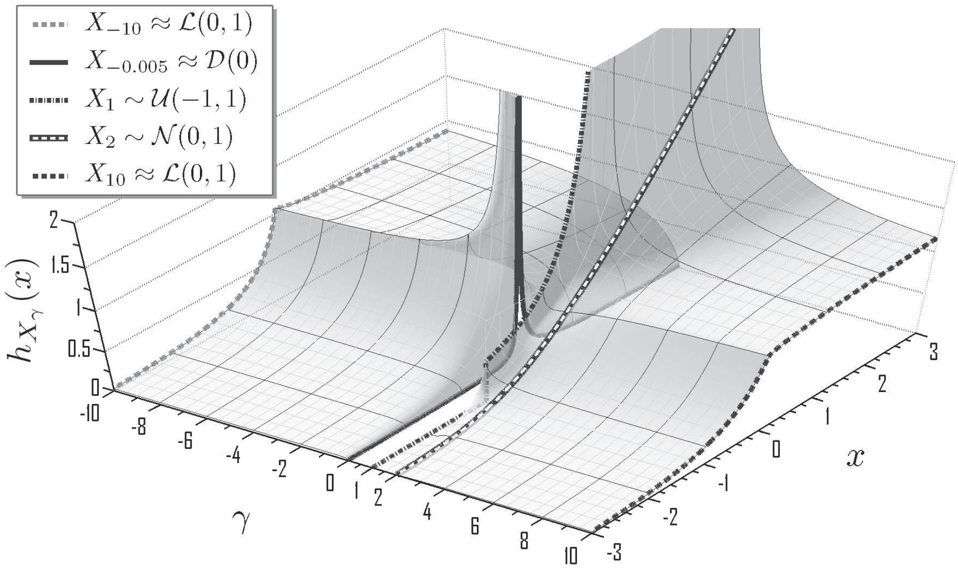

Figure 2. Surface graph of all the hazard rates $h_{X_\gamma}(x)$ along x-axis and γ-axis, where $X_\gamma \sim \mathcal{N}_\gamma(0,1)$.

p.d.f. f_X^+ when

$$f_X^+(x;\ \rho) = \begin{cases} 0, & \text{if} \quad x > \rho, \\ \dfrac{f_X(x)}{F_X(\rho)} = \dfrac{C_\gamma^1 \sigma}{F_X(\rho)} \exp\left\{ -\frac{\gamma-1}{\gamma} \left| \frac{x-\mu}{\sigma} \right|^{\frac{\gamma}{\gamma-1}} \right\}, & \text{if} \quad x \le \rho. \end{cases} \tag{30}$$

Similarly, the γ-GND r.v. X is a *left-truncated* γ*-GND r.v. at* $x = \tau$, when

$$f_X^-(x;\ \tau) = \begin{cases} 0, & \text{if} \quad x < \tau, \\ \dfrac{f_X(x)}{1 - F_X(\tau)} = \dfrac{C_\gamma^1 \sigma}{1 - F_X(\tau)} \exp\left\{ -\frac{\gamma-1}{\gamma} \left| \frac{x-\mu}{\sigma} \right|^{\frac{\gamma}{\gamma-1}} \right\}, & \text{if} \quad x \ge \tau. \end{cases} \tag{31}$$

The Lognormal distribution can be also nicely extended to the γ-order Lognormal distribution, or γ-GLND, in the sense that if $X \sim \mathcal{N}_\gamma^1(\mu, \sigma^2)$ then e^X will follow the γ-GLND, i.e., $e^X \sim \mathcal{LN}_\gamma(\mu, \sigma)$ is a γ-order lognormally distributed r.v. The p.d.f. of $X_\gamma \sim \mathcal{LN}_\gamma(\mu, \sigma)$ is then given by

$$g_{X_\gamma}(x) = \tfrac{1}{x} f_{\log X_\gamma}(\log x) = C_\gamma^1 \sigma x^{-1} \exp\left\{ -\tfrac{\gamma-1}{\gamma} \left| \tfrac{\log x - \mu}{\sigma} \right|^{\frac{\gamma}{\gamma-1}} \right\}, \quad x \in \mathbb{R}_+^*, \tag{32}$$

as $\log X_\gamma \sim \mathcal{N}_\gamma(\mu, \sigma^2)$.

