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ON MELANCHOLIC MAGIC SQUARES[1]

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Abstract

Starting with Dürer's magic square which appears in the well-known copper plate engraving *Melencolia* we consider the class of melancholic magic squares. Each member of this class exhibits the same 86 patterns of Dürer's magic square and is magic again. Special attention is paid to the eigenstructure of melancholic magic squares, their group inverse and their Moore-Penrose inverse. It is seen how the patterns of the original Dürer square to a large extent are passed down also to the inverses of the melancholic magic squares.

Keywords: magic squares, patterns, group inverse, Moore-Penrose inverse, eigenvalues and eigenvectors.

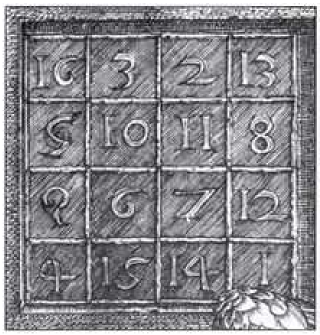
2010 Mathematics Subject Classification: 00A08, 15A09, 15A18.

[1]Dedicated to George Styan on the occasion of his 75th birthday.

1. Introduction

In [5] the authors of this paper gave an introduction to magic squares from the matrix point of view. To prepare such an article they were encouraged by George Styan, then Editor-in-Chief of *IMAGE, The Bulletin of the International Linear Algebra Society*, who also took over the burden of editing a somewhat poor manuscript. We dedicate the present paper to this outstanding scientist and friend, who after his retirement also became more and more interested in the wonderland of magic squares.

As in [5], the copper-plate engraving *Melencolia* plays the central role in our paper. It was created by the German painter and printmaker Albrecht Dürer (1471–1528), whose residence was Nuremberg, and who is regarded as one of the greatest artists of the Renaissance. George Styan and the first author had the opportunity to see prints of two versions of *Melencolia* at Museum Otto Schäfer in Schweinfurt, Germany, on March 15th, 2007. *Melencolia* was completed in 1514, which is the year when the artist's mother died. It is the subject of various interpretations of its symbols which still have not come to an end.



The engraving *Melencolia* and the magic square beneath the bell.

The following wonderful description is taken from Andrews [1, p. 146]:

> *The symbolism of this engraving has interested to a marked degree almost every observer. The figure of the brooding genius sitting listless and dejected amid her uncompleted labors, the scattered tools, the*

swaying balance, the flowing sands of the glass, and the magic square of 16 beneath the bell — these and other details reveal an attitude of mind and a connection of thought, which the great artist never expressed in words, but left for every beholder to interpret for himself.

