Discussiones Mathematicae
Probability and Statistics 28 (2008) 157–165

ON COMMUTATIVITY OF PROJECTORS

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Dedicated to the memory of Jerzy K. Baksalary

Abstract

It is shown that commutativity of two oblique projectors is equivalent with their product idempotency if both projectors are not necessarily Hermitian but orthogonal with respect to the same inner product.

Keywords: oblique projector, orthogonal projector, commutativity.

2000 Mathematics Subject Classification: Primary, 15A27; Secondary 15A03.

1. Introduction

For a given subspace $\mathcal{L}$ of the complex vector space $\mathcal{C}^n$ let $\mathcal{L}^c$ denote a complement of $\mathcal{L}$ in $\mathcal{C}^n$, i.e., $\mathcal{L} \cap \mathcal{L}^c = \{\mathbf{0}\}$ and $\mathcal{L} + \mathcal{L}^c = \mathcal{C}^n$. An operator $\mathbf{P}$ such that $\mathbf{Px} = \mathbf{x}$ for $\mathbf{x} \in \mathcal{L}$ and $\mathbf{Px} = \mathbf{0}$ for $\mathbf{x} \in \mathcal{L}^c$ is called a projector on $\mathcal{L}$ along $\mathcal{L}^c$ and is denoted by $\mathbf{P}_{\mathcal{L}|\mathcal{L}^c}$. Such operators are characterized by the idempotency condition

$$\mathbf{P}^2 = \mathbf{P}. \tag{1.1}$$

It is well known that if (1.1) holds, then $\mathbf{P}$ is a projector on $R(\mathbf{P})$, the range of $\mathbf{P}$, along $R(\mathbf{Q})$, where $\mathbf{Q} = \mathbf{I} - \mathbf{P}$. Thus, any idempotent operator we will call a projector or oblique projector and write $\mathbf{P}$ instead of $\mathbf{P}_{R(\mathbf{P})|R(\mathbf{Q})}$.

Now let us assume that $\mathcal{C}^n$ is equipped with the inner product $<,>$ defined with the use of a positive definite matrix $\mathbf{V}$. Then any two vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be $\mathbf{V}$-orthogonal if $\mathbf{x}^*\mathbf{V}\mathbf{y} = 0$, where the star superscript denotes the conjugate transposing operation. Moreover, all vectors in $\mathcal{C}^n$ that are $\mathbf{V}$-orthogonal to every vector in a given subspace $\mathcal{L}$ form the $\mathbf{V}$-orthogonal complement of $\mathcal{L}$. Since such complement is determined exclusively by the subspace $\mathcal{L}$ and the matrix $\mathbf{V}$, the projector on $\mathcal{L}$ along the $\mathbf{V}$-orthogonal complement of $\mathcal{L}$ is termed as $\mathbf{V}$-orthogonal. Such projectors are characterized (see e.g. [1] p. 268) by adding to the condition (1.1) the second requirement in the form

$$\mathbf{V}\mathbf{P} = \mathbf{P}^*\mathbf{V}. \tag{1.2}$$

An operator fulfilling (1.1) and (1.2) we will denote by $\mathbf{P}^{\mathbf{V}}$.

When $\mathbf{V} = \mathbf{I}$, i.e., the inner product is standard, we will use the term orthogonal instead of $\mathbf{V}$-orthogonal and write $\mathbf{P}$ instead of $\mathbf{P}^{\mathbf{I}}$. It does not lead to any confusion, since $R(\mathbf{Q})$ is indeed the $\mathbf{I}$-orthogonal complement of $R(\mathbf{P})$, if $\mathbf{P}$ is idempotent and Hermitian.

Attention of many authors was focussed on the problem of commutativity of orthogonal projectors. Baksalary [2] collected forty-five conditions equivalent to the equality

$$\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 \tag{1.3}$$

and presented them with some specific statistical implications. However, the most elegant equivalency condition states that (1.3) takes place if and only if the product $\mathbf{P}_1\mathbf{P}_2$ is a projector itself, i.e.,

$$(\mathbf{P}_1\mathbf{P}_2)^2 = \mathbf{P}_1\mathbf{P}_2.$$

More precisely, in such a case, the product $\mathbf{P}_1\mathbf{P}_2$ is the orthogonal projector. In the present paper it is shown that this equivalency can be directly extended also to specific but non-Hermitian projectors.

