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SMALL PERTURBATIONS WITH LARGE EFFECTS ON VALUE-AT-RISK

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Abstract

We show that in the *delta-normal* model there exist perturbations of the Gaussian multivariate distribution of the returns of a portfolio such that the initial marginal distributions of the returns are statistically undistinguishable from the perturbed ones and such that the perturbed V@R is close to

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the worst possible V@R which, under some reasonable assumptions, is the sum of the V@Rs of each of the portfolio assets.

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1. Introduction

The aim of this note is to prove that small perturbations of the returns of each individual asset in a portfolio, for example non detectable under the Kolmogorov-Smirnov test, can change the global *Value-at-Risk* of the portfolio up to a value near the worst V@R possible according to formula (4) below. In an apparently well diversified portfolio with hundreds of assets, if the correlations between the assets' returns are changed by non-noticeable perturbations of the returns of each particular asset, the *Value-at-Risk* of the portfolio can suffer great changes for the worst. This new V@R shortcoming gives reinforced importance, in the *delta-normal* model, to a rigorously estimated and realistic correlation matrix between the assets' returns.

Despite the more recent capital adequacy standards for banking and insurance the *delta-normal* model for V@R computation is still widely used in the industry. In [5] attention is drawn to the fact that switching from a finite to an infinite mean model alters the additivity properties for V@R in case the dependence is Archimedean. In [14] and we quote: "explicit lower and upper bounds on the value-at-risk (VaR) for the sum of possibly dependent risks are derived when only partial information is available about the dependence structure and the individual behaviors.". Results on worst case V@R scenarios in higher dimensions for dependent risks are given in [7] with an analytical solution in case of uniform marginals. The first noticeable study on worst case V@R scenarios is [6] with complementary results and extensions given in [12] and also in [11].

Let us detail these preliminary remarks. Following McNeil, Frey and Embrechts [13, p. 38], given some confidence level $\alpha \in]0,1[$ and a fixed time horizon T the *Value-at-Risk* $\mathrm{V@R}_\alpha^X$ of a portfolio, with unit value and profit and loss X, is given by the smallest number l such that the probability that the loss of the portfolio exceeds l, during the time interval T, is no larger than $1-\alpha$. Hence the *Value-at-Risk* is a quantile of the distribution of the portfolio's profit and loss distribution. Both in theoretical and applied literature, the *Value-at-Risk* can be computed either using the distribution of the profit and loss (P&L) or using the distribution of losses taken with positive values.

For a portfolio with unit total value, for which the P&L random variable Z has normal distribution with mean μ and variance σ^2, given Φ the standard normal's distribution, it is straightforward (see [13, p. 38]) to check that:

$$\text{V@R}_\alpha^Z = \mu + \sigma\Phi^{-1}(\alpha), \tag{1}$$

with, usually, $\alpha \in \{0.05, 0.01, 0.003\}$. In practice (see [13, p. 38] and [10, p. 111]), one often use the *mean V@R* defined by $\text{V@R}_{\alpha,\text{ mean}}^Z := \text{V@R}_\alpha^Z - \mu$ where μ is the mean value of the P&L's distribution. When this distribution is normal, we have:

$$\text{V@R}_{\alpha,\text{ mean}}^Z = \sigma\Phi^{-1}(\alpha)\ . \tag{2}$$

The *delta-normal* model has two main assumptions; firstly, the joint distribution of the individual returns is taken to be multivariate normal, so that each individual return is itself normally distributed (see [10, p. 162]). Secondly, it is assumed that the P&L is given by the product of the returns by the capital invested either in the asset or in the whole portfolio (see Section 2); given that the capital is deterministic it follows that the P&L's distribution is also normal. A general setting to compute the *Value-at-Risk* in the *delta-normal* model is given by the variance-covariance method one can find in [3, p. 37] or [13, p. 48]. In the benchmark applications of this method, let the column vector $\mathbf{V}$ have as components the *mean V@Rs* of each individual asset. Using the correlation matrix $\not{\mathbf{R}}$ between the returns of each individual assets then the portfolio's *mean V@R* is given by the following striking formula (proved, for completeness, in Section 2),

$$\left(\text{V@R}_{\alpha,\text{ mean}}^X\right)^2 = \mathbf{V}^t \not{\mathbf{R}} \mathbf{V}, \tag{3}$$

that one can find, for example, in [2, p. 23].

Observation 1. *One simple but important remark following formula* (3) *is that if* $\mathbf{V}^t = (V_1, \ldots, V_N)$, $\not{\mathbf{R}} = [\rho_{ij}]_{i,j\in\{1,\ldots,N\}}$ *and if* $V_i \geq 0$ *(or if* $V_i \leq 0$*) for all* $i \in \{1, \ldots, N\}$ *and* $\rho_{ij} \in [-1, 1]$ *for* $i, j \in \{1, \ldots, N\}$ *then, we always have:*

$$\mathbf{V}^t \not{\mathbf{R}} \mathbf{V} = \sum_{i,j=1}^N V_i V_j \rho_{ij} \leq \sum_{i,j=1}^N V_i V_j = \left(\sum_{i=1}^N V_i\right)^2 . \tag{4}$$

This observation is suggested in [10, p. 164] in the case where the portfolio has two assets. Let us stress that in the *delta-normal* model and under the assumption that the *Value-at-Risk* of each asset is nonnegative then the maximum *Value-at-Risk* of the portfolio is given by the sum of the *Values-at-Risk* of each asset.

2. Computing the V@R in the *delta-normal* model

For completeness, we now present the well known *Value-at-Risk* computation under the *delta-normal* model's assumptions. We will consider a portfolio with N assets. Let $\mathbf{W} = (W_1, \dots, W_N)$ be the column vector with components given by the capital invested in each asset then $W_\Sigma = \sum_{i=1}^{N} W_i$ is the total capital invested in the portfolio. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_N)$ be the column vector with components being the ratio of the capital invested in each asset, that is for all $i \in \{i, \dots, N\}$ we have $\omega_i = W_i / W_\Sigma$. Finally let $\mathbf{X} = (X_1, \dots, X_N)$ be the column vector having as components the additive returns of each asset of the portfolio. The first assumption of the *delta-normal* model is that $\mathbf{X}$ is multivariate normal, noted $\mathbf{X} \frown \mathcal{N}\left(\boldsymbol{\mu}(\mathbf{X}), \not{\Sigma}(\mathbf{X})\right)$ where $\boldsymbol{\mu}(\mathbf{X})$ is the mean value vector and $\not{\Sigma}(\mathbf{X})$ is the variance-covariance matrix. As the next proposition will show, we can compute the distributions of both the return and the P&L of the portfolio.