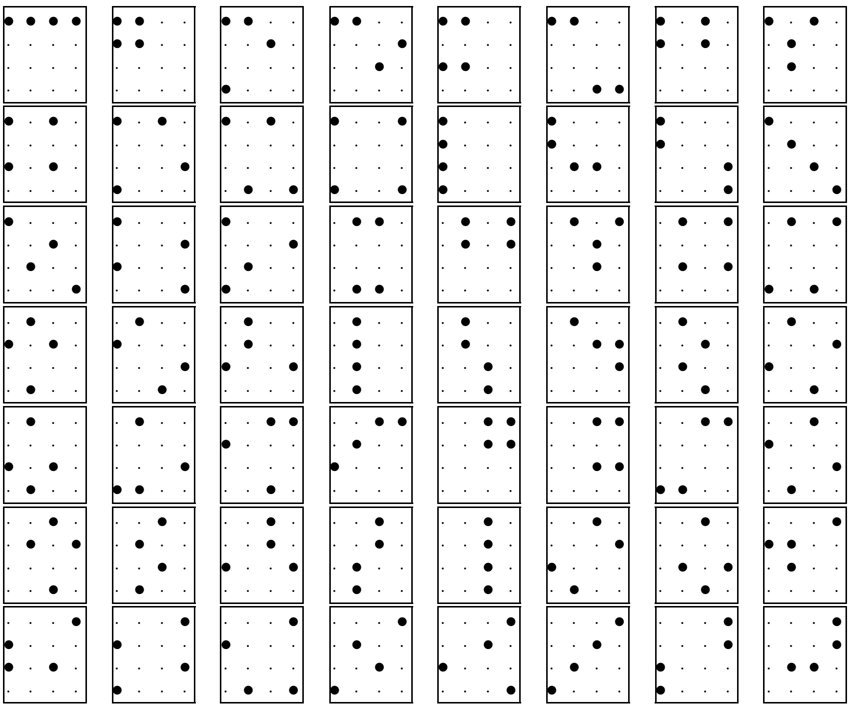
2. Melancholic magic squares

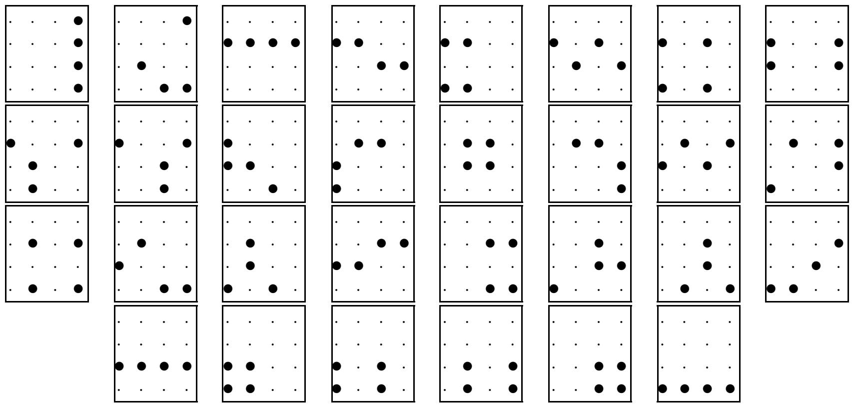
Subsequently we pay more attention to the magic square from the engraving in *Melencolia*, located beneath the bell. As a matrix it can be written as

$$\mathbf{D} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}.$$

Its columns, rows and two diagonal sums are all equal to $s(\mathbf{D}) = 34$ which is called the magic number. In the square there are still more patterns of numbers with sum 34, for instance the four $2{\times}2$ corner subsquares.

The following display describes the 86 patterns of 4 numbers with sum 34 in Dürer's magic square.





The matrix $\mathbf{D}$ is a classic magic square since its entries are the positive integers $1, 2, \ldots, 4^2 = 16$. In general, an $n \times n$ classic magic square $\mathbf{M}$ has magic number $s(\mathbf{M}) = n(n^2+1)/2$.

The patterns in $\mathbf{D}$ correspond to 86 linear equations in 16 unknowns. With the support of *Mathematica* it can be shown that there is an infinite number of magic squares with these patterns and magic number $s = 34$, given by

$$\mathbf{M} = 34\mathbf{J}_4 + \alpha\mathbf{M}_0\,,$$

where $\mathbf{J}_4 = \frac{1}{4}\mathbf{1}_4\mathbf{1}_4'$, $\mathbf{1}_4 = (1,1,1,1)'$, $\alpha \in \mathbb{R}$ and

$$\mathbf{M}_0 = \begin{pmatrix} -15 & 11 & 13 & -9 \\ 7 & -3 & -5 & 1 \\ -1 & 5 & 3 & -7 \\ 9 & -13 & -11 & 15 \end{pmatrix}.$$

This class also comprises magic squares with entries that are not necessarily an integer. Of course, the "basis matrix" $\mathbf{M}_0$ and every $\mathbf{M}$ displays the original patterns of the Dürer matrix $\mathbf{D}$.

In the following we consider the more general class $\mathfrak{M}$ of "melancholic magic squares", where $\mathbf{M} \in \mathfrak{M}$ can be written as

$$\mathbf{M} = s\mathbf{J}_4 + \alpha\mathbf{M}_0\,, \quad s \in \mathbb{R}\,, \alpha \in \mathbb{R}\,.$$

Each $\mathbf{M}$ is a magic square with magic number $s(\mathbf{M}) = s$ and the 86 patterns of $\mathbf{D}$. Thus $\mathbf{M}$ inherits this property. Note that $\mathbf{D} = 34\mathbf{J}_4 - \frac{1}{2}\mathbf{M}_0$ belongs to $\mathfrak{M}$, of course.