3. Maximum likelihood estimation

Let $X = \{X_1, X_2, \dots, X_n\}$ be a random sample drawn from (1) with n different values. The log-likelihood function $\ell(\mu, \Sigma, \gamma;\ X)$ is then given by

$$\ell(\mu, \Sigma, \gamma;\ X) = \sum_{i=1}^{n} \log f_X(X_i;\ \mu, \Sigma) = \sum_{i=1}^{n} \left\{ \log C_\gamma^p - \tfrac{1}{2} \log |\,\Sigma\,| - \tfrac{\gamma-1}{\gamma} Q(X_i)^{\frac{\gamma}{2(\gamma-1)}} \right\}$$

$$= n \log C_\gamma^p - \tfrac{n}{2} \log |\Sigma| - \tfrac{\gamma-1}{\gamma} \sum_{i=1}^{n} \left[(X_i - \mu)^{\mathrm{T}}\, \Sigma^{-1} (X_i - \mu) \right]^{\frac{\gamma}{2(\gamma-1)}} . \tag{33}$$

3.1. Univariate case

For the univariate case $\mathcal{N}_\gamma^1(\mu, \sigma^2)$ with known γ, it is

$$\ell(\mu, \sigma^2;\ X) \ = \ n \log C_\gamma^1 - \tfrac{n}{2} \log \sigma^2 - (\tfrac{\gamma-1}{\gamma}) \sigma^{-\frac{\gamma}{\gamma-1}} \sum_{i=1}^{n} |X_i - \mu|^{\frac{\gamma}{\gamma-1}}$$

$$= \ n \log \left\{ \tfrac{1}{2} \sigma^{-1} [\Gamma(\tfrac{\gamma-1}{\gamma} + 1)]^{-1} (\tfrac{\gamma-1}{\gamma})^{\frac{\gamma-1}{\gamma}} \right\} - \tfrac{\gamma-1}{\gamma} \sum_{i=1}^{n} \left| \tfrac{X_i - \mu}{\sigma} \right|^{\frac{\gamma}{\gamma-1}} . \tag{34}$$

The partial derivatives are

$$\frac{\partial \ell}{\partial \mu}(\mu, \sigma^2) \ = \ \sigma^{-\frac{\gamma}{\gamma-1}} \sum_{i=1}^{n} |X_i - \mu|^{\frac{2-\gamma}{\gamma-1}} (X_i - \mu)$$

$$= \ \sigma^{-\frac{\gamma}{\gamma-1}} \sum_{i=1}^{n} \operatorname{sgn}(X_i - \mu)\, (X_i - \mu)^{\frac{1}{\gamma-1}} , \tag{35}$$

$$\frac{\partial^2 \ell}{\partial \mu^2}(\mu, \sigma^2) \ = \ -\tfrac{1}{\gamma-1} \sigma^{-\frac{\gamma}{\gamma-1}} \sum_{i=1}^{n} |X_i - \mu|^{\frac{2-\gamma}{\gamma-1}} , \tag{36}$$

$$\frac{\partial \ell}{\partial \sigma^2}(\mu, \sigma^2) \ = \ -\tfrac{n}{2\sigma^2} + \tfrac{1}{2} \sigma^{\frac{2-3\gamma}{\gamma-1}} \sum_{i=1}^{n} |X_i - \mu|^{\frac{\gamma}{\gamma-1}} , \tag{37}$$

$$\frac{\partial^2 \ell}{\partial (\sigma^2)^2}(\mu, \sigma^2) \ = \ \tfrac{n}{2\sigma^4} + \tfrac{2-3\gamma}{4(\gamma-1)} \sigma^{\frac{4-5\gamma}{\gamma-1}} \sum_{i=1}^{n} |X_i - \mu|^{\frac{\gamma}{\gamma-1}} , \tag{38}$$

$$\frac{\partial^2 \ell}{\partial \mu \partial \sigma^2}(\mu, \sigma^2) \ = \ -\tfrac{\gamma}{2(\gamma-1)} \sigma^{\frac{2-3\gamma}{\gamma-1}} \sum_{i=1}^{n} |X_i - \mu|^{\frac{2-\gamma}{\gamma-1}} (X_i - \mu)$$

$$= \ -\tfrac{\gamma}{2(\gamma-1)} \sigma^{\frac{2-3\gamma}{\gamma-1}} \sum_{i=1}^{n} \operatorname{sgn}(X_i - \mu)\, (X_i - \mu)^{\frac{1}{\gamma-1}} . \tag{39}$$

