Discussiones Mathematicae
Probability and Statistics 33 (2013) 47–64
doi:10.7151/dmps.1150

EXTREMAL (IN)DEPENDENCE OF A MAXIMUM AUTOREGRESSIVE PROCESS

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Abstract

Maximum autoregressive processes like MARMA (Davis and Resnick, [5] 1989) or power MARMA (Ferreira and Canto e Castro, [12] 2008) have singular joint distributions, an unrealistic feature in most applications. To overcome this pitfall, absolute continuous versions were presented in Alpuim and Athayde [2] (1990) and Ferreira and Canto e Castro [14] (2010b), respectively. We consider an extended version of absolute continuous maximum autoregressive processes that accommodates both asymptotic tail dependence and independence. A full characterization of the bivariate lag-m tail dependence is presented. This will be useful in an adjustment procedure of the model to real data. An illustration with financial data is presented at the end.

Keywords: extreme value theory, autoregressive processes, tail dependence, asymptotic tail independence.

2010 Mathematics Subject Classification: Primary: 60G70; Secondary: 62G32.

1. Introduction

Many areas of environment, geophysics, engineering or finance demand statistical tools to analyze the extreme events. The classical extreme value theory (EVT), firstly developed for independent and identically distributed (i.i.d.) random variables (r.v.'s), has been expanded under the more realistic assumption of dependence. Linear autoregressive series like ARMA are perhaps the best known and used in the modeling. A heavy-tailed distribution function (d.f.) F can be represented through

$$1 - F(x) = x^{-1/\gamma} L(x), \tag{1}$$

where L is a slowly varying function at ∞, i.e, $L(ax)/L(x) \to 1$ for any fixed $a > 0$, as $x \to \infty$, and $\gamma > 0$ is the tail index that rules the "heaviness" of the tail (the larger the γ the heavier the tail). Condition (1) means that $1 - F$ is a regularly varying function with index $-1/\gamma$ and models with this formulation are also denoted Pareto-type. In domains of applications of EVT, taking the maxima or the sum of two heavy-tailed r.v.'s is basically equivalent for the tail behavior (see Embrechts *et al.* [4] 1997, Chap. 2). The max-autoregressive moving average processes MARMA (Davis and Resnick, [5] 1989) are derived from ARMA by replacing summation by the maximum operator. The particular case MAR(1) or ARMAX, given by

$$X_i = cX_{i-1} \vee Z_i,\ 0 < c < 1, \tag{2}$$

with $\{Z_i\}_{i\geq 1}$ an i.i.d. innovations sequence having non-negative support and X_i independent of Z_j, for any integers $i < j$, has been widely studied in the literature (Alpuim [1] 1989, Lebedev [19] 2008, Ferreira [10, 11] 2011/2013, Ferreira and Canto e Castro [12, 13] 2008/2010a, among others). Since MARMA finite-dimensional distributions can easily be written explicitly, they are more convenient for analysis than heavy-tailed linear ARMA. We remark that second or even first moments may not exist for heavy-tailed r.v.'s and the lag-m auto-correlation function (ACF), with m positive integer, cannot be obtained to assess temporal dependence. Alternative measures within EVT framework have been considered in the literature, like the *lag-m tail dependence coefficient*, in short *lag-m TDC* (Zhang *et al.* [20] 2005),

$$\lambda_m = \lim_{t\downarrow 0} P(F(X_{1+m}) > 1 - t | F(X_1) > 1 - t), \tag{3}$$

where F is the marginal d.f. of a stationary process $\{X_i\}_{i\geq 1}$. Loosely stated, λ_m is the probability of one variable being extreme given that another lag-m apart is extreme too. If $\lambda_m > 0$ then the random pair (X_1, X_{1+m}) is tail dependent and $\lambda_m = 0$ means tail independence. This latter case encompasses exact independence and asymptotically vanishing dependence, i.e., the degree of dependence between exceedances decreases as $t \downarrow 0$, with the extremal behavior of $\{X_i\}_{i\geq 1}$ increasingly approaching an i.i.d. series at higher thresholds. This phenomenon has been noticed in practical and theoretical applications which include the well-known autoregressive Gaussian processes. The degree of asymptotically vanishing dependence can be measured through the Ledford and Tawn coefficient at lag-m, η_m (see Heffernan *et al.* [17] 2007). More precisely, as $t \downarrow 0$,

$$P(F(X_1) > 1-t, F(X_{1+m}) > 1-t) \sim t^{1/\eta_m} L(1/t), \tag{4}$$

where L is a slowly varying function at ∞ and $\eta_m \in (0,1]$. It is easily seen that X_1 and X_{1+m} are tail dependent when $\eta_m = 1$ and $L(1/t) \not\to 0$ as $t \downarrow 0$, and are asymptotically tail independent otherwise. Moreover, the r.v.'s are positively associated when $1/2 < \eta_m < 1$, (nearly) independent when $\eta_m = 1/2$ and negatively associated when $0 < \eta_m < 1/2$. Observe that (4) can be formulated as

$$P\left(\min\left(\frac{1}{1-F(X_1)}, \frac{1}{1-F(X_{1+m})}\right) > t\right) \sim t^{-1/\eta_m} L(t), \tag{5}$$

which, according to (1), means that $P(\min((1-F(X_1))^{-1}, (1-F(X_{1+m}))^{-1}) > t)$ is a regularly varying function with index $-1/\eta_m$ and thus η_m corresponds to the tail index of

$$\min\left(\frac{1}{1-F(X_1)}, \frac{1}{1-F(X_{1+m})}\right). \tag{6}$$

In the sequel, coefficient η_m shall be shortly referred *lag-m LTC*.

MARMA processes are tail dependent. In particular, the ARMAX model in (2) with Pareto-type marginals given in (1) has $\lambda_m = c^{m/\gamma}$ (Ferreira, [10] 2011) and so $\eta_m = 1$. Thus it is not suitable for data presenting asymptotic tail independence. This gap was bridged with other processes, e.g., power MARMA (Ferreira and Canto e Castro, [12] 2008) and extended moving maxima (Heffernan *et al.*, [17] 2007). For instance, the power ARMAX, in short pARMAX, obtained under the same assumptions of ARMAX by just replacing the factor parameter c by an exponent p, i.e.,

$$X_i = X_{i-1}^p \vee Z_i, \tag{7}$$

with $0 < p < 1$, has $\eta_m = \max(1/2, p^m)$ (and of course $\lambda_m = 0$).