Proposition 2. *Let* $\mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \dots, \mathbb{E}[X_N])$. *Considering additive returns, the total return of the portfolio* X_Σ *satisfies:*

$$X_\Sigma = \boldsymbol{\omega}^t \cdot \mathbf{X} \quad \text{and } X_\Sigma \frown \mathcal{N}\left(\boldsymbol{\omega}^t \cdot \mathbb{E}[\mathbf{X}], \sqrt{\boldsymbol{\omega}^t \cdot \not{\Sigma}(\mathbf{X}) \cdot \boldsymbol{\omega}}\right).$$

hence the P&L of the portfolio, $P\&G_\Sigma$*, satisfies:*

$$P\&G_\Sigma = W \times X_\Sigma = W \times \left(\boldsymbol{\omega}^t \cdot \mathbf{X}\right). \tag{5}$$

Furthermore,

$$P\&G_\Sigma \frown \mathcal{N}\left(W \times \left(\boldsymbol{\omega}^t \cdot \mathbb{E}[\mathbf{X}]\right), W \times \sqrt{\boldsymbol{\omega}^t \cdot \not{\Sigma}(\mathbf{X}) \cdot \boldsymbol{\omega}}\right). \tag{6}$$

Proof. One can see that after a certain time interval, the new total value of the porfolio is the sum of all the new values of each individual assets. Let W_Σ^t be the value of the portfolio at time t. Using the definition of additive returns we have:

$$\begin{aligned} W_\Sigma^{t+1} &= \sum_{i=1}^{N} W_i^{t+1} = \sum_{i=1}^{N} W_i^t \times (1 + X_i) = W_\Sigma^t + W_\Sigma^t \times \sum_{i=1}^{N} X_i \times \frac{W_i^t}{W_\Sigma^t} \\ &= W_\Sigma^t \times (1 + \sum_{i=1}^{N} X_i \times \omega_i) = W_\Sigma^t \times (1 + X_\Sigma), \end{aligned}$$

this shows that $X_\Sigma = \boldsymbol{\omega}^t \cdot \mathbf{X}$. From this, it follows immediately that X_Σ being a linear combination of the components of a normal vector it is a random variable with the above normal distribution (see [15, p. 414]). The P&L is then obtained by the difference between the value of the portfolio after the time interval and the initial value, which is:

$$\text{P\&G}_\Sigma = W_\Sigma^{t+1} - W_\Sigma^t = W_\Sigma^t \times (1 + X_\Sigma) - W_\Sigma^t = W_\Sigma^t \times X_\Sigma$$

hence immediately, we get formula (5). Standard results about normal variables lead to the desired P&L distribution stated in (6). ■

Consequently due to formulae (1) and (2) we have the following fundamental result.

Corollary 3. *In the delta-normal model, given some confidence level* $\alpha \in]0,1[$:

$$V@R_\alpha^{P\&G_\Sigma} = W \times \left(\boldsymbol{\omega}^t \cdot \mathbb{E}\left[\mathbf{X}\right]\right) + W \times \sqrt{\boldsymbol{\omega}^t \cdot \not\Sigma(\mathbf{X}) \cdot \boldsymbol{\omega}} \times \Phi^{-1}(\alpha)$$

and

$$(7)\quad V@R_{\alpha,\ mean}^{P\&G_\Sigma} = W \times \sqrt{\boldsymbol{\omega}^t \cdot \not\Sigma(\mathbf{X}) \cdot \boldsymbol{\omega}} \times \Phi^{-1}(\alpha) = \sqrt{\mathbf{W}^t \cdot \not\Sigma(\mathbf{X}) \cdot \mathbf{W}} \times \Phi^{-1}(\alpha).$$

Finally, we have the proof of (3) used in practical applications. Let $\not\Sigma(\mathbf{X}) = [\mathrm{cov}(X_i, X_j)]_{i,j\in\{1,\ldots,N\}}$ and let the correlation matrix of the assets returns be $\not\mathbf{R}(\mathbf{X}) = [\rho(X_i, X_j)]_{i,j\in\{1,\ldots,N\}}$. For a better understanding of the next proposition, we will make a preliminary observation on variance-covariance matrices.

Observation 4. *As for any real column vector* $\mathbf{W} = (W_1, \ldots, W_N)$, *we have that*

$$\mathbf{W}^t \cdot \not\Sigma(\mathbf{X}) \cdot \mathbf{W} = \mathbb{E}\left[\left(\sum_i W_i \left(X_i - \mathbb{E}\left[X_i\right]\right)\right)^2\right] \geq 0,$$

it then follows that the sign of $V@R_{\alpha,\ mean}^{P\&G_\Sigma}$ *in formula* (7) *is given by the sign of* $\Phi^{-1}(\alpha)$.

Corollary 5. *In the delta-normal model, given some confidence level* $\alpha \in]0,1[$, *if* $\mathbf{V}$ *is the column vector having as components the mean V@Rs of each individual asset then:*

$$V@R_{\alpha,\ mean}^{P\&G_\Sigma} = \begin{cases} +\sqrt{\mathbf{V}^t \not\mathbf{R}(\mathbf{X}) \mathbf{V}} & \text{if } \alpha \in]\frac{1}{2}, 1[\\ 0 & \text{if } \alpha = \frac{1}{2} \\ -\sqrt{\mathbf{V}^t \not\mathbf{R}(\mathbf{X}) \mathbf{V}} & \text{if } \alpha \in]0, \frac{1}{2}[. \end{cases}$$