A full rank decomposition of $\mathbf{M}_0$ is given by $\mathbf{M}_0 = \mathbf{G}\mathbf{H}$, where

$$\mathbf{G} = \begin{pmatrix} -15 & 11 \\ 7 & -3 \\ -1 & 5 \\ 9 & -13 \end{pmatrix} \quad \text{and} \quad \mathbf{H} = \begin{pmatrix} 1 & 0 & -0.5 & -0.5 \\ 0 & 1 & 0.5 & -1.5 \end{pmatrix}.$$

It follows that $\operatorname{rank}(\mathbf{G}) = \operatorname{rank}(\mathbf{H}) = \operatorname{rank}(\mathbf{M}_0) = 2$. Consequently, for $\mathbf{M} \in \mathfrak{M}$ we get

$$\operatorname{rank}(\mathbf{M}) = \begin{cases} 0, & \text{if } \alpha = 0,\ s = 0; \\ 1, & \text{if } \alpha = 0,\ s \neq 0; \\ 2, & \text{if } \alpha \neq 0,\ s = 0; \\ 3, & \text{if } \alpha \neq 0,\ s \neq 0. \end{cases}$$

This implies that the Dürer matrix has $\operatorname{rank}(\mathbf{D}) = 3$. Note that $\mathbf{J}_4\mathbf{M}_0 = \mathbf{M}_0\mathbf{J}_4 = \mathbf{0}$.

Some straightforward calculations yield $\mathbf{HGHG} = 256\mathbf{I}_2$, where $\mathbf{I}_n$ is the $n \times n$ identity matrix. It follows that

$$\mathbf{M}_0^3 = (\mathbf{GH})^3 = \mathbf{G}(\mathbf{HGHG})\mathbf{H} = 256\mathbf{M}_0\,.$$

Hence the third power of a melancholic magic square $\mathbf{M} = s\mathbf{J}_4 + \alpha\mathbf{M}_0$ is melancholic again:

$$\mathbf{M}^3 = s^3\mathbf{J}_4 + \alpha^3\mathbf{M}_0^3 = s^3\mathbf{J}_4 + 256\alpha^3\mathbf{M}_0$$

with $s(\mathbf{M}^3) = s^3$.

It can be easily seen that $\mathbf{M}^2$ is semimagic, i.e., column and rows add up to s^2, also the entries of the second diagonal have sum s^2, but not those of the first.

3. Group inverse

Since $\mathbf{M}_0^3 = 256\mathbf{M}_0$, it follows that $\operatorname{rank}(\mathbf{M}_0) = \operatorname{rank}(\mathbf{M}_0^2) = 2$. Hence $\mathbf{M}_0$ has a group inverse $\mathbf{M}_0^{\#}$ which is characterized by the three conditions

(i) $\mathbf{M}_0\mathbf{M}_0^{\#}\mathbf{M}_0 = \mathbf{M}_0$,

(ii) $\mathbf{M}_0^{\#}\mathbf{M}_0\mathbf{M}_0^{\#} = \mathbf{M}_0^{\#}$,

(iii) $\mathbf{M}_0\mathbf{M}_0^{\#} = \mathbf{M}_0^{\#}\mathbf{M}_0$,

see [2, Section 4.4].

The group inverse is unique and can be calculated by means of Theorem 3 in [2, Section 4.4] as

$$\mathbf{M}_0^{\#} = \mathbf{G}(\mathbf{HG})^{-2}\mathbf{H} = \frac{1}{256}\mathbf{M}_0\,,$$

since $(\mathbf{HG})^2 = 256\mathbf{I}_2$. Hence the group inverse of $\mathbf{M}_0$ is also a melancholic magic square.

It is readily established that $\operatorname{rank}((s\mathbf{J}_4+\alpha\mathbf{M}_0)^2) = \operatorname{rank}(s\mathbf{J}_4+\alpha\mathbf{M}_0)$ and due to $\mathbf{J}_4\mathbf{M}_0 = \mathbf{M}_0\mathbf{J}_4 = \mathbf{0}$, for $\mathbf{M} \in \mathfrak{M}$, we have

$$\mathbf{M}^{\#} = (s\mathbf{J}_4+\alpha\mathbf{M}_0)^{\#} = s^{\dagger}\mathbf{J}_4 + \alpha^{\dagger}\mathbf{M}_0^{\#} \; = s^{\dagger}\mathbf{J}_4 + \frac{1}{256}\alpha^{\dagger}\mathbf{M}_0\,,$$

where $\lambda^{\dagger} = 1/\lambda$ if $\lambda \neq 0$ and $\lambda^{\dagger} = 0$ if $\lambda = 0$ for any real number λ. It follows that the group inverse of $\mathbf{D}$ is

$$\mathbf{D}^{\#} = \frac{1}{34}\mathbf{J}_4 - \frac{1}{128}\mathbf{M}_0\,,$$

see [5].