Now we call the attention for the singular feature of ARMAX to possibly generate runs of values in exact geometric progression. A long sample path of the process would also allow us to know the value of p. This behavior might be found in some specific scenarios (e.g. economic ones) but seems implausible in most of the practical cases. A minor modification of the process will be free of this defect. Alpuim and Athayde [2] (1990) presented a model that, instead of a fixed value of p, considers a randomly selected value of p at each stage of the process. More precisely, the absolutely continuous version of ARMAX is given by

$$X_i = U_i X_{i-1} \vee Z_i, \tag{8}$$

where $\{U_i\}_{i \geq 1}$ is an i.i.d. sequence with support in $(0,1)$ and U_j independent of $\{Z_i\}_{i \geq 1}$ for all integer j.

An analogous phenomenon was noticed in the study of the logarithm transform of pARMAX. An absolutely continuous version was latter presented in Ferreira and Canto e Castro [14] (2010b),

$$X_i = U_i X_{i-1}^p \vee Z_i, \tag{9}$$

with $0 < p < 1$ and denoted pRARMAX.

Here we consider model (9) with $0 < p \leq 1$, and thus denoted *extended pRARMAX*, in short EpRARMAX. A sample path of this process with $p = 1$ and with $p = 0.7$ can be seen in Figure 1 (left and right, respectively).

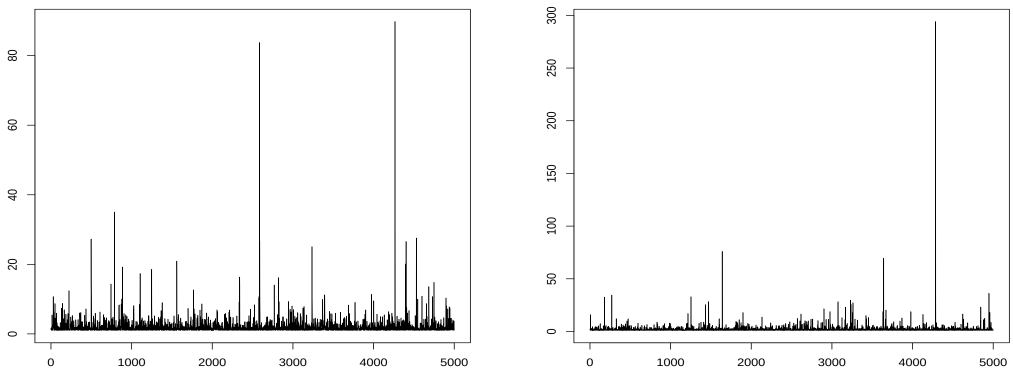

Figure 1. Five thousand realizations of an EpRARMAX process with $U_i \frown \mathrm{U}(0,1)$ and $Z_i \frown \mathrm{Pareto}(0.7)$, $i = 1, \ldots, 5000$, considering $p = 1$ (left) and $p = 0.7$ (right).

It is evident the occurrence of sudden large peaks, also observed in heavy-tailed ARMA and MARMA processes (Davis and Resnick, [5] 1989). We compute the lag-m tail measures TDC in (3) and LTC in (4) and find that the EpRARMAX process comprises both lag-m tail independence and dependence (Section 2). The estimation of these measures will be addressed and allow us to estimate the parameter p. In Section 3, this will be used in an adjustment methodology of the EpRARMAX process to a dataset. An application to financial data illustrates the procedure.

2. Extremal (in)dependence of EpRARMAX

Consider nonnegative r.v.'s, $\{Z_i\}_{i\geq 1}$ and $\{U_i\}_{i\geq 1}$, i.i.d. copies of Z and U, respectively, U_j independent of $\{Z_i\}_{i\geq 1}$, for all integer j. A sequence $\{X_i\}_{i\geq 1}$ is said to be an *EpRARMAX* process if it satisfies the recursion,

$$X_i = U_i X_{i-1}^p \vee Z_i\ , \quad 0 < p \leq 1. \tag{10}$$

Admit that Z has non-degenerate d.f. F_Z. The existence of an unique stationary solution of equation (10) is assured by considering U with support $\mathcal{U} \subseteq (0,1)$. Denote F_U the d.f. of U and K the marginal d.f. of the EpRARMAX. The stationarity equation of the process is thus given by

$$K(x) = F_Z(x) \int_{\mathcal{U}} K((x/u)^{1/p}) dF_U(u). \tag{11}$$

For details, see Ferreira and Canto e Castro [14] (2010b).

Observe that if U is degenerate in 1 we obtain the pARMAX in (7), taking $p = 1$ leads to model (8) and, in addition, if U is degenerate in c, $0 < c < 1$, we obtain the ARMAX process in (2).

In the sequel we will consider the case of an EpRARMAX process with Pareto-type marginal d.f. K, i.e.,

$$K(x) = 1 - x^{-1/\gamma} L_K(x), \tag{12}$$

with tail index $\gamma > 0$. According to (11), we thus have the following definition of F_Z:

$$F_Z(x) = K(x)\left[1 - x^{-1/(p\gamma)} L_K\big(x^{1/p}\big) \int_{\mathcal{U}} u^{1/(p\gamma)} \frac{L_K\big(\big(\frac{x}{u}\big)^{1/p}\big)}{L_K\big(x^{1/p}\big)} dF_U(u)\right]^{-1}.$$

Note that, $L_K\big(\big(\frac{x}{u}\big)^{1/p}\big)/L_K\big(x^{1/p}\big) \to 1$, as $x \to \infty$, and thus by the Dominated Convergence Theorem,

$$F_Z(x) \sim K(x)\Big[1 - x^{-1/(p\gamma)} L_K\big(x^{1/p}\big) E\big(U^{1/(p\gamma)}\big)\Big]^{-1}, \tag{13}$$

where notation $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$.

It is not difficult to conclude that the tail behavior of the process is similar to the one of the innovations Z, sharing both the same tail index γ.

Now consider

$$a_t = K^{-1}(1-t) = t^{-\gamma} L_{K^{-1}}(t), \tag{14}$$

where K is the marginal d.f. in (12) and the convention $\prod_{k=i}^{j} U_k = 1$ for $i > j$.

Applying (13) and (14) we have, for each positive integer k and as $t \downarrow 0$,

$$
\begin{aligned}
F_Z(a_t^{1/p^k}) &\sim \frac{K(a_t^{1/p^k})}{1-(a_t^{1/p^k})^{-1/(\gamma p)} L_K(a_t^{1/p^{k+1}}) E(U^{1/(\gamma p)})} \\
&\sim 1 - t^{1/p^k} \mathcal{L}(a_t^{1/p^k}) + t^{1/p^{k+1}} \mathcal{L}(a_t^{1/p^{k+1}}) E(U^{1/(\gamma p)}),
\end{aligned} \tag{15}
$$

where

$$
\mathcal{L}\left(a_t^{1/p^k}\right) = L_K\left(a_t^{1/p^{k+1}}\right) L_{K^{-1}}(t)^{-1/(\gamma p^{k+1})}. \tag{16}
$$

The following Lemmas will be used in the proof of the main result below (Proposition 3).

