

ABOUT THE DENSITY OF SPECTRAL MEASURE OF THE TWO-DIMENSIONAL $S_{\alpha S}$ RANDOM VECTOR

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Abstract

In this paper, we consider a symmetric α -stable p -sub-stable two-dimensional random vector. Our purpose is to show when the function $\exp \left\{ -(|a|^p + |b|^p)^{\alpha/p} \right\}$ is a characteristic function of such a vector for some p and α . The solution of this problem we can find in [3], in the language of isometric embeddings of Banach spaces. Our proof is based on simple properties of stable distributions and some characterization given in [4].

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1. INTRODUCTION

Let $X = (X_1, X_2)$ be a symmetric random vector. It is well known that X is α -stable, $\alpha \in (0, 2)$ if and only if there exists a finite symmetric measure μ on the unit sphere $S_1 \subset \mathbb{R}^2$ such that

$$(1) \quad \mathbf{E} \exp \{i(aX_1 + bX_2)\} = \exp \left\{ - \int_{S_1} |ax_1 + bx_2|^\alpha \mu(dx_1, dx_2) \right\}.$$

An $S\alpha S$ random vector X is p -sub-stable, $\alpha < p \leq 2$, if there exists a symmetric p -stable random vector Y such that

$$X \stackrel{d}{=} Y\Theta^{1/p},$$

where $\Theta \geq 0$ is α/p -stable random variable with the Laplace transform $\exp\{-t^{\alpha/p}\}$, Y and Θ are independent. We say that an $S\alpha S$ random vector is *maximal* if it is not p -sub-stable for any $p > \alpha$.

In the paper, we will use the very well known theorem stating that *every two-dimensional Banach space embeds isometrically into some $L_1((0, 2\pi], \mu)$ space*. This theorem has been proved by several authors (see e.g. Misiewicz and Ryll-Nardzewski [6]).

Thus if $c : \mathbb{R}^2 \rightarrow [0, \infty)$ is a norm on \mathbb{R}^2 , then there exists a positive finite measure μ_1 on the unit sphere $S_1 \subset \mathbb{R}^2$, or equivalently there exists a positive finite measure μ on $(0, 2\pi]$ such that

$$c(a, b) = \int_{S_1} |ax_1 + bx_2| \mu_1(dx_1, dx_2) = \int_0^{2\pi} |a \cos \varphi + b \sin \varphi| \mu(d\varphi).$$

The proof of this theorem given in [6] implies that for the norm c smooth enough the measure μ is absolutely continuous with respect to the Lebesgue measure and it has the density function given by:

$$Q(\varphi) = \frac{1}{4} \left(q''\left(\varphi - \frac{\pi}{2}\right) + q\left(\varphi - \frac{\pi}{2}\right) \right),$$

where $q(\varphi) = c(\cos \varphi, \sin \varphi)$. Convexity of every norm in \mathbb{R}^2 implies that $q''(\varphi) + q(\varphi) \geq 0$.

Example 1. For $p > 1$ let us consider the following norm on \mathbb{R}^2 :

$$c(a, b) = (|a|^p + |b|^p)^{1/p}.$$

Laborious, but straightforward calculations show that

$$(2) \quad Q_p(\varphi) = \frac{1}{4} (p-1) |\cos \varphi \sin \varphi|^{p-2} (|\cos \varphi|^p + |\sin \varphi|^p)^{\frac{1}{p}-2}.$$

According to the theorem on isometric embeddings we obtain

$$(|a|^p + |b|^p)^{1/p} = \int_0^{2\pi} |a \cos \varphi + b \sin \varphi| Q_p(\varphi) d\varphi.$$

Using now the characterization (1) we have that $\exp \left\{ -(|a|^p + |b|^p)^{1/p} \right\}$ is a characteristic function of some 1-stable random vector $Y = (Y_1, Y_2)$. This fact was very well known for $p \in (0, 2]$, for $p > 2$ this is a part of the main result in [6].