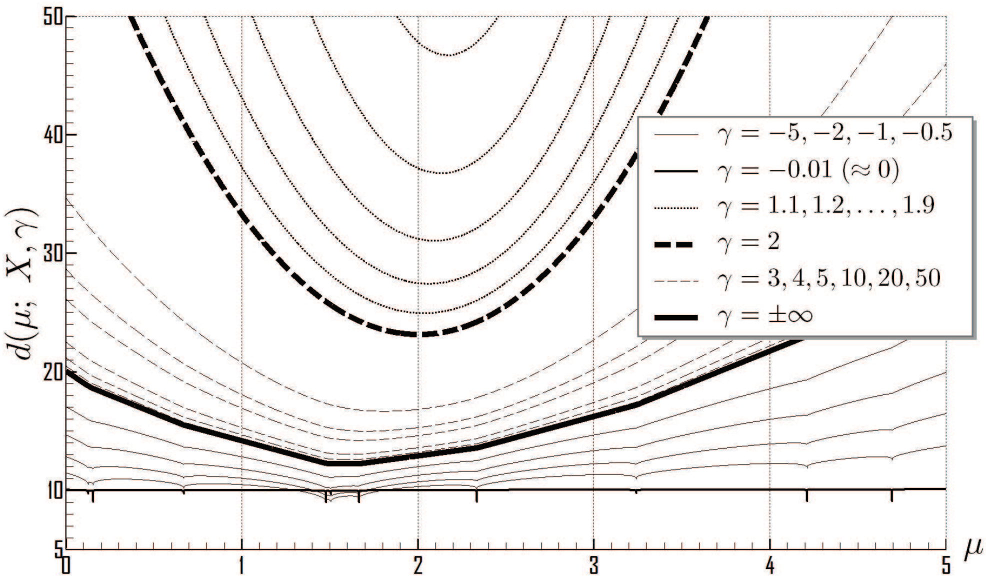

Figure 3. Graphs of the $d(\mu;\ X, \gamma)$ values along $\mu \in [0, 5]$ for various γ values for the same random sample X.

For $\gamma < 0$ and $\gamma > 2$, (35) suggests that the log likelihood has m points of non-differentiality. In general, (35) does not have an explicit solution. Nevertheless, there are examples of estimates that can be found explicitly.

Example 10. *For the Laplace distributed random variable* $X \sim \mathcal{N}_{\pm\infty}(\mu, \sigma^2) = \mathcal{L}(\mu, \sigma)$, *the MLE of* μ *is* $\hat{\mu} = \mathrm{Med}\{X_i\}$.

Example 11. *For the Normal distributed random variable* $X \sim \mathcal{N}_2(\mu, \sigma^2) = \mathcal{N}(\mu, \sigma)$, *the MLE of* μ *is* $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

Figure 3 illustrates part of the function $d(\mu;\ X, \gamma) = \sum_{i=1}^{n} |X_i - \mu|^{\frac{\gamma}{\gamma-1}}$ for a random sample $X = \{X_1, X_2, \ldots, X_{10}\}$ and various values of γ. One can see the point of non-differentiality and that as γ goes to infinity the line tends to a polygonal one. For $\gamma < 1$, from (36), we derive that $\ell(\mu, \sigma^2;\ X)$ is a union of convex curves and this suggest that maximum is at one of the X_i's.
On the other hand, (37) is always explicitly solved and the MLE of σ^2, when γ is known, is given by,

$$\hat{\sigma}^2 = \left(\frac{1}{n} \sum_{i=1}^{n} |X_i - \mu|^{\frac{\gamma}{\gamma-1}} \right)^{\frac{2(\gamma-1)}{\gamma}}. \tag{40}$$