Lemma 1. *For each positive integer m,*

$$
\begin{aligned}
&\int \cdots \int_{\mathcal{U}^m} K\left(\left(a_t / \prod_{j=0}^{m-1} u_{m-j}^{p^j}\right)^{1/p^m}\right) \\
&\cdot \prod_{j=0}^{m-1} F_Z\left(\left(a_t / \prod_{i=0}^{j-1} u_{m-i}^{p^i}\right)^{1/p^j}\right) dF_U(u_1) \ldots dF_U(u_m) = K(a_t).
\end{aligned} \tag{17}
$$

Proof. Just observe that, by (11),

$$
\int_{\mathcal{U}} K\left(\left(a_t / \prod_{j=0}^{m-1} u_{m-j}^{p^j}\right)^{1/p^m}\right) dF_U(u_1) = \frac{K((a_t/\prod_{j=0}^{m-2} u_{m-j}^{p^j})^{1/p^m})}{F_Z((a_t/\prod_{j=0}^{m-2} u_{m-j}^{p^j})^{1/p^m})} \tag{18}
$$

and hence

$$
\begin{aligned}
&\int_{\mathcal{U}} K\left(\left(a_t / \prod_{j=0}^{m-1} u_{m-j}^{p^j}\right)^{1/p^m}\right) dF_U(u_1) F_Z\left(\left(a_t / \prod_{j=0}^{m-2} u_{m-j}^{p^j}\right)^{1/p^m}\right) \\
= \; & K\left(\left(a_t / \prod_{j=0}^{m-2} u_{m-j}^{p^j}\right)^{1/p^m}\right).
\end{aligned} \tag{19}
$$

Therefore, the left-hand side of (17) simplifies to

$$
\begin{aligned}
&\int \cdots \int_{\mathcal{U}^{m-1}} K\left(\left(a_t / \prod_{j=0}^{m-2} u_{m-j}^{p^j}\right)^{1/p^{m-1}}\right) \\
&\cdot \prod_{j=0}^{m-2} F_Z\left(\left(a_t / \prod_{i=0}^{j-1} u_{m-i}^{p^i}\right)^{1/p^j}\right) dF_U(u_2) \ldots dF_U(u_m).
\end{aligned}
$$

The result follows by applying successively the same reasoning. ∎

Lemma 2. *For each positive integer m,*

$$\begin{aligned}(20)\quad &\int_{\mathcal{U}^m}\cdots\int \prod_{j=0}^{m-1} F_Z\Bigg(\Bigg(a_t/\prod_{i=0}^{j-1} u_{m-i}^{p^i}\Bigg)^{1/p^j}\Bigg)dF_U(u_1)...dF_U(u_m)\\ &\qquad\sim 1-t+t^{1/p^m}\mathcal{L}\left(a_t^{1/p^m}\right)\prod_{j=1}^{m-1}E(U^{1/(\gamma p^j)})E(U^{1/(\gamma p)}).\end{aligned}$$

Proof. For simplicity, we do the calculations for $m=3$, but the same reasoning can be applied to any value of m. Observe that, by (15) and the *Dominated Convergence Theorem*,

(21)

$$\begin{aligned}&\iiint_{\mathcal{U}^3} F_Z((a_t/u_3)^{1/p})F_Z((a_t/(u_3u_2^p))^{1/p^2})dF_U(u_1)dF_U(u_2)dF_U(u_3)\\ &=\iint_{\mathcal{U}^2} F_Z((a_t/u_3)^{1/p})F_Z((a_t/(u_3u_2^p))^{1/p^2})dF_U(u_2)dF_U(u_3)\\ &\sim\iint_{\mathcal{U}^2}[1-t^{1/p}\mathcal{L}(a_t^{1/p})u_3^{1/(\gamma p)}+t^{1/p^2}\mathcal{L}(a_t^{1/p^2})u_3^{1/(\gamma p^2)}E(U^{1/(\gamma p)})][1-t^{1/p^2}\mathcal{L}(a_t^{1/p^2})\\ &\cdot u_3^{1/(\gamma p^2)}u_2^{1/(\gamma p)}+t^{1/p^3}\mathcal{L}(a_t^{1/p^3})u_3^{1/(\gamma p^2)}u_2^{1/(\gamma p)}E(U^{1/(\gamma p)})]dF_U(u_2)dF_U(u_3).\end{aligned}$$

Thus, after some simplifications, we have

$$\begin{aligned}(22)\quad &\iint_{\mathcal{U}^2} F_Z((a_t/u_3)^{1/p})F_Z((a_t/(u_3u_2^p))^{1/p^2})dF_U(u_2)dF_U(u_3)\\ &\sim\iint_{\mathcal{U}^2}[1-t^{1/p}\mathcal{L}(a_t^{1/p})u_3^{1/(\gamma p)}\\ &\qquad+t^{1/p^3}\mathcal{L}(a_t^{1/p^3})u_3^{1/(\gamma p^2)}u_2^{1/(\gamma p)}E(U^{1/(\gamma p)})]dF_U(u_2)dF_U(u_3)\\ &=1-t^{1/p}\mathcal{L}(a_t^{1/p})E(U^{1/(\gamma p)})+t^{1/p^3}\mathcal{L}(a_t^{1/p^3})E(U^{1/(\gamma p^2)})E(U^{1/(\gamma p)})^2.\end{aligned}$$

Now we just need to multiply the last expression by $F_Z(a_t)$. By considering (15) with $k=1$, we obtain the result, i.e.,

$$(23)\qquad 1-t+t^{1/p^3}\mathcal{L}(a_t^{1/p^3})E(U^{1/(\gamma p^2)})E(U^{1/(\gamma p)})^2.$$

∎

Proposition 3. *For each positive integer m,*

$$P\left(X_1 > a_t, X_{1+m} > a_t\right)$$

(24)
$$\sim t^2 \mathbb{1}_{\{p^m \leq 1/2\}} + t^{1/p^m} \mathcal{L}\left(a_t^{1/p^m}\right) \prod_{j=1}^{m-1} E(U^{1/(\gamma p^j)}) E(U^{1/(\gamma p)}) \mathbb{1}_{\{p^m > 1/2\}}.$$