Proof. Following (2) applied to each individual asset one has that:

$$\mathbf{W}^t \cdot \not{\Sigma}(\mathbf{X}) \cdot \mathbf{W} \times \left(\Phi^{-1}(\alpha)\right)^2 = \sum_{i,j} \left(W_i \Phi^{-1}(\alpha)\right) \operatorname{cov}(X_i, X_j) \left(W_j \Phi^{-1}(\alpha)\right)$$

$$= \sum_{i,j} \left(W_i \Phi^{-1}(\alpha)\sigma(X_i)\right) \frac{\operatorname{cov}(X_i, X_j)}{\sigma(X_i)\sigma(X_j)} \left(W_j \Phi^{-1}(\alpha)\sigma(X_j)\right)$$

$$= \sum_{i,j} \mathrm{V@R}_{\alpha,\ \mathrm{mean}}^{(W_i X_i)}\, \rho(X_i, X_j)\, \mathrm{V@R}_{\alpha,\ \mathrm{mean}}^{(W_j X_j)} = \mathbf{V}^t \not{\mathbf{R}}(\mathbf{X})\mathbf{V},$$

as was expected. ∎

Observation 6 (Fundamental)**.** *Corollary* 5 *permits to go even further on Observation* 1. *In the delta-normal model, under the assumption that the Value-at-Risk of each individual asset have the same sign, the worst Value-at-Risk of the portfolio is given by the sum of the Values-at-Risk of all the assets. One can note that in the delta-normal model this happens in* (2)*, with the obvious assumption that all Value-at-Risk of the assets are taken given the same confidence level* α*.*

3. Small Gaussian perturbations

In this section we show how to produce small perturbations, with respect to the Kolmogorov distance, of Gaussian random variables so that the perturbed distribution remains Gaussian and the mean value is preserved. We will start with the univariate case and then present the multivariate case.

3.1. Small perturbations of random univariate normal variables

We will use the following notations. Let $X \frown \mathcal{N}(\mu, \sigma)$ denote X to be a normal variable with mean value μ and variance σ^2, its density function being given by

$$F'_X(x) = \frac{1}{\sqrt{2\sigma^2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right],$$

and its characteristic function by:

$$\phi_X(t) = \exp\left(i\mu t - \frac{t^2\sigma^2}{2}\right). \tag{8}$$

The existence of small univariate perturbations of Gaussian random variables is detailed by the following result.

Theorem 7. *Let* $X \frown \mathcal{N}(\mu, \sigma)$ *and* $E \frown \mathcal{N}(0, \epsilon)$ *be such that* $X \perp\!\!\!\perp E$ *and so* $X + E \frown \mathcal{N}(\mu, \sqrt{\sigma^2 + \epsilon^2})$*. Assuming furthermore that* $\epsilon < \sigma$*, we obtain the following bound for the Kolmogorov distance between the distribution functions of* $X + E$ *and* X*:*

$$\sup_{x \in \mathbb{R}} |F_{X+E}(x) - F_X(x)| \leq \frac{1}{\pi} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \left(\frac{\epsilon}{\sigma}\right)^{2n} = \frac{1}{\pi} \ln\left(1 + \left(\frac{\epsilon}{\sigma}\right)^2\right). \tag{9}$$

In particular, as in (9) *the series being an alternating series with terms converging to* 0 *monotonically, we have the following estimate:*

$$\sup_{x \in \mathbb{R}} |F_{X+E}(x) - F_X(x)| \leq \frac{1}{\pi} \left(\frac{\epsilon}{\sigma}\right)^2. \tag{10}$$

Proof. Given that the sum of two independent Gaussian random variables is still Gaussian, the proof is a consequence of the Esseen inequality (see [8], [9, p. 538] or [18, p. 296]), stating that for all $T > 0$:

$$\sup_x |F_{X+E}(x) - F_X(x)| \leq \frac{2}{\pi} \int_0^T \left|\frac{\phi_{X+E}(t) - \phi_X(t)}{t}\right| dt + \frac{24}{\pi T} \sup_x \left|F_X'(x)\right|.$$

Using (8), the fact that F_X' is bounded, the independency of X and E (so that $\phi_{X+E} = \phi_X \cdot \phi_E$), we have:

$$\sup_{x \in \mathbb{R}} |F_{X+E}(x) - F_X(x)| \leq \frac{2}{\pi} \int_0^{+\infty} e^{-\frac{\sigma^2 t^2}{2}} \frac{1 - e^{-\frac{\epsilon^2 t^2}{2}}}{t} dt.$$

The exponential series being uniformly convergent on compact sets, we have:

$$\begin{aligned}
&\frac{2}{\pi} \int_0^{+\infty} e^{-\frac{\sigma^2 t^2}{2}} \frac{1 - e^{-\frac{\epsilon^2 t^2}{2}}}{t} dt \\
&\quad = \frac{2}{\pi} \lim_{N \in \mathbb{N} \to +\infty} \int_0^N e^{-\frac{\sigma^2 t^2}{2}} \left(\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n!} \frac{\epsilon^{2n} t^{2n-1}}{2^n}\right) dt \\
&\quad = \frac{2}{\pi} \lim_{N \in \mathbb{N} \to +\infty} \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} \epsilon^{2n}}{n!\, 2^n} \left(\int_0^N t^{2n-1} e^{-\frac{\sigma^2 t^2}{2}} dt\right).
\end{aligned}$$

Using the change of variable $\sigma t = u$ followed by integration by parts, an induction argument allows us to conclude:

$$\int_0^{+\infty} t^{2n-1} e^{-\frac{\sigma^2 t^2}{2}} dt = \frac{1}{\sigma^{2n}} \int_0^{+\infty} u^{2n-1} e^{-\frac{u^2}{2}} du = \frac{1}{\sigma^{2n}} 2^{n-1} (n-1)!\,.$$

We then have:

(11)
$$\int_0^N t^{2n-1}e^{-\frac{\sigma^2t^2}{2}}dt = \int_0^{+\infty} t^{2n-1}e^{-\frac{\sigma^2t^2}{2}}dt - \int_N^{+\infty} t^{2n-1}e^{-\frac{\sigma^2t^2}{2}}dt \leq \frac{2^{n-1}(n-1)!}{\sigma^{2n}},$$