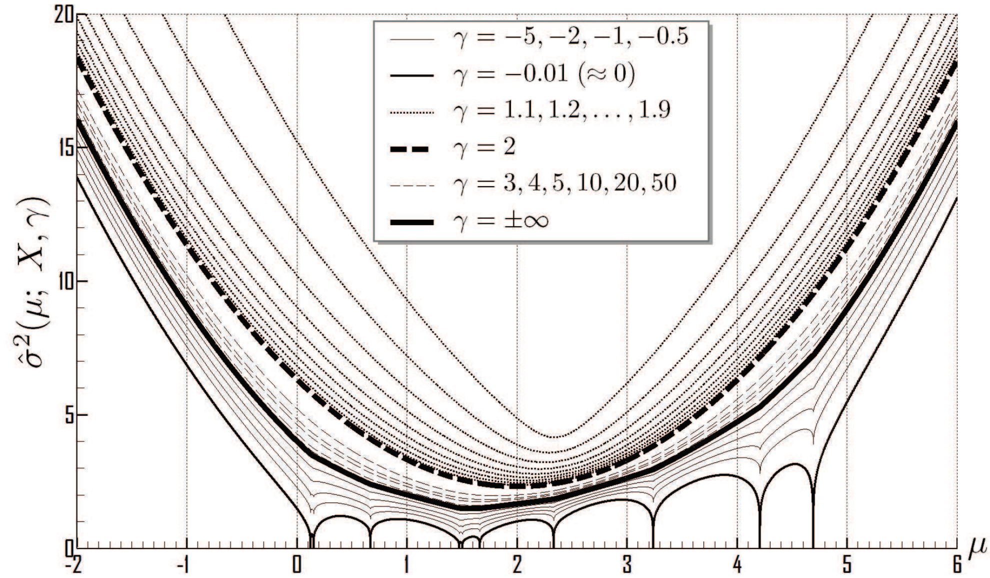

Figure 4. Graphs of $\hat{\sigma}^2(\mu;\ X, \gamma)$ along $\mu \in [0, 5]$ for various γ values for the same random sample X.

Figure 4 illustrates $\hat{\sigma}^2(\mu;\ X, \gamma)$ values as in (40).

The MLE of μ is asym-ptotically unbiased, see [4] and [7], and its asymptotic variance is given by,

$$\text{Var}\,\hat{\mu} = \tfrac{1}{n}(\tfrac{\gamma-1}{\gamma})^{2/\gamma}\frac{\Gamma(\frac{\gamma-1}{\gamma})}{\Gamma(\frac{\gamma+1}{\gamma})}\hat{\sigma}^2. \tag{41}$$

When μ is unknown, Chiodi in [4] gives an unbiased estimate for the $\sigma^{\frac{\gamma}{\gamma-1}}$ which is given by

$$\hat{\sigma}^{\frac{\gamma}{\gamma-1}} = \frac{\sum_{i=1}^{n} |X_i - \hat{\mu}|^{\frac{\gamma}{\gamma-1}}}{n - \frac{\gamma}{2(\gamma-1)}}, \tag{42}$$

and its asymptotic sampling distribution is given by

$$f(x) = \frac{\lambda^c}{\Gamma(c)} x^{c-1} e^{-\lambda x}, \tag{43}$$

i.e., a Gamma distribution with

$$\lambda = \frac{n(\gamma - 1)}{\gamma \sigma^{\frac{\gamma}{\gamma-1}}} \ \text{ and } \ c = n\tfrac{\gamma-1}{\gamma} - \tfrac{1}{2}.$$

When μ is known, Lunetta in [13] gives the asymptotic sampling distribution

$$f(x) = \frac{\lambda^{c'}}{\Gamma(c')} x^{c'-1} e^{-\lambda x}, \tag{44}$$

i.e., a Gamma distribution with the same λ and $c' = c + 1/2$.

The asymptotic matrix of variance of the maximum likelihood estimators ($\hat{\mu}$, $\hat{\sigma}$), i.e., the inverse of the Fisher's information matrix is given by

$$\mathrm{I}^{-1} = \begin{bmatrix} \sigma^2 \frac{(\gamma-1)\,\Gamma(1-1/\gamma)}{\Gamma(1/\gamma)} \left(\frac{\gamma}{\gamma-1}\right)^{\frac{\gamma-2}{\gamma}} & 0 \\ 0 & \sigma^2 \frac{\gamma-1}{\gamma} \end{bmatrix}. \tag{45}$$

This implies, that the parameters μ and σ are orthogonal, according to the Fisher's information matrix.