Proof. We have, successively,

$$P(X_1 > a_t, X_{1+m} > a_t)$$

$$= P\Bigg(X_1 > a_t, \bigvee_{j=0}^{m-1} \prod_{i=0}^{j-1} U_{1+m-i}^{p^i} Z_{1+m-j}^{p^j} \vee \prod_{i=0}^{m-1} U_{1+m-i}^{p^i} X_1^{p^m} > a_t\Bigg)$$

$$= 1 - K(a_t) - P\Bigg(X_1 > a_t, \bigvee_{j=0}^{m-1} \prod_{i=0}^{j-1} U_{1+m-i}^{p^i} Z_{1+m-j}^{p^j} < a_t, \prod_{i=0}^{m-1} U_{1+m-i}^{p^i} X_1^{p^m} < a_t\Bigg)$$

$$= 1 - K(a_t) - \int_{\mathcal{U}^m} \cdots \int \left[K\Bigg(\bigg(a_t / \prod_{j=0}^{m-1} u_{1+m-j}^{p^j}\bigg)^{1/p^m}\Bigg) - K(a_t)\right]$$

$$\cdot \prod_{j=0}^{m-1} F_Z\Bigg(\bigg(a_t / \prod_{i=0}^{j-1} u_{1+m-i}^{p^i}\bigg)^{1/p^j}\Bigg) dF_U(u_2) \ldots dF_U(u_{m+1}).$$

Now the result is straightforward from Lemmas 1 and 2. ■

Corollary 4. *For each positive integer m,*

(25)
$$\lambda_m = \begin{cases} 0 & , 0 < p < 1 \\ E(U^{1/\gamma})^m & , p = 1. \end{cases}$$

Corollary 5. *For each positive integer m,*

(26)
$$\eta_m = \begin{cases} \max(1/2, p^m) & , 0 < p < 1 \\ 1 & , p = 1. \end{cases}$$

Observe that if U is degenerated equal to some constant $c \in (0, 1)$, i.e., corresponding to the ARMAX process defined in (2), then $E(U^{1/\gamma}) = c$ and thus $\lambda_m = c^{m/\gamma}$ as expected. If $0 < p < 1$, the lag-m tail coefficients coincide with the ones of the pARMAX in (7), indicating that the dependence behavior in the tail is controlled by the exponent p and that the random coefficients U_i $(i \geq 1)$ are "innocuous" in this case. We conclude that for $p = 1$ the process is lag-m tail dependent whilst $0 < p < 1$ leads to a lag-m asymptotic tail independent process. The exponent parameter p of the process can therefore be estimated through the lag-1 LTC η_1 (details on this topic can be seen in Ferreira and Canto e Castro, [13, 14] 2010a,b). The results are illustrated in Figures 2, 3, 4, 5, based on the generation of samples of extended pRARMAX processes $\{X_i\}_{i\geq 1}$ of size $n = 5000$. The horizontal lines correspond to the true values. For the lag-m TDC (Figures 2, 4) we have used estimator

$$\widehat{\lambda}_m = 2 - \frac{\log \widehat{C}_m\,(1-k/n, 1-k/n)}{\log\,(1-k/n)}, \quad 1 \leq k < n, \tag{27}$$

with $\widehat{C}_m$ denoting the empirical copula

$$\widehat{C}_m(u,v) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{R_i<u, R_{i+m}<v\}}, \tag{28}$$

where $\mathbb{1}$ denotes the indicator function and R_i the rank of X_i (see e.g. Frahm et al. [16] 2005). In what concerns the lag-m LTC, since it coincides with the tail index of (6), we consider estimator

$$\widehat{\eta}_m = \frac{1}{k}\sum_{i=1}^{k} \log \frac{T^{(n,m)}_{n,n-i+1}}{T^{(n,m)}_{n,n-k}}, \quad 1 \leq k < n, \tag{29}$$

corresponding to the Hill estimator (Hill, [18] 1975) of $\{T_i^{(n,m)}\}_{i\geq 1}$, which is based on the k upper order statistics $T^{(n,m)}_{n,n} \geq \ldots \geq T^{(n,m)}_{n,n-k}$, with

$$T_i^{(n,m)} = \min\left(\frac{1}{1-R_i/(n+1)}, \frac{1}{1-R_{i+m}/(n+1)}\right). \tag{30}$$

For more details, see Draisma et al. [8] (2004) and Ferreira and Canto e Castro [13, 14] (2010a,b).

In the lag-m LTC sample paths (Figures 3 and 5), i.e., the sample paths of the Hill estimator of $\{T_i^{(n,m)}\}_{i\geq 1}$, plotted against the respective k upper order statistics, it is evident a larger variance for small values of k and a larger bias whenever k increases. This bias-variance trade-off is a feature of the tail index estimators in general and of several TDC estimators (this phenomena is also observed in the lag-m TDC plots of Figures 2 and 4). An optimal choice of k is not easy to derive and, in practice, the estimate usually corresponds to a "plateau" region of the plot. Observe that the information of both TDC and

LTC, as well as considering more than lag-1, are important in order to conclude whether we have tail dependence or asymptotic independence. For instance, in the case $p = 0.7$ (Figures 2 and 3), the lag-1 graph of TDC points out tail dependence but the lag-1 graph of LTC indicates asymptotic tail independence and the next lags confirm this latter. However, going further than lags-3, 4 may lead to wrongly infer asymptotic tail independence if we have $p = 1$ (Figures 4 and 5).

3. Fitting an EpRARMAX: an application to financial data

In Ferreira and Canto e Castro [14, 15] (2010b,c) it was considered an adjustment method of pRARMAX to data. Here we shall widen this latter to the EpRARMAX model by including lag-m TDC and LTC plots, in order to decide wether to fit an EpRARMAX with $p = 1$ (tail dependence case) or with $p < 1$ (asymptotic tail independence case corresponding to a pRARMAX). This will be illustrated with an application to the financial time series Nasdaq index, for the period February 1971 to July 2013. More precisely, we analyze the volatility of this index through the squared log-returns $R_i^2 = (\log P_{i+1}/\log P_i)^2$, $i = 1, \ldots, 10700$, corresponding to a sample of size $n = 10700$. The values of R_i and R_i^2 are plotted in Figure 6 (left and right, respectively). Observe the sudden large peaks of the $\{R_i^2\}_i$ series, similar to the EpRARMAX plots in Figure 1. In order to the marginals be modeled by Pareto distributions, we implement a robust regression leading to the transformed data $X_i = aR_i^2 + b$, with $a = 7150.04$ and $b = 1.12$. In the following we denote sequence $\{X_i\}_{i\geq 1}$ as X.