The sum of a series can be interpreted as an integral relative to the integer counting measure μ_c, so that:

$$\lim_{N\to+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}\epsilon^{2n}}{n!\;2^n}\left(\int_0^N t^{2n-1}e^{-\frac{\sigma^2t^2}{2}}dt\right)$$
$$= \lim_{N\to+\infty}\int_{\{1,2,\ldots,n,\ldots\}}\frac{(-1)^{n+1}\epsilon^{2n}}{n!\;2^n}\left(\int_0^N t^{2n-1}e^{-\frac{\sigma^2t^2}{2}}dt\right)d\mu_c(n)\,.$$

One can note that for $\epsilon < \sigma$, the equation (11) implies that:

$$\left|\frac{(-1)^{n+1}\epsilon^{2n}}{n!\;2^n}\left(\int_0^N t^{2n-1}e^{-\frac{\sigma^2t^2}{2}}dt\right)\right| \leq \frac{1}{n}\left(\frac{\epsilon}{\sigma}\right)^{2n}.$$

The series with general term given by the righthand side of this equation is convergent, hence:

$$\int_{\{1,2,\ldots,n,\ldots\}}\frac{1}{n}\left(\frac{\epsilon}{\sigma}\right)^{2n}d\mu_c(n) = \sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{\epsilon}{\sigma}\right)^{2n} < +\infty.$$

Finally by Lebesgue's dominated convergence theorem, we obtain:

$$\frac{2}{\pi}\int_0^{+\infty} e^{-\frac{\sigma^2t^2}{2}}\,\frac{1-e^{-\frac{\epsilon^2t^2}{2}}}{t}dt$$
$$= \frac{2}{\pi}\lim_{N\in\mathbb{N}\to+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}\epsilon^{2n}}{n!\;2^n}\left(\int_0^N t^{2n-1}e^{-\frac{\sigma^2t^2}{2}}dt\right)$$
$$= \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}\epsilon^{2n}}{n!\;2^n}\lim_{N\in\mathbb{N}\to+\infty}\left(\int_0^N t^{2n-1}e^{-\frac{\sigma^2t^2}{2}}dt\right) = \frac{1}{\pi}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{\epsilon}{\sigma}\right)^{2n}.$$

The convergence of the series obtained in the last inequality permits to conclude the proof of (9). ■

Observation 8. *Perturbing data with Gaussian distribution by summing a zero mean Gaussian random variable with small variance (with respect to the original variable) does not change significantly the V@R. Indeed, using the notations* *of the Theorem* 7, *we have for* $\alpha \in]0,1[$ *that* $V@R_\alpha^{X+E} = \mu + \sqrt{\sigma^2+\epsilon^2}\,\Phi^{-1}(\alpha)$ *and* $V@R_\alpha^X = \mu + \sigma\,\Phi^{-1}(\alpha)$ (*as seen in* (1)), *so that for* $\epsilon \ll \sigma$ *we have:*

$$\left| V@R_\alpha^{X+E} - V@R_\alpha^X \right| = \left| 1 - \sqrt{1+\left(\frac{\epsilon}{\sigma}\right)^2} \right| \sigma \left|\Phi^{-1}(\alpha)\right| \approx \left(\frac{\epsilon}{\sigma}\right)^2 \frac{\sigma \left|\Phi^{-1}(\alpha)\right|}{2}. \tag{12}$$

We can observe that, as should be expected, the approximation of the pertubated V@R by the non perturbated V@R is of the same order as the approximation error between the distribution functions in Theorem 7, *that is, of the order of* $(\epsilon/\sigma)^2$.

Observation 9. *Let* $F_{X+E}^{(n)}$ *be the empirical distribution function of a dimension* n *sample of the variable* $X+E$. *As we have:*

$$\begin{aligned} \sup_{x\in\mathbb{R}} \left| F_{X+E}^{(n)}(x) - F_{X+E}(x) \right| &\leq \sup_{x\in\mathbb{R}} \left| F_{X+E}^{(n)}(x) - F_X(x) \right| \\ &\quad + \sup_{x\in\mathbb{R}} \left| F_{X+E}(x) - F_X(x) \right|, \end{aligned} \tag{13}$$

using the estimates of Theorem 7, *one can argue that for* $\epsilon \ll \sigma$ *and for a sufficiently large sample, the distribution functions of* X *and* $X+E$ *are nearly indistinguishable in the sense of the Kolmogorov-Smirnov test. More precisely, suppose that we are in a situation were the proposed model is described by the random variable* X *but that a more appropriate model would be given by the perturbed model* $X+E$. *Hence, this would imply that the observed distribution function is* $F_{X+E}^{(n)}$ *and not* $F_X^{(n)}$. *As a consequence when performing the Kolmogorov-Smirnov test we calculate* $\sup_{x\in\mathbb{R}} \left| F_{X+E}^{(n)}(x) - F_X(x) \right|$ *instead of* $\sup_{x\in\mathbb{R}} \left| F_X^{(n)}(x) - F_X(x) \right|$. *For example* (*see Section* 4), *lets consider a sample of* 36 (*additive*) *returns, given monthly, of an PSI-20 asset such that* $\sup_{x\in\mathbb{R}} \left| F_{X+E}^{(36)}(x) - F_X(x) \right| = 0.262$. *If* $\epsilon \ll \sigma$ *is chosen so that* $\sup_{x\in\mathbb{R}} \left| F_{X+E}(x) - F_X(x) \right| < 0.0001$ *then, for a significance level of 1% the model* X *is not rejected. Furthermore by* (13) *and with the same significance level, the model* $X+E$ *is also not rejected* (*notations and numerical values are taken from* [15, pp. 284–286, 536]).

3.2. Small perturbations of random multivariate Gaussian variables

From now on, we will assume the hypothesis of the *delta-normal* model. We are going to show how to build the variance-covariance matrix of an independent Gaussian multivariate perturbation of the return's vector, so that the V@R of the portfolio is as close as possible of the maximum V@R (according to Observations 1 and 6).

We recall our previously partial conclusions on the univariate Gaussian perturbations, to be used next. The result in Theorem 7 suggests that we can define a small Gaussian perturbation of a Gaussian random variable as any independent Gaussian random variable with strictly smaller variance. Observation 8 in Subsection 3.1 shows the small impact on the V@R of a small normal perturbation. Similarly, in Observation 9 we saw that, given some confidence level, if the variance of the perturbation is sufficiently small this perturbation can't be detected by a test based on the Kolmogorov distance. Our method has the following steps.