2. Results

First let us recall that any oblique projector can be treated as $\mathbf{V}$-orthogonal one with a special choice of $\mathbf{V}$. This possibility follows from the following

Lemma 1. *Let $\mathcal{L}$ and $\mathcal{M}$ be any two complementary subspaces of $\mathcal{C}^n$. Then, for any positive scalars α and β, the matrix*

$$\mathbf{V} = \alpha \mathbf{P}^*_{\mathcal{L}|\mathcal{M}} \mathbf{P}_{\mathcal{L}|\mathcal{M}} + \beta \mathbf{P}^*_{\mathcal{M}|\mathcal{L}} \mathbf{P}_{\mathcal{M}|\mathcal{L}}$$

is positive definite and the subspaces $\mathcal{L}$ and $\mathcal{M}$ are $\mathbf{V}$-orthogonal.

The crucial point of this early result of Baksalary and Kala [3] states that there always exist a positive definite $\mathbf{V}$ such that

$$\mathbf{P}_{\mathcal{L}|\mathcal{M}} = \mathbf{P}^{\mathbf{V}}_{\mathcal{L}} = \mathbf{P}^{\mathbf{V}},$$

the last equality follows because $\mathcal{L} = R(\mathbf{P}^{\mathbf{V}})$. Although the matrix $\mathbf{V}$ is not unique, the projector $\mathbf{P}^{\mathbf{V}}_{\mathcal{L}}$ is. It admits a representation

$$\mathbf{P}^{\mathbf{V}}_{\mathcal{L}} = \mathbf{L}(\mathbf{L}^*\mathbf{V}\mathbf{L})^-\mathbf{L}^*\mathbf{V},$$

where $\mathbf{L}$ is any matrix such that $\mathcal{L} = R(\mathbf{L})$ and the minus superscript denotes a g-inverse of the matrix involved. The matrix $\mathbf{V}$ in the above representation need not to be positive definite. It can be non-negative definite only, but such that $\mathcal{L} \subset R(\mathbf{V})$. For a definition of a special projection in such case see Rao [4].

The operator $\mathbf{P}^{\mathbf{V}}$ is idempotent but not Hermitian. However, the following properties can easily be checked.

Lemma 2. *For any two $\mathbf{V}$-orthogonal projectors $\mathbf{P}^{\mathbf{V}}_1$ and $\mathbf{P}^{\mathbf{V}}_2$ the matrices*

$$\mathbf{V}\mathbf{P}^{\mathbf{V}}_1, \quad \mathbf{V}\mathbf{P}^{\mathbf{V}}_1\mathbf{P}^{\mathbf{V}}_2\mathbf{P}^{\mathbf{V}}_1, \quad \mathbf{V}^{1/2}\mathbf{P}^{\mathbf{V}}_1\mathbf{P}^{\mathbf{V}}_2\mathbf{P}^{\mathbf{V}}_1\mathbf{V}^{-1/2}$$

are Hermitian and non-negative definite.

Properties of the first and second matrix in Lemma 2 play a key role in the proof of the following theorem extending the results contained in [2].

Theorem 1. *For any two* $\mathbf{V}$*-orthogonal projectors* $\mathbf{P}_1^{\mathbf{V}}$ *and* $\mathbf{P}_2^{\mathbf{V}}$ *the following two conditions:*

$$\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}} = \mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}, \quad \textit{commutativity}, \tag{2.1}$$

$$\left(\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\right)^2 = \mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}, \quad \textit{idempotency}, \tag{2.2}$$

are equivalent.

Proof. The sufficiency of commutativity condition, in view of idempotency of both projectors $\mathbf{P}_1^{\mathbf{V}}$ and $\mathbf{P}_2^{\mathbf{V}}$, is obvious. For the necessity, first observe that by the property of the first matrix in Lemma 2, we have

$$\left(\mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\right)^* = \mathbf{V}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}. \tag{2.3}$$

Now consider the product

$$\begin{aligned}
&\left(\mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}} - \mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)\mathbf{V}^{-1}\left(\mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}} - \mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)^* \\
&= \left(\mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}} - \mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)\left(\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}} - \mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right) \\
&= \mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}} - \mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}},
\end{aligned} \tag{2.4}$$

which, in view of (2.2), reduces to the zero matrix. In consequence

$$\mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}} = \mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}.$$