Observe in Figure 7 the Pareto qq-plot as well as the empirical mean excess function plot, which indicate a Pareto-type model (see, e.g., Beirlant *et al.* [3] 2004).

We also test if X is in the domain of attraction of a heavy tail (see, e.g., Dietrich *et al.* [7] 2002). The sample path of this latter, plotted against k (the number of upper order statistics of X) corresponds to the left graphic of Figure 8. The horizontal line is the critical value above which we reject the heavy tail domain of attraction. Observe that we do not reject this assumption for $k \lesssim 990$, which is a plausible result to keep it.

In order to estimate the tail index γ of X, we use estimators in the EVT literature that can be applied to several time series models (Drees, [9] 2003) and in particular to the pRARMAX (see Ferreira and Canto e Castro [13, 14] 2010a,b). Here we will use the Hill estimator (Hill, [18] 1975) and moments estimator (Dekkers *et al.*, [6] 1989), which are plotted in Figure 8 (center and right, respectively), also against k upper order statistics of X. The value where both sample paths yield approximately a flat line is at about $\widehat{\gamma} = 0.5$.

The lag-m TDC and LTC plots can be seen in Figures 9 and 10, respectively.

For the lag-m LTC, besides the Hill estimator (first line plots of Figure 10) given in (29), we also consider the moments estimator (second line plots). From left to right we have the lags-$1, 2, 3$ respective estimator plots. Observe that the TDC sample paths present some stability around zero and the ones of LTC are away from one, indicating asymptotic tail independence. Observe that the lag-1 LTC sample path is approximately stable around 0.65 (this is more evident in the moments estimator) and thus we have $\widehat{p} = 0.65$. Observe also that the lags-$2, 3$ LTC plots are approximately stable around 0.5, which is consistent with the η_m formula of the EpRARMAX given in Corollary 5.

Now, in order to evaluate if a pRARMAX recursion is present in data, we are going to apply to X the algorithm considered in Ferreira and Canto e Castro [14, 15] (2010b,c), based on the theory of multiple hypothesis tests and on classification theory. It is quite detailed in these latter references and therefore we only give a brief description of its steps:

a. According to the pRARMAX recursion in (9), whenever $X_i > X_{i-1}^p$ the maximum component surely comes from the innovation Z_i (observe that if $X_i < X_{i-1}^p$, we cannot know for sure from which component X_i comes from). Thus we apply this criterion (replacing p by the estimate $\widehat{p}$ already obtained) to separate the innovations and test if this sample is also in the Fréchet domain of attraction and compute its tail index (recall that the innovations and the process must have the same tail index).

b. Capture the observations corresponding to the random factors U, through the following criterion:

   if $X_i < X_{i-1}^{\widehat{p}}$ and X_i belongs to the significance region

$$\mathcal{B}_\upsilon = \left\{ t : \frac{\pi_0 t^{-1/(\widehat{\gamma}\widehat{p})} f_Z(t)}{\pi_0 t^{-1/(\widehat{\gamma}\widehat{p})} f_Z(t) + (1-\pi_0) t^{-1/(\widehat{\gamma}\widehat{p})-1} F_Z(t)} \leq \upsilon \right\}, [1]$$

   with f_Z the density function of of the innovations sequence, $\pi_0 = 1 - \pi_1$ with $\pi_1 = P(U_i X_{i-1}^{\widehat{p}} > Z_i)/P(X_{i-1}^{\widehat{p}} > Z_i)$, then we consider that X_i is given by $U_i X_{i-1}^{\widehat{p}}$ i.e., $U_i = X_i / X_{i-1}^{\widehat{p}}$; υ cannot be too large (larger errors) nor too small (not enough observations to carry out the test of the next step). For more details, see Ferreira and Canto e Castro [14] 2010 and references therein.

[1] $\mathcal{B}_\upsilon$ is obtained based on classification theory and a Bayesian solution that minimizes the risk of possible wrong decisions, i.e., consider that X_i corresponds to the innovation Z_i component when it actually comes from the autoregressive part $U_i X_{i-1}^p$ or the other way round, whenever we have the ambiguous case $X_i < X_{i-1}^p$.

c. Test whether the sample of random variables U captured in step b. has distribution $\text{Beta}(1/(\widehat{\gamma}\widehat{p}) + 1, 1)$, using e.g. the Kolmogorov-Smirnov test (since we have $U_i|_{X_i \in \mathcal{B}_\upsilon, X_i = U_i X_{i-1}^{\widehat{p}}} \frown \text{Beta}(1/(\gamma p) + \alpha, \beta)$, assuming that $U_i \frown \text{Beta}(\alpha, \beta)$; observe that if $\alpha = \beta = 1$, then $U_i \frown \text{U}(0, 1)$).

From Figure 11, we conclude that the innovations Z captured on step a. do not reject a Fréchet domain of attraction and the tail index is also $\widehat{\gamma}_Z \approx 0.5$.

We implement step b. and capture the values corresponding to the random coefficients U, by considering $\upsilon = 0.05, \ldots, 0.5$. In applying the Kolmogorov-Smirnov test of step c., we do not reject the adjustment hypothesis to the model $\text{Beta}(1/(0.5 * 0.65) + 1, 1)$ for $\upsilon = 0.25$ (we are considering $U \frown \text{U}(0, 1)$), with p-value$= 0.0502$ (see Figure 12), which is consistent with the simulation results in Ferreira and Canto e Castro [14] (2010b). Therefore, we conclude that the EpRARMAX model, with $p = 0.65$ and $\gamma = 0.5$, can be considered for the modeling of X.