For $p \in (1, 2]$ we can obtain here something interesting. We know that $Y \stackrel{d}{=} X\Theta^{1/p}$, where X has the characteristic function $\exp \{-|a|^p - |b|^p\}$, $\Theta \geq 0$ independent of X is $1/p$ stable with the Laplace transform $\exp \{-t^{1/p}\}$. It was shown in the paper of Misiewicz and Takenaka [4] that the density of the spectral measure for Y is given by

$$g(\cos \varphi, \sin \varphi) = c \int_0^\infty r^2 f_p(r \cos \varphi) f_p(r \sin \varphi) dr,$$

where f_p is the density function for the random variable X_0 with the characteristic function $\exp\{-|t|^p\}$, and $c^{-1} = \mathbf{E}|X_0|$. Thus for every $p \in (1, 2]$ we obtain the following formula:

$$\int_0^\infty r^2 f_p(r \cos \varphi) f_p(r \sin \varphi) dr = \frac{p-1}{4c} |\cos \varphi \sin \varphi|^{p-2} (|\cos \varphi|^p + |\sin \varphi|^p)^{\frac{1}{p}-2}.$$

The question whether or not there exists a number $\alpha \in (0, 2]$, $\alpha > p$ such that $\exp \left\{ -(|a|^\alpha + |b|^\alpha)^{\alpha/p} \right\}$ is a characteristic function was considered in [3] in the language of isometric embeddings of Banach spaces, and the answer is negative. We give here another proof of this fact. The proof is based only on properties of stable distributions and the characterization given in [4].

Theorem 1. *Let $p > 0$ and $\alpha > 0$. The function $\exp \left\{ -(|a|^\alpha + |b|^\alpha)^{\alpha/p} \right\}$ is a characteristic function of a two-dimensional $S\alpha S$ random vector X if and only if one of the following conditions holds:*

- a) $p \in (0, 2]$ and $\alpha \leq p$;
- b) $p > 2$ and $\alpha \leq 1$.

Moreover, X is maximal if and only if

- a) $p \in (0, 2]$ and $\alpha = p$;
- b) $p > 2$ and $\alpha = 1$.

Proof. The canonical construction of β -stable α -sub-stable random vector shows that if $\exp\left\{-(|a|^p + |b|^p)^{\alpha/p}\right\}$ is a characteristic function, then for every $\beta < \alpha$ function $\exp\left\{-(|a|^p + |b|^p)^{\beta/p}\right\}$ is also a characteristic function. Example 1 shows that for every $p > 2$ $\exp\left\{-(|a|^p + |b|^p)^{1/p}\right\}$ is a characteristic function. We also know that for $p \in (0, 2]$ $\exp\left\{-(|a|^p + |b|^p)\right\}$ is a characteristic function. Thus, the only untrivial part of the proof is to show that the function $\psi(a, b) = \exp\left\{-(|a|^p + |b|^p)^{\alpha/p}\right\}$ cannot be positive definite on \mathbb{R}^2 in the following two cases: 1) $p > 2, \alpha \in (1, 2]$; 2) $\alpha, p \in (0, 2], \alpha > p$.

- 1) Assume the opposite, i.e., assume that $\psi(a, b)$ is a characteristic function of a random vector X for $p > 2$, and $\alpha \in (1, 2]$. It is easy to see that X is symmetric α -stable. Consider now the random vector $Y = X\Theta^{1/\alpha}$, where Θ independent of X has the Laplace transform $\exp\{-t^{1/\alpha}\}$. Then we obtain

$$\begin{aligned} \mathbf{E} \exp\{i(aY_1 + bY_2)\} &= \mathbf{E} \exp\left\{-(|a|^p + |b|^p)^{\alpha/p} \Theta\right\} \\ &= \exp\left\{-(|a|^p + |b|^p)^{1/p}\right\}. \end{aligned}$$

This would mean that the symmetric 1-stable random vector Y with the characteristic function $\exp\left\{-(|a|^p + |b|^p)^{1/p}\right\}$ is α -sub-stable for some $\alpha > 1$. However we know (see Example 1) that the spectral density $Q_p(\varphi)$ of Y has the property $Q_p(k\pi/2) = 0$, $k = 1, 2, 3, 4$. Thus we obtained a contradiction since Theorem 1 in [4] implies that the spectral density of 1-stable, α -sub-stable random vector has to be positive everywhere.

- 2) Assume that $\psi(a, b)$ is a characteristic function of a random vector X for some $\alpha, p \in (0, 2], \alpha > p$. Of course, X is symmetric α -stable. Then, it is easy to see that the function $\exp\{-|a|^p - |b|^p\}$ is the characteristic function of p -stable, α -sub-stable random vector Y . According to Theorem 1 in [4] we have that the spectral measure of Y is absolutely continuous with respect to the Lebesgue measure. This is in contradiction to the fact that Y with the characteristic function $\exp\left\{-(|a|^p + |b|^p)\right\}$ has the spectral measure of the form $\frac{1}{2}(\delta_{(0,1)} + \delta_{(0,-1)} + \delta_{(1,0)} + \delta_{(-1,0)})$.

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