4. Moore-Penrose inverse

We use the full rank decomposition to $\mathbf{M}_0 = \mathbf{G}\mathbf{H}$ to calculate the Moore-Penrose inverse $\mathbf{M}_0^{\dagger}$. It is uniquely determined by the four conditions

(i) $\mathbf{M}_0\mathbf{M}_0^{\dagger}\mathbf{M}_0 = \mathbf{M}_0$,

(ii) $\mathbf{M}_0^{\dagger}\mathbf{M}_0\mathbf{M}_0^{\dagger} = \mathbf{M}_0^{\dagger}$,

(iii) $\mathbf{M}_0\mathbf{M}_0^{\dagger}$ is symmetric,

(iv) $\mathbf{M}_0^{\dagger}\mathbf{M}_0$ is symmetric.

Since $\mathbf{G}$ is of full column and $\mathbf{H}$ of full row rank, $\mathbf{M}_0^{\dagger}$ can be derived from $\mathbf{G}^{\dagger}$ and $\mathbf{H}^{\dagger}$, namely as $\mathbf{M}_0^{\dagger} = \mathbf{H}^{\dagger}\mathbf{G}^{\dagger}$. Some easy calculations yield

$$\mathbf{G}^{\dagger} = \frac{1}{320}\begin{pmatrix} -23 & 21 & 19 & -17 \\ -11 & 17 & 23 & -29 \end{pmatrix} \quad \text{and} \quad \mathbf{H}^{\dagger} = \frac{1}{10}\begin{pmatrix} 7 & -1 \\ -1 & 3 \\ -4 & 2 \\ -2 & -4 \end{pmatrix}.$$

From this we get

$$\mathbf{M}_0^{\dagger} = \frac{1}{320}\begin{pmatrix} -15 & 13 & 11 & -9 \\ -1 & 3 & 5 & -7 \\ 7 & -5 & -3 & 1 \\ 9 & -11 & -13 & 15 \end{pmatrix}.$$

It is seen that $\mathbf{M}_0^{\dagger}$ is magic, but not melancholic. Nevertheless it has also 86 patterns of four numbers adding to zero. Actually, $\mathbf{M}_0$ and $\mathbf{M}_0^{\dagger}$ have 52 patterns in common, and there are 34(!) patterns occurring only in $\mathbf{M}_0$ and 34 patterns to be seen only in $\mathbf{M}_0^\dagger$ (see [5]). The number of patterns in every melancholic magic square is surprisingly high, but can be explained by the subsequent considerations. Other 4×4 magic squares occasionally display a similar structure, but less distinct, see e.g. [3].

Using again $\mathbf{J}_4\mathbf{M}_0 = \mathbf{M}_0\mathbf{J}_4 = \mathbf{0}$, we obtain

$$\mathbf{M}^\dagger = s^\dagger \mathbf{J}_4 + \alpha^\dagger \mathbf{M}_0$$

for every member $\mathbf{M} = s\mathbf{J}_4 + \alpha\mathbf{M}_0$ of $\mathfrak{M}$.

The Dürer matrix $\mathbf{D} = 34\mathbf{J}_4 - \frac{1}{2}\mathbf{M}_0$ has the Moore-Penrose inverse

$$\begin{aligned}\mathbf{D}^\dagger &= \frac{1}{34}\mathbf{J}_4 - 2\mathbf{M}_0^\dagger \\ &= \frac{1}{34\cdot 80}\begin{pmatrix} 275 & -201 & -167 & 173 \\ 37 & -31 & -65 & 139 \\ -99 & 105 & 71 & 3 \\ -133 & 207 & 241 & -235 \end{pmatrix}.\end{aligned}$$

Up to the factor $1/34\cdot 80$ it contains only odd numbers!