1. Start by defining the matrix's diagonal coefficients. Let $\delta_1, \delta_2 > 0$ be, respectively, the upper bounds for the maximum error admissible in the distance between the V@R of the inicial distribution and the perturbed one, and the maximum error admissible for the Kolmogorov distance between the same distributions. This choice will hold for each component of the return and perturbation vectors. Relatively to the distance between V@Rs, if:

$$\left|\mathrm{V@R}_\alpha^{X_i+E_i} - \mathrm{V@R}_\alpha^{X_i}\right| = \left|\sqrt{\sigma(X_i)^2 + \sigma(E_i)^2} - \sigma(X_i)\right| \left|\Phi^{-1}(\alpha)\right| = \delta_1,$$

then the standard deviation of the perturbation E_i is:

$$\sigma(E_i) = \sqrt{\frac{2\delta_1\sigma(X_i)}{|\Phi^{-1}(\alpha)|} + \left(\frac{\delta_1}{|\Phi^{-1}(\alpha)|}\right)^2} \approx \sqrt{\frac{2\delta_1\sigma(X_i)}{|\Phi^{-1}(\alpha)|}},$$

the approximation estimate being obtained for $\delta_1 \ll \left|\Phi^{-1}(\alpha)\right|$; one can note that this is the same approximation obtained using directly the bound given by formula (12). For the Kolmogorov distance we have, using the bound given by (9), that:

$$\frac{1}{\pi}\ln\left(1 + \left(\frac{\sigma(E_i)}{\sigma(X_i)}\right)^2\right) = \delta_2,$$

hence the variance of the perturbation E_i is given by:

$$\sigma(E_i) = \sigma(X_i)\sqrt{e^{\delta_2\pi} - 1} \approx \sigma(X_i)\sqrt{\delta_2\pi}, \tag{14}$$

with the last approximation holding for $\delta_2\pi \ll 1$; as above, this is the same approximation that we would get using directly formula (10). Therefore each component E_i of the perturbation must satisfy:

$$\sigma(E_i) = \min\left(\sqrt{\frac{2\delta_1\sigma(X_i)}{|\Phi^{-1}(\alpha)|} + \left(\frac{\delta_1}{|\Phi^{-1}(\alpha)|}\right)^2}, \sigma(X_i)\sqrt{e^{\delta_2\pi} - 1}\right). \tag{15}$$

2. Non diagonal terms of the variance-covariance matrix can be defined noticing that the independency between the returns $\mathbf{X}$ and the multivariate perturbation $\mathbf{E}$ imply that the variance-covariance matrix of $\mathbf{X}+\mathbf{E}$ satisfies: $\Sigma(\mathbf{X}+\mathbf{E}) = \Sigma(\mathbf{X})+\Sigma(\mathbf{E})$. Now, consider the matrix $\Sigma(\mathbf{E}) = [\mathrm{cov}(E_i, E_j)]_{i,j\in\{1,\dots,N\}}$ and let the correlation matrix of the perturbation be given by $R(\mathbf{X}+\mathbf{E}) = [\rho(X_i+E_i, X_j+E_j)]_{i,j\in\{1,\dots,N\}}$. The Cauchy-Schwarz inequality:

$$|\mathrm{cov}(E_i, E_j)| \leq \sigma(E_i)\sigma(E_j), \tag{16}$$

implies that the variance-covariance matrix of the perturbations can't be chosen arbitrarly: it must, at least, satisfy this bound. Therefore to complete our construction, one first solution is to chose an upper bound satisfying the Cauchy-Schwarz inequality. Unfortunately, choosing arbitrarily variance-covariances satisfying this bound will generally lead to a singular matrix. To overcome this difficulty one can choose randomly the non diagonal coefficients $\mathrm{cov}(E_i, E_j)$ of the variance-covariance matrix in the intervals $[\sigma(E_i)\sigma(E_j) - \gamma, \sigma(E_i)\sigma(E_j)]$ for a small γ. If the matrix obtained by this method is non positive definite, one can obtain one that is, altering slightly the matrix, using a transformation method inspired by the one given in [16] (see also [17]).

Such a transformation is done in the following way. Let the matrix $C = [c_{ij}]_{i,j\in\{1,\dots,N\}}$, the initially estimated variance-covariance matrix of size N, be nonsingular; in practice, this matrix is made to be symmetric but can have non-positive eigenvalues hence being non positive definite. Let Λ be a diagonal matrix having as diagonal entries $\lambda_1, \dots, \lambda_N$ the eigenvalues of C, ordered by decreasing absolute value. Let M be an orthogonal matrix with columns given by the eigenvectors of M, i.e. satisfying: $C \cdot M = M \cdot \Lambda$. Observe that, given that the eigenvectors are chosen so to form an orthonormal base, M is orthogonal and $C = M\Lambda M^{-1} = M\Lambda M^t$.

(a) Let $|\Lambda|$ be the matrix with coefficients given by the absolute values of the eigenvalues, ordered in the same way as Λ.

(b) Let $M = [m_{ij}]_{i,j\in\{1,\dots,N\}}$ and consider $T = [t_{ij}]_{i,j\in\{1,\dots,N\}}$ the diagonal matrix defined by:

$$t_{ii} := \frac{c_{ii}}{\sum_{k=1}^{N} m_{ik}^2 \, |\lambda_k|}. \tag{17}$$

(c) Let $\sqrt{T}$ and $\sqrt{|\Lambda|}$ be the diagonal matrices which coefficients are the square root of those in T and Λ. Let $B := \sqrt{T}M\sqrt{|\Lambda|}$ then:

$$C_{\text{alt}} := B \cdot B^t = \left(\sqrt{T} M \sqrt{|\Lambda|}\right)\left(\sqrt{T} M \sqrt{|\Lambda|}\right)^t \tag{18}$$
$$= \sqrt{T} M \, |\Lambda| \, M^t \sqrt{T}$$

is the transformed matrix given by a variation of the Rebonato method (cf. Theorem 10 below).