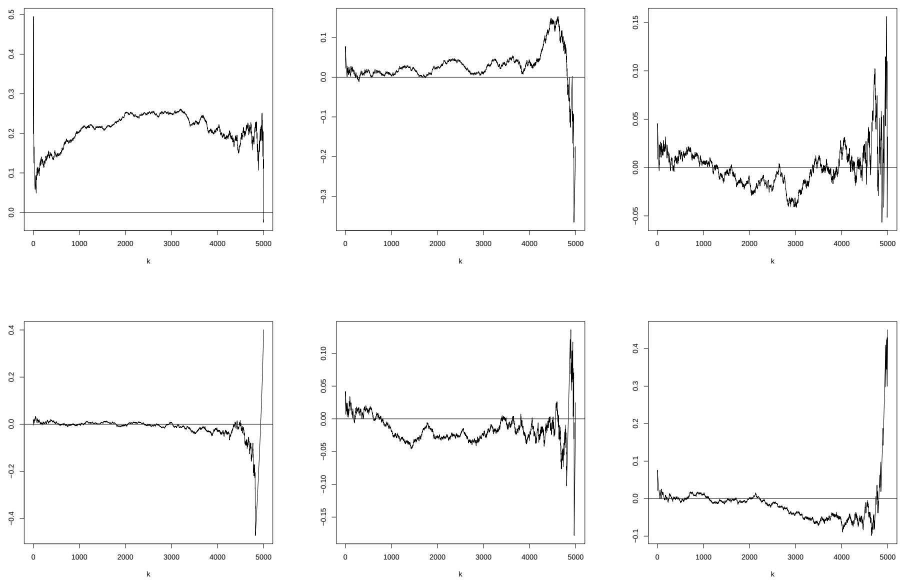

Figure 2. TDC estimates $\widehat{\lambda}_m$ plotted against k upper order statistics of an EpRARMAX process with $U \frown \text{U}(0, 1)$, $p = 1$ and $\gamma = 1$, and for lag-m with (left-to-right and top-to-bottom) $m = 1, \ldots, 6$, respectively.

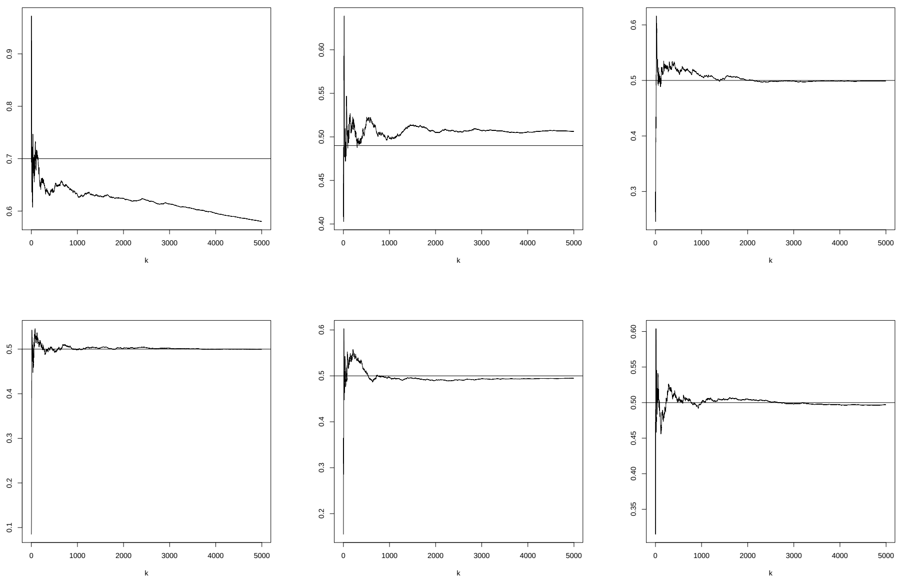

Figure 3. LTC estimates $\widehat{\eta}_m$ plotted against k upper order statistics of an EpRARMAX process with $U \frown \mathrm{U}(0,1)$, $p = 0.7$ and $\gamma = 1$, and for lag-m with (left-to-right and top-to-bottom) $m = 1, \ldots, 6$, respectively.

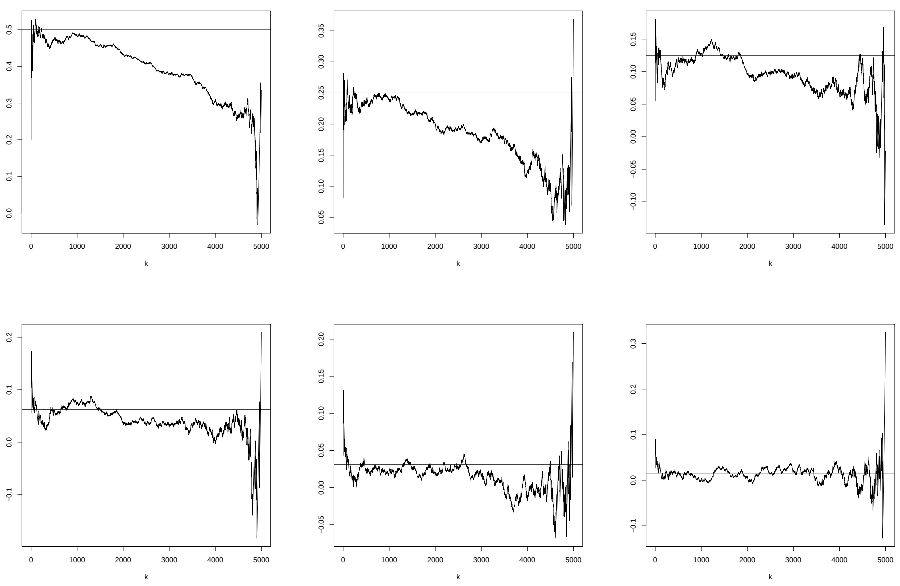

Figure 4. TDC estimates $\widehat{\lambda}_m$ plotted against k upper order statistics of an EpRARMAX process with $U \frown \mathrm{U}(0,1)$, $p = 1$ and $\gamma = 1$, and for lag-m with (left-to-right and top-to-bottom) $m = 1, \ldots, 6$, respectively.

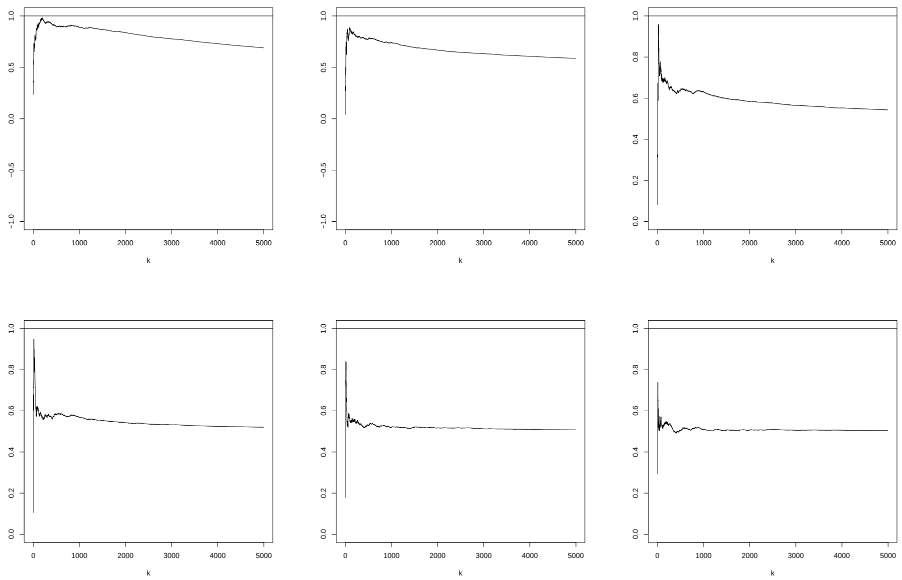

Figure 5. LTC estimates $\widehat{\eta}_m$ plotted against k upper order statistics of an EpRARMAX process with $U \frown \mathrm{U}(0,1)$, $p = 1$ and $\gamma = 1$, and for lag-m with (left-to-right and top-to-bottom) $m = 1, \ldots, 6$, respectively.