If the shape parameter γ is unknown, then

$$\begin{aligned} \frac{\partial \ell(\mu,\sigma,\gamma)}{\partial \gamma} &= \tfrac{n}{\gamma^2} \left[\log \tfrac{\gamma}{\gamma-1} + \psi\left(2 - \tfrac{1}{\gamma}\right) - 1 \right] - \tfrac{1}{\gamma^2 \sigma^{\frac{\gamma}{\gamma-1}}} \sum_{i=1}^{n} |X_i - \mu|^{\frac{\gamma}{\gamma-1}} \\ &- \tfrac{1}{\gamma(\gamma-1)\sigma^{\frac{\gamma}{\gamma-1}}} \left(\log \sigma \sum_{i=1}^{n} |X_i - \mu|^{\frac{\gamma}{\gamma-1}} - \sum_{i=1}^{n} |X_i - \mu|^{\frac{\gamma}{\gamma-1}} \log |X_i - \mu| \right) \end{aligned}$$

where $\psi(x) = \frac{d}{dx} \log \Gamma(x)$ the digamma function. It is obvious that the latter can not be solved explicitly. The asymptotic variance/covariance matrix of the MLE's of (μ, σ, γ) is given by,

$$\mathrm{I}^{-1} = \begin{bmatrix} \sigma^2 \frac{(\gamma-1)\,\Gamma(1-1/\gamma)}{\Gamma(1/\gamma)} \left(\frac{\gamma}{\gamma-1}\right)^{\frac{\gamma-2}{\gamma}} & 0 & 0 \\ 0 & \sigma^2 \frac{(\gamma-1)^5}{\gamma} \left(1 + \frac{A_\gamma}{B_\gamma}\right) & \sigma\gamma(\gamma-1)^3 \frac{A_\gamma}{B_\gamma} \\ 0 & \sigma\gamma(\gamma-1)^3 \frac{A_\gamma}{B_\gamma} & \frac{\gamma(\gamma-1)}{B_\gamma} \end{bmatrix}, \tag{46}$$

where $A_\gamma = [-\log(1 - \frac{1}{\gamma}) + \psi(2 - \frac{1}{\gamma})]^2$, $B_\gamma = (2 - \frac{1}{\gamma})\psi'(2 - \frac{1}{\gamma}) - 1$ and $\psi'(x)$ is the trigamma function, [1]. This implies, that the γ parameter is orthogonal to μ but not to σ, according to the Fisher's information matrix. The proof follows the one found in [2].

Mineo and Ruggieri in [8], has presented the useful *normalp* R package which among others, contains the paramp($\cdot$) function that estimates the location parameter μ and the scale parameter σ by means of the maximum likelihood method, by considering the two cases when γ is known and when it is unknown. Nevertheless, when it is unknown, the estimate of $p_\gamma = \gamma/(\gamma - 1)$ is obtained through the index of kurtosis VI, [7].

3.2. Multivariate case

For the multivariate case $\mathcal{N}_\gamma^p(\mu, \Sigma)$ with known γ, we obtain

$$\frac{\partial \ell(\mu, \Sigma)}{\partial \mu} = \sum_{i=1}^{n} Q(x_i)^{\frac{2-\gamma}{2(\gamma-1)}} \Sigma^{-1}(X_i - \mu), \tag{47}$$

$$\begin{aligned}\frac{\partial \ell(\mu, \Sigma)}{\partial \Sigma} &= -n\Sigma^{-1} + \sum_{i=1}^{n} R(X_i)[Q(X_i)]^{\frac{2-\gamma}{2(\gamma-1)}} + \tfrac{n}{2}\Sigma^{-1} \circ \mathbb{I}_p \\ &\quad -\tfrac{1}{2}\sum_{i=1}^{n} R(X_i)[Q(X_i)]^{\frac{2-\gamma}{2(\gamma-1)}} \circ \mathbb{I}_p,\end{aligned}$$

where $R(x) = \Sigma^{-1}(x - \mu)(x - \mu)^{\mathrm{T}}\Sigma^{-1}$ and $\circ$ being the element-wise (Hadamard) product on matrices, see [10] and [5]. For $\gamma = 2$ we obtain

$$\frac{\partial \ell(\mu, \Sigma)}{\partial \Sigma} = -n\Sigma^{-1} + \sum_{i=1}^{n} R(X_i) + \tfrac{n}{2}\,\Sigma^{-1} \circ \mathbb{I}_p - \tfrac{1}{2}\sum_{i=1}^{n} R(X_i) \circ \mathbb{I}_p,$$

with solution

$$\Sigma = n^{-1}\sum_{i=1}^{n} X_i X_i^{\mathrm{T}},$$

and coincides with the one for the Normal distribution, [9].

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