The next theorem, on our variant of Rebonato's method, will describe the main proprieties of C_{alt} giving an estimate of the distance between the initial matrix C and the altered matrix C_{alt}depending only on C.

Theorem 10 (Variation of the Rebonato method)**.** *Let C be the initial variance-covariance matrix, assumed to be nonsingular, symmetric with some negative eigenvalues. By construction, the matrix C_{alt} given by* (18) *is symmetric, positive definite and has diagonal entries identical to those of C. Let $|||\cdot|||_F$ denote the Frobenius norm for matrices. Then C_{alt} satisfies the following estimate:*

$$|||C - C_{alt}|||_F \leq \left(\sum_{i,j=1}^{N} \left(1 - \sqrt{t_{ii}}\sqrt{t_{jj}}\right)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} \lambda_i^2\right)^{\frac{1}{2}} + 2\left(\sum_{i:\lambda_i<0} \lambda_i^2\right)^{\frac{1}{2}}, \tag{19}$$

where the t_{ii} are given by (17) *and $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of C.*

Proof. If $C_{\text{alt}} := B \cdot B^t$ with $B := \sqrt{T} M \sqrt{|\Lambda|}$ invertible then C_{alt} is necessarly symmetric and positive definite. Given $C_{\text{alt}} = [c_{ik}^{\text{alt}}]_{i,k \in \{1,\ldots,N\}}$, the right equality in (18) implies that:

$$c_{ik}^{\text{alt}} = \sqrt{t_{ii}} \left(\sum_{j=1}^{N} m_{ij} \, |\lambda_j| \, m_{kj}\right) \sqrt{t_{kk}}.$$

Together with formula (17) this implies that for $i = k$, the diagonal coefficients c_{ii}^{alt} of C_{alt} are the same as the ones of C:

$$c_{ii}^{\text{alt}} = t_{ii} \left(\sum_{j=1}^{N} m_{ij} \, |\lambda_j| \, m_{ij}\right) = c_{ii}.$$

To prove the estimate (19), consider the following matrix $|C| := M \, |\Lambda| \, M^{-1} = M \, |\Lambda| \, M^t$. The triangular inequality implies:

$$|||C - C_{\text{alt}}|||_{\text{F}} \leq |||C_{\text{alt}} - |C|\,|||_{\text{F}} + |||\,|C| - C|||_{\text{F}}\,.$$

The Frobenius norm being invariant for orthogonal transformation, we have:

$$\begin{aligned}|||\,|\Lambda| - \Lambda|||_{\mathrm{F}} &= \left|\left|\left|M\left(|\Lambda| - \Lambda\right)M^t\right|\right|\right|_{\mathrm{F}} = \left|\left|\left|\left(M\,|\Lambda| - M\Lambda\right)M^t\right|\right|\right|_{\mathrm{F}} \\ &= \left|\left|\left|M\,|\Lambda|\,M^t - M\Lambda M^t\right|\right|\right|_{\mathrm{F}} = |||\,|C| - C|||_{\mathrm{F}}\,.\end{aligned}$$

Since $|\Lambda| - \Lambda$ is a diagonal matrix for which the only positive terms are the ones of the form $2\,|\lambda_i|$ for $\lambda_i < 0$, the Frobenius norm property $|||A|||_{\mathrm{F}} = \sqrt{\mathrm{Tr}(AA^t)}$, implies that:

$$|||\,|C| - C|||_{\mathrm{F}} = |||\,|\Lambda| - \Lambda|||_{\mathrm{F}} = \sqrt{\mathrm{Tr}((|\Lambda| - \Lambda)^2)} = 2\left(\sum_{i:\lambda_i<0} \lambda_i^2\right)^{\frac{1}{2}}.$$

Let $\mathbf{1}$ be the $N \times N$ matrix with all entries equal to one and the matrix $\tilde{T} := [\sqrt{t_{ii}}\sqrt{t_{kk}}]_{i,k\in\{1,\ldots,N\}}$. Consider $A \circ B$ the Hadamard product between A and B. Using proprieties of the Hadamard product, the fact that the Frobenius norm is sub-multiplicative and again the invariance of orthogonal transformations under the Hadamard norm, we have:

$$\begin{aligned}|||C_{\mathrm{alt}} - |C|\,|||_{\mathrm{F}} &= \left|\left|\left||C| - \tilde{T} \circ |C|\right|\right|\right|_{\mathrm{F}} = \left|\left|\left|\mathbf{1} \circ |C| - \tilde{T} \circ |C|\right|\right|\right|_{\mathrm{F}} \\ &= \left|\left|\left|\left(\mathbf{1} - \tilde{T}\right) \circ |C|\right|\right|\right|_{\mathrm{F}} \le \left|\left|\left|\left(\mathbf{1} - \tilde{T}\right)\right|\right|\right|_{\mathrm{F}} |||\,|C|\,|||_{\mathrm{F}} \\ &= \left|\left|\left|\left(\mathbf{1} - \tilde{T}\right)\right|\right|\right|_{\mathrm{F}} |||\,|\Lambda|\,|||_{\mathrm{F}}\,.\end{aligned}$$

Proof of the estimate (19) is obtained by noting that:

$$|||\,|\Lambda|\,|||_{\mathrm{F}} = \sqrt{\mathrm{Tr}(|\Lambda|\,|\Lambda|^t)} = \sqrt{\sum_{i=1}^{N} \lambda_i^2},$$

and after some calculations, that we have:

$$\left|\left|\left|\left(\mathbf{1} - \tilde{T}\right)\right|\right|\right|_{\mathrm{F}} = \sqrt{\mathrm{Tr}\left(\left(\mathbf{1} - \tilde{T}\right)\left(\mathbf{1} - \tilde{T}\right)^t\right)} = \sqrt{\sum_{i,j=1}^{N} \left(1 - \sqrt{t_{ii}}\sqrt{t_{jj}}\right)^2}.$$

■

Observation 11. *It would be interesting to have, for the spectral norm, a finer estimate (and if possible, optimal) than the one we just obtained in (19) for the Frobenius norm. Note that if* $|||\cdot|||$ *is the spectral norm then for an invertible matrix* C*:* $|||C||| \le |||C|||_F \le N\,|||C|||$.