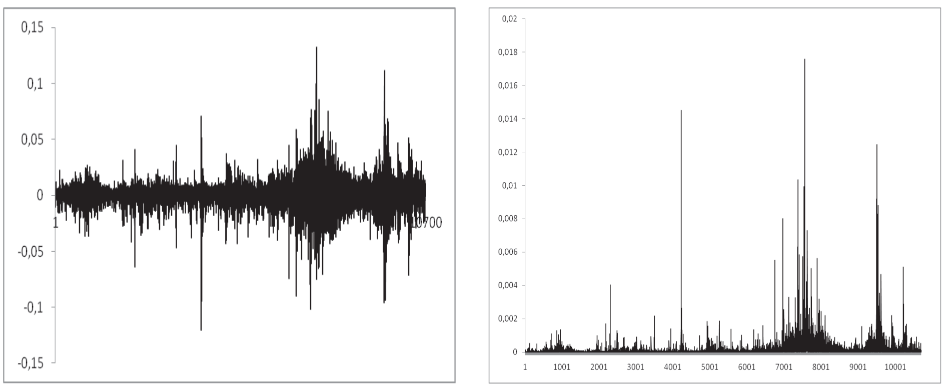

Figure 6. Log-returns (R_i) (left) and volatility (R_i^2) (right) of the Nasdaq index, from February 1971 to July 2013.

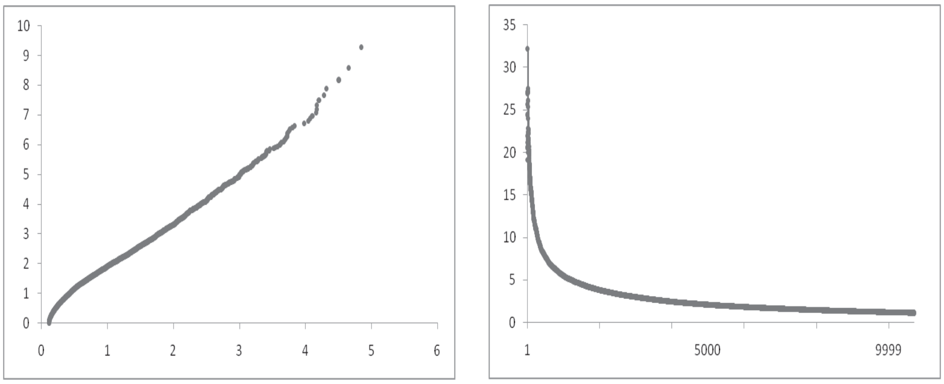

Figure 7. Pareto qq-plot of X (left) and the empirical mean excess function against k upper order statistics (right).

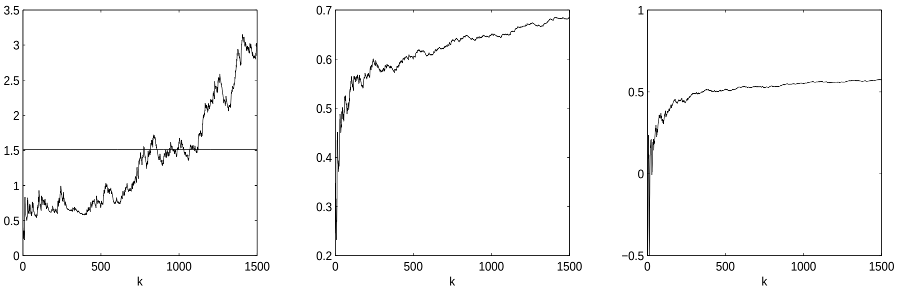

Figure 8. Left: sample path of the heavy tail domain of attraction test statistic for X with the horizontal line corresponding to the critical value above which we reject the extreme value condition. Center/Right: Hill/momments estimates for X against k upper order statistics, respectively.

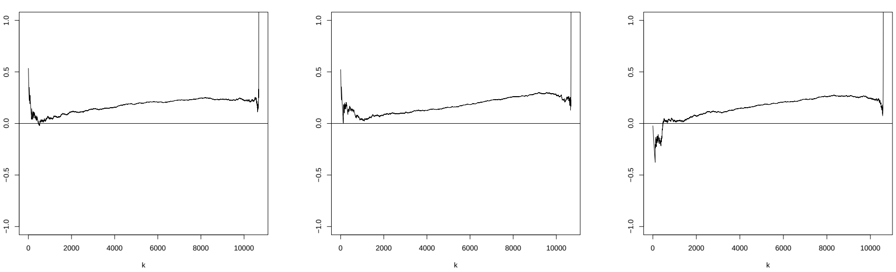

Figure 9. TDC estimates $\widehat{\lambda}_m$ plotted against k upper order statistics of the Nasdaq series X, for (left-to-right) lags $m = 1, 2, 3$, respectively.