But the matrix on the right hand side is Hermitian, as stated in Lemma 2. Thus, by (2.3) again,

$$\mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}} = \mathbf{V}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}},$$

which completes the proof, since $\mathbf{V}$ is non-singular. ■

The result above can be supplemented by forty-four equivalent conditions collected by Baksalary [2]. Of course, some minor modifications are indispensable. For example, the condition (A2) of his Theorem 1, in our notation

$$\mathbf{A}^*\mathbf{B} = \mathbf{A}^*\mathbf{P}_{\mathcal{B}}\mathbf{P}_{\mathcal{A}}\mathbf{B},$$

where $\mathcal{A} = R(\mathbf{A})$ and $\mathcal{B} = R(\mathbf{B})$, now takes the form

$$\mathbf{A}^*\mathbf{V}\mathbf{B} = \mathbf{A}^*\mathbf{V}\mathbf{P}_{\mathcal{B}}^{\mathbf{V}}\mathbf{P}_{\mathcal{A}}^{\mathbf{V}}\mathbf{B}.$$

In a similar way the recent results of Baksalary and Baksalary [5] can also be extended. It is also possible to rewrite the most general result on this area. It is established in [6], with the original proof based on a simple property of powers of Hermitian and non-negative definite matrices. The appropriate modification of this result, which, however, has only purely theoretical character, is presented bellow together with a direct adaptation of the proof.

Theorem 2. *For any two* $\mathbf{V}$*-orthogonal projectors* $\mathbf{P}_1^{\mathbf{V}}$ *and* $\mathbf{P}_2^{\mathbf{V}}$*, let* $\mathbf{P}_{(m;i)}^{\mathbf{V}}$ *denote an m-factor product of* $\mathbf{P}_1^{\mathbf{V}}$ *and of* $\mathbf{P}_2^{\mathbf{V}}$*, with* $\mathbf{P}_i^{\mathbf{V}}$ *being the first factor and* $\mathbf{P}_i^{\mathbf{V}}$, $\mathbf{P}_j^{\mathbf{V}}$ *occurring alternately,* $i, j = 1, 2; i \neq j$*. Then the commutativity condition* (2.1) *is equivalent with any statement of the following form:*

$$\mathbf{P}_{(p;i)}^{\mathbf{V}} = \mathbf{P}_{(q;j)}^{\mathbf{V}} \text{ for some } p, q \geq 2 \text{ and } i, j = 1, 2,$$

except for the trivial case $p = q$ *and* $i = j$*.*

Proof. As it was observed in [6], for any non-negative definite matrix $\mathbf{W}$, the equality $\mathbf{W}^k = \mathbf{W}^l$ for some $k < l$ is equivalent with idempotency of $\mathbf{W}$. Applying this result to the matrix $\mathbf{V}^{1/2}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\mathbf{V}^{-1/2}$, which, by the property of the third matrix in Lemma 2, is non-negative definite, and using the obvious equality

$$\left(\mathbf{V}^{1/2}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\mathbf{V}^{-1/2}\right)^k = \mathbf{V}^{1/2}\left(\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)^k\mathbf{V}^{-1/2},$$

we have that

$$\left(\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)^k = \left(\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)^l \text{ for some } k < l$$

implies

$$\left(\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)^2 = \mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}} = \mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}.$$

This, in turn, implies that the product (2.4) reduces to the zero matrix, which leads to commutativity of $\mathbf{P}_1^{\mathbf{V}}$ and $\mathbf{P}_2^{\mathbf{V}}$. Having the equivalence

$$\left(\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)^k = \left(\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}}\right)^l \text{ for some } k < l \iff \mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}} = \mathbf{P}_2^{\mathbf{V}}\mathbf{P}_1^{\mathbf{V}},$$

the rest of proof follows the lines exactly as in [6]. ■

3. Comments

Under the commutativity condition (2.1) the product $\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}$ is the $\mathbf{V}$-orthogonal projector on $R(\mathbf{P}_1^{\mathbf{V}}) \cap R(\mathbf{P}_2^{\mathbf{V}})$. It is so, because in that case $\mathbf{V}\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{V}}$ is a Hermitian matrix. This conclusion corresponds to the early result of Rao and Mitra [7]. Their Theorem 5.1.4 states that if two oblique projectors, in our notation $\mathbf{P}_1^{\mathbf{V}}$ and $\mathbf{P}_2^{\mathbf{W}}$ with $\mathbf{V}$ not necessarily equal to $\mathbf{W}$, commute, then the product $\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{W}}$ is a projector and it projects on $R(\mathbf{P}_1^{\mathbf{V}}) \cap R(\mathbf{P}_2^{\mathbf{W}})$ along $N(\mathbf{P}_1^{\mathbf{V}}) + N(\mathbf{P}_2^{\mathbf{W}})$, where $N(\mathbf{P}_i^{\mathbf{V}})$ is the null space of the matrix $\mathbf{P}_i^{\mathbf{V}}$.