Observation 12. *One consequence of the inequality* (16) *and independency is the following. Since:*

$$\rho(X_i + E_i, X_j + E_j) = \frac{cov(X_i, X_j) + cov(E_i, E_j)}{\sqrt{\sigma(X_i)^2 + \sigma(E_i)^2}\sqrt{\sigma(X_j)^2 + \sigma(E_j)^2}},$$

for $i \neq j$ we have the following upper bound of the correlations of the perturbated returns:

$$\rho(X_i + E_i, X_j + E_j) \leq \frac{cov(X_i, X_j) + \sigma(E_i)\sigma(E_j)}{\sqrt{\sigma(X_i)^2 + \sigma(E_i)^2}\sqrt{\sigma(X_j)^2 + \sigma(E_j)^2}}. \tag{20}$$

Formula (20) *shows that the correlation matrix of the perturbed data can't be arbitrarily close to the matrix with all entries equal to one, which corresponds to maximal V@R. Furthermore formula* (14) *gives:*

$$\begin{aligned}\rho(X_i + E_i, X_j + E_j) &\leq \rho(X_i, X_j) + \rho(E_i, E_j)\frac{\sigma(E_i)}{\sigma(X_i)}\frac{\sigma(E_j)}{\sigma(X_j)} \\ &\leq \rho(X_i, X_j) + \frac{\sigma(E_i)}{\sigma(X_i)}\frac{\sigma(E_j)}{\sigma(X_j)} \approx \rho(X_i, X_j) + \delta_2\pi,\end{aligned}$$

so that any correlation of the perturbed data is bounded from above by the sum of the corresponding correlation of the initial data and an error term identical to the one considered for each perturbed distribution.

The next theorem shows that if the diagonal terms of the correlation matrix of the perturbed data are given by (20) then this condition determines the maximal V@R of the portfolio of perturbed returns.

Theorem 13. *For $i = 1, \ldots, N$, let the standard deviation of the pertubation $\sigma(E_i)$ be defined by* (15). *Also let for $i = j$ $\rho_{ii}^{max} = 1$ and, for $i \neq j$ let the non diagonal terms of the maximal correlation matrix be given by*

$$\rho_{ij}^{max} := \rho^{max}(X_i + E_i, X_j + E_j) := \frac{cov(X_i, X_j) + \sigma(E_i)\sigma(E_j)}{\sqrt{\sigma(X_i)^2 + \sigma(E_i)^2}\sqrt{\sigma(X_j)^2 + \sigma(E_j)^2}}, \tag{21}$$

assuming for $i \neq j$ that we have $\rho_{ij}^{max} \in [-1, 1]$ and that the matrix $\mathbf{R}^{max} = [\rho_{ij}^{max}]_{i,j \in \{i,\ldots,N\}}$ is positive definite. Then for $\alpha \in]0, \frac{1}{2}[$ the mean V@R of the perturbed portfolio is given by

$$V@R_{\alpha,\ mean}^{P\&G_\Sigma} = -\left(\sum_{i,j=1}^{N} \sqrt{\sigma(X_i)^2 + \sigma(E_i)^2}\sqrt{\sigma(X_j)^2 + \sigma(E_j)^2}\left(\Phi^{-1}(\alpha)\right)^2 \rho_{ij}^{max}\right)^{\frac{1}{2}}.$$

Proof. This is a direct consequence of corollary 5 and formula (2) since one can see that for each asset $i \in \{1, \ldots, N\}$, the corresponding perturbed V@R is given by: $\sqrt{\sigma(X_i)^2 + \sigma(E_i)^2}\,\Phi^{-1}(\alpha)$. ∎

Observation 14. *For example, if* δ_1 *e* δ_2 *are chosen such that in formula* (15) *we have:*

$$\sigma(E_i) = \sqrt{\frac{2\delta_1 \sigma(X_i)}{|\Phi^{-1}(\alpha)|} + \left(\frac{\delta_1}{|\Phi^{-1}(\alpha)|}\right)^2},$$

then the mean V@R of the perturbed portfolio, given $\alpha \in]0, \frac{1}{2}[$, *is:*

$$V@R^{P\&G_\Sigma}_{\alpha,\, mean} = -\left(\sum_{i=1}^{N} \left(\left| V@R^{X_i}_{\alpha,\, mean}\right| + \delta_1\right)^2 + 2\sum_{i<j} cov(X_i, X_j)\left|\Phi^{-1}(\alpha)\right| \right.$$
$$\left. + \sqrt{\left(2\delta_1 \left| V@R^{X_i}_{\alpha,\, mean}\right| + \delta_1^2\right)\left(2\delta_1 \left| V@R^{X_j}_{\alpha,\, mean}\right| + \delta_1^2\right)}\right)^{\frac{1}{2}}.$$

4. An application with real data

We present next a real world application showing that for a standard small portfolio there exist a large class of statistically undetectable perturbations of the individual returns which change significantly the V@R of the portfolio. This practical application of this paper results was independently developed in [4]. The 10 assets used in the portfolio and their respective portfolio weights are MilleniumBCP (0.16), SEMAPA (0.075), BES (0.11), EDP (0.05), Teixeira Duarte (0.075), Brisa (0.09), Mota-Engil (0.125), Portucel (0.14), SonaeCom (0.075) and ZonMultimédia (0.1). The monthly closing prices were obtained from May 2007 to May 2010, that is, 37 observations for each asset. All the additive returns were tested for normality and for none of them the hypothesis of a Gaussian distribution was rejected. Again, one can see from (7) that the mean V@R is proportional to the total value of the portfolio, so that, from now on we will assume this value is set to one. This allow us to look at the values of the mean V@R as percentage of the total value of the portfolio.

Observation 15. *According to Section* 2 *all the returns considered are additive instead of multiplicative. In this case, tests are needed to check for normality and independency. All of our additive returns were tested and the hypothesis was rejected in none of them.*

We present next a summary of the results of our application to real data, done with *Mathematica* 7 *TM* software [2]. We will first present deterministic results and then go on to the results obtained in a simulation of perturbed returns.