From the preceding derivations it is obvious that group and Moore-Penrose inverse of $\mathbf{M}_0$ differ. The same is true for every nontrivial $\mathbf{M}\in\mathfrak{M}$, and thus for the Dürer matrix $\mathbf{D}$. Hence these squares are not EP, i.e. row and column spaces are not equal, see [4]. Having a closer look at $\mathbf{M}_0$ and $\mathbf{M}_0^\dagger$, a surprising similarity emerges. Indeed,

$$\mathbf{M}_0^\dagger = \frac{1}{320}\mathbf{T}\mathbf{M}_0\mathbf{T}\,,$$

where

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the permutation matrix resulting from $\mathbf{I}_4$ by interchanging second and third column. This implies for example that

$$\mathbf{D}^\dagger = \frac{1}{34}\mathbf{J}_4 - \frac{1}{160}\mathbf{T}\mathbf{M}_0\mathbf{T}\,.$$

5. EIGENVALUES AND EIGENVECTORS

The eigenvalues λ_j of every melancholic magic square $\mathbf{M} = s\mathbf{J}_4 + \alpha\mathbf{M}_0$ can be easily computed and are displayed in the following table along with corresponding eigenvectors.

Eigenvalues	Eigenvectors
$\lambda_1 = s$	$\mathbf{v}_1 = (1, 1, 1, 1)'$
$\lambda_2 = 0$	$\mathbf{v}_2 = (-1, 3, -3, 1)'$
$\lambda_3 = 16\alpha$	$\mathbf{v}_3 = (-1, 0, -1, 2)'$
$\lambda_4 = -16\alpha$	$\mathbf{v}_3 = (-2, 1, 0, 1)'$

For the Dürer matrix $\mathbf{D} = 34\mathbf{J}_4 - \frac{1}{2}\mathbf{M}_0$ we have $\lambda_1 = 34$, $\lambda_2 = 0$, $\lambda_1 = -8$ and $\lambda_1 = 8$ with the eigenvectors from the table above.

Note that the characteristic polynomial of $\mathbf{M} = s\mathbf{J}_4 + \alpha\mathbf{M}_0$ can be written as

$$p(\lambda) = \det(\lambda\mathbf{I}_4 - \mathbf{M}) = \lambda(\lambda - s)(\lambda^2 - 256\alpha^2)\,.$$

Following [4], we see that a melancholic magic square $\mathbf{M}$ is "keyed" with magic key $\kappa = 256\alpha^2$, a term coined by George Styan. In general, an $n \times n$ magic square $\mathbf{N}$ with magic number $s(\mathbf{N})$ is keyed whenever its characteristic polynomial is of the form $\det(\lambda\mathbf{I}_n - \lambda\mathbf{N}) = \lambda^{n-3}(\lambda - s(\mathbf{N}))(\lambda^2 - \kappa)$ where κ is the magic key of $\mathbf{N}$. The magic key of the Dürer matrix $\mathbf{D}$ is $\kappa = 64$.

Since $\mathbf{M}$ is not EP (see Section 4), it also follows that $\mathbf{M}^2$ is not symmetric (see [4] again). This fact can alternatively be confirmed by direct determination of $\mathbf{M}^2$. Observe that $\mathbf{M}_0^3 = \kappa_0\mathbf{M}_0$, where $\kappa_0 = 256$ is the magic key of $\mathbf{M}_0$.

6. CONCLUDING REMARKS

A melancholic magic square $\mathbf{M} = s\mathbf{J}_4 + \alpha\mathbf{M}_0$ is associated, i.e., $\mathbf{M} + \mathbf{FMF} = 2s\mathbf{J}_4$, where

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is the so-called flip matrix (see [3, p. 8] and [4]). Other names for this notion can be found in the literature like counteridentity matrix or exchange matrix. We finally note that higher powers of $\mathbf{M}$ can be easily calculated, mainly by using the simple formula for $\mathbf{M}^3$ of Section 2 and the fact that $\mathbf{M}_0\mathbf{J}_4 = \mathbf{J}_4\mathbf{M}_0 = \mathbf{0}$, which we used before several times.

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