It is well known that the reverse implications, in general, are not true. Considering the following projectors:

$$\mathbf{P}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{P}^{\mathbf{V}}, \text{ with } \mathbf{V} = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{P}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \mathbf{P}^{\mathbf{U}}, \text{ with } \mathbf{U} = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\mathbf{P}_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \mathbf{P}^{\mathbf{W}}, \text{ with } \mathbf{W} = \begin{pmatrix} 3 & -3 & -1 \\ -3 & 5 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$

it can be checked that:

$$(\mathbf{P}_1\mathbf{P}_2)^2 = \mathbf{P}_1\mathbf{P}_2, \text{ but } \mathbf{P}_1\mathbf{P}_2 \neq \mathbf{P}_2\mathbf{P}_1 \quad (= \mathbf{P}_2),$$

$$(\mathbf{P}_2\mathbf{P}_3)^2 = \mathbf{P}_3, \text{ but } \mathbf{P}_2\mathbf{P}_3 \neq \mathbf{P}_3\mathbf{P}_2 \quad (= \mathbf{P}_2),$$

$$(\mathbf{P}_3\mathbf{P}_1)^2 = \mathbf{P}_3\mathbf{P}_1 = \mathbf{0} = \mathbf{P}_{\{\mathbf{0}\}|\mathcal{C}^n}, \text{ but } \mathbf{P}_3\mathbf{P}_1 \neq \mathbf{P}_1\mathbf{P}_3 \quad (= \mathbf{P}_3).$$

The commutativity of two oblique projectors can be judged by the use of the result of Gross and Trenkler [8]. According to their criterion, two projectors $\mathbf{P}_1^{\mathbf{V}}$ and $\mathbf{P}_2^{\mathbf{W}}$ commute if and only if the product $\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{W}}$ is a projector on $R(\mathbf{P}_{\mathbf{1}}^{\mathbf{V}}) \cap R(\mathbf{P}_2^{\mathbf{W}})$ along $N(\mathbf{P}_1^{\mathbf{V}}) + N(\mathbf{P}_2^{\mathbf{W}})$ and rank $(\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{W}}) = \text{rank}(\mathbf{P}_2^{\mathbf{W}}\mathbf{P}_1^{\mathbf{V}})$.

In the example above the product $\mathbf{P}_1\mathbf{P}_2$ is a projector, but not on $R(\mathbf{P}_1) \cap R(\mathbf{P}_2) = \{\mathbf{0}\}$. The products $\mathbf{P}_2\mathbf{P}_3$ as well as $\mathbf{P}_3\mathbf{P}_2$ are both projectors on $R(\mathbf{P}_{\mathbf{1}}) = R(\mathbf{P}_2)$, but not along $N(\mathbf{P}_2) + N(\mathbf{P}_3) = \mathcal{C}^n$. Finally, the product $\mathbf{P}_3\mathbf{P}_1$ fulfils the first requirement, but not the second, since $\text{rank}(\mathbf{P}_3\mathbf{P}_1) = 0 \neq 1 = \text{rank}(\mathbf{P}_1\mathbf{P}_3)$.

Many others necessary and sufficient conditions for commutativity of two oblique projectors are delivered in [9] and also in [5]. One that links the commutativity with idempotency property states that $\mathbf{P}_1^{\mathbf{V}}$ and $\mathbf{P}_2^{\mathbf{W}}$ commute if and only if all four products:

$$\mathbf{P}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{W}}, \quad \mathbf{P}_1^{\mathbf{V}}\mathbf{Q}_2^{\mathbf{W}}, \quad \mathbf{Q}_1^{\mathbf{V}}\mathbf{P}_2^{\mathbf{W}}, \quad \mathbf{Q}_1^{\mathbf{V}}\mathbf{Q}_2^{\mathbf{W}},$$

where $\mathbf{Q