- The mean V@Rs of the portfolio are given by

BCP	SEMAPA	BES	EDP	TD
−0.0294738	−0.00813451	−0.0219616	−0.00524454	−0.0239367
BRISA	ME	PORTUCEL	SC	ZONM
−0.0116766	−0.0236536	−0.0181462	−0.0157889	−0.0158876

  We may observe that they are all negative.

- We computed the correlation matrix of the portfolio's returns. To this correlation matrix corresponds a mean V@R of −0.13238 and a maximum mean V@R of −0.173904 (according to Observation 6).

- Given that $\delta_1 = \delta_2 = 0.1$, according to formula (15) one can compute the standard deviation for each of the perturbed assets and obtain the following results for their variance:

BCP	SEMAPA	BES	EDP	TD
0.00462948	0.00160487	0.00543804	0.00150098	0.0138965
BRISA	ME	PORTUCEL	SC	ZONM
0.0022964	0.00488509	0.00229201	0.00604614	0.00344364

- Next, using Theorem 13 and the standard deviations of each perturbed asset according to formula (15), the correlation matrix of the perturbed returns was computed. As announced we have the following important result. The maximum mean V@R of the portfolio computed using Theorem 13 is −0.169373 which is close to the maximum mean V@R computed according to Observation 6, −0.173904.

Next, we detail a simulation example. We simulate Gaussian perturbation of the returns of the portfolio according to Step 2 of Section 3.2. We compute the corresponding mean V@Rs for each asset and the V@R of the portfolio. Finally, we tested the equality between the distributions of the perturbed returns and the distributions of the initial, non perturbed, returns.

- We computed an instance of a variance-covariance matrix of the perturbation, built according to the Step 2 of Section 3.2. The first matrix obtained using this method has non diagonal coefficients $\mathrm{cov}(E_i, E_j)$ with uniform distribution in the intervals $[\sigma(E_i)\sigma(E_j) - \gamma, \sigma(E_i)\sigma(E_j)]$ with $\gamma = 0.0003$.

[2] All the data and computational results used in our application can be obtained at the following address: http://ferrari.dmat.fct.unl.pt/personal/mle/pps/pm-mle2009a.html.

This matrix being non positive definite, we had to apply Rebonato's method. Using the spectral norm, the distance between this first matrix and the one computed with Rebonato's method is:

$$|||C - C_{\text{alt}}||| = 0.000966904. \tag{22}$$

This distance, computed using the Frobenius norm, is:

$$|||C - C_{\text{alt}}|||_{\text{F}} = 0.00118581. \tag{23}$$

On the other hand, the value obtained from the upper bound of inequality (19) is 0.018035. Using the L^2 norm, the distance between the eigenvectors of the first matrix and the ones of the altered matrix is 0.00093739. These values show that the process transforming a singular matrix to a positive definite one, does not change the matrix significantly.

- The next table shows the mean V@Rs of the assets of the perturbed portfolio which can be compared to the mean V@Rs of the assets in the non perturbed portfolio. The norm of the difference between the vector of initial mean V@Rs and the vector of mean V@Rs of the perturbed portfolio is 0.0128048. This shows that even a small perturbation can alter significantly all the V@Rs.

BCP	SEMAPA	BES	EDP	TD
−0.0318634	−0.0086518	−0.0249565	−0.0056083	−0.0260704
BRISA	ME	PORTUCEL	SC	ZONM
−0.0122701	−0.0251029	−0.0202031	−0.0166392	−0.0175773

- We then have the second most important result of this application to real data. In our simulation, the mean V@R of the perturbed portfolio is −0.1766 which is even greater than the maximum V@R of the original portfolio (according to Observation 6) which, as noted above, is −0.173904.

- Given the levels of confidence 0.05 and 0.01, we used the Kolmogorov and the Kolmogorov-Smirnov tests to check, for each asset, that the perturbed distribution is statistically indistinguishable from the original one. With the Kolmogorov test, for each perturbed asset, the hypothesis of normality with mean value and variance estimated from the original data was not rejected. With the Kolmogorov-Smirnov test, the hypothesis that both perturbed and original distributions are equal was, also, not rejected.

5. Conclusion and open questions

In the benchmark applications of the *delta-normal* model, an a priori estimation of the correlation matrix is generally given. Our results show that in the case that the returns of the assets suffer a statistically undetectable perturbation, with no significant impact on the correlation matrix, the V@R of the portfolio may change to the point it is close to the maximum V@R possible. Therefore, as a measure of the risk associated to the portfolio, the mere V@R is insufficient. If, nevertheless, we have to use the V@R methodology, it is fundamental that the initial correlation matrix and its estimation method are explicitly given, together with the portfolio V@R.

To complement our study, it would be interesting to have a statistical test allowing to decide that both the initial and the perturbed variance-covariance matrices are equal.

In Section 3.2 we saw that given a variance-covariance matrix C and C_{alt} the matrix obtained after the applying the Rebonato's method then the spectral norm $|||C - C_{\text{alt}}|||$ is small. This property, which was confirmed in our practical application, lacks a proper justification (see formula (22) and Observation 11). According to our numerical results, we propose the following conjecture: the spectral norm is comparable to the norm obtained by the Euclidian norm of the difference between the eigenvectors obtained after Rebonato's modification and the original eigenvectors [3].

One way to compare the correlation matrix of the perturbed returns $\not{\mathbf{R}}_{\text{est}}(\mathbf{X} + \mathbf{E})$ with the original estimated matrix $\not{\mathbf{R}}_{\text{est}}(\mathbf{X})$, is to define a perturbation index. So consider the ratio of the Frobenius norm of the difference of the matrices to the norm of the worst possible case (this being is the difference between the matrix with all entries equal to one and the identity matrix). Using our numerical example this gives:

$$I_{\text{per}} := \frac{|||\not{\mathbf{R}}_{\text{est}}(\mathbf{X} + \mathbf{E}) - \not{\mathbf{R}}_{\text{est}}(\mathbf{X})|||}{\sqrt{N^2 - N}} = \frac{1.3423}{90} = 0.01491444 \ ,$$

that is, a perturbation with index inferior to 1,5%. This confirms our claim that it is indeed a small perturbation with large effects. A deeper study of this index should bring interesting results.

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