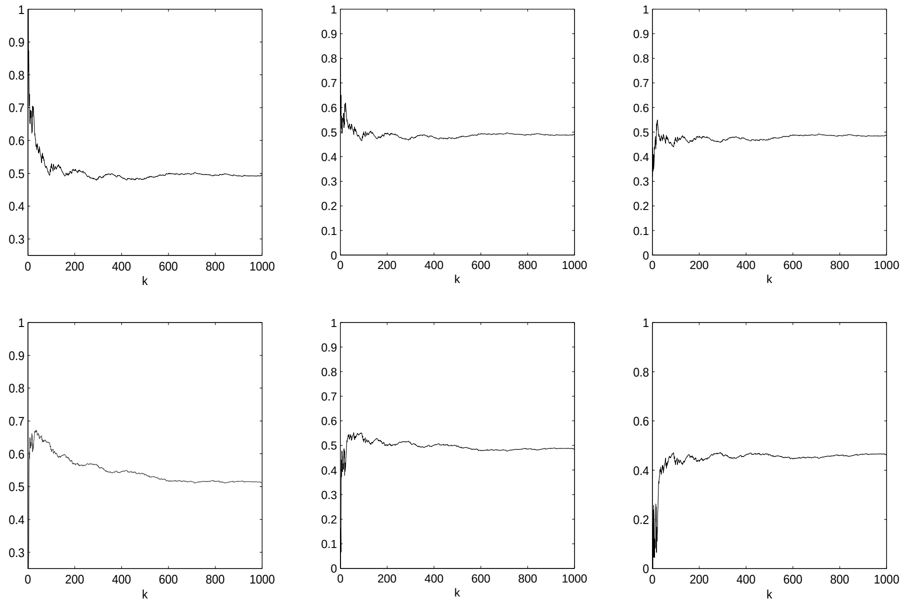

Figure 10. LTC estimates $\widehat{\eta}_m$ plotted against k upper order statistics of the Nasdaq series X, for (left-to-right) lags $m = 1, 2, 3$, respectively. The first line corresponds to the Hill estimator sample paths and the second line to the moments estimator sample paths.

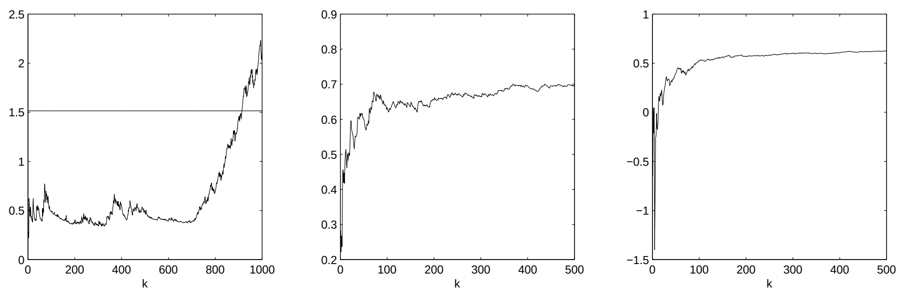

Figure 11. Left: sample path of the heavy tail domain of attraction test statistic for the captured innovations Z in step a., with the horizontal line corresponding to the critical value above which we reject the extreme value condition. Center/Right: Hill/momments estimates for Z against k upper order statistics, respectively.

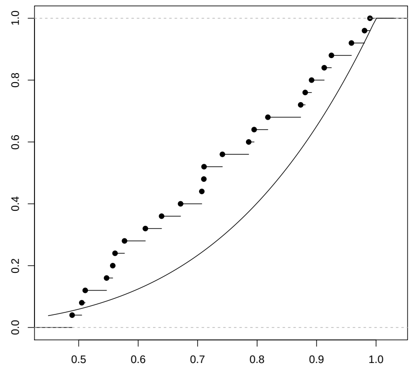

Figure 12. Empirical and theoretical d.f. of the random coefficients U captured through step b. (with $\widehat{p} = 0.65$), for a significance region with $\upsilon = 0.25$.

Acknowledgements

The author was partially supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program and by PTDC/FEDER grants from Portugal.

References

[1] M.T. Alpuim, *An extremal Markovian sequence*, J. Appl. Probab. **26** (1989) 219–232.

[2] M.T. Alpuim and E. Athayde, *On the stationary distribution of some extremal Markovian sequences*, J. Appl. Probab. **27** (1990) 291–302.

[3] J. Beirlant, Y. Goegebeur, J. Segers and J. Teugels, Statistics of Extremes: Theory and Application (John Wiley, Chichester, UK, 2004).

[4] P. Embrechts, C. Klüppelberg and T. Mikosch, Modelling Extremal Events for Insurance and Finance (Springer-Verlag, Berlin, 1997).

[5] R. Davis and S. Resnick, *Basic properties and prediction of max-ARMA processes*, Adv. Appl. Probab. **21** (1989) 781–803.

[6] A.L.M. Dekkers, J.H.J. Einmahl and L. de Haan, *A moment estimator for the index of an extreme value distribution*, Ann. Statist. **17** (1989) 1833–1855.

[7] D. Dietrich, L. de Haan and J. Hüsler, *Testing extreme value conditions*, Extremes **5** (2000) 71–85.

[8] G. Draisma, H. Drees, A. Ferreira and L. de Haan, *Bivariate tail estimation: dependence in asymptotic independence*, Bernoulli **10** (2004) 251–280.

[9] H. Drees, *Extreme quantile estimation for dependent data with applications to finance*, Bernoulli **9** (2003) 617–657.

[10] M. Ferreira, *On tail dependence: a characterization for first-order max-autoregressive processes*, Math. Notes **90** (2011) 103–114.

[11] M. Ferreira, *Parameter estimation and dependence characterization of the MAR(1) process*, ProbStat Forum **5** (2012) 107–111.

[12] M. Ferreira and L. Canto e Castro, *Tail and dependence behaviour of levels that persist for a fixed period of time*, Extremes **11** (2008) 113–133.

[13] M. Ferreira and L. Canto e Castro, *Asymptotic and pre-asymptotic tail behavior of a power max-autoregressive model*, ProbStat Forum **3** (2010a) 91–107.

[14] M. Ferreira and L. Canto e Castro, *Modeling rare events through a pRARMAX process*, J. Statist. Plann. Inference **140** (2010b) 3552–3566.

[15] M. Ferreira and L. Canto e Castro, *A Method For Fitting A pRARMAX Model: An Application To Financial Data*, in: Proceedings of The World Congress on Engineering 2010 Vol. III, Lecture Notes in Engineering and Computer Science (Ed(s)), (London, U.K., 2010c) 2022–2026.

[16] G. Frahm, M. Junker and R. Schmidt, *Estimating the tail-dependence coefficient: properties and pitfalls*, Insurance Math. Econom. **37** (2005) 80–100.

[17] J.E. Heffernan, J.A. Tawn and Z. Zhang, *Asymptotically (in)dependent multivariate maxima of moving maxima processes*, Extremes **10** (2007) 57–82.

[18] B.M. Hill, *A simple general approach to inference about the tail of a distribution*, Ann. Statist. **3** (1975) 1163–1174.

[19] A.V. Lebedev, *Statistical analysis of first-order MARMA processes*, Math. Notes **83** (2008) 506–511.

[20] Z. Zhang, *A New Class of Tail-Dependent Time Series Models and Its Applications in Financial Time Series*, Advances in Econometrics **20B** (2005) 323–358.

Received 4 March 2013
Revised 16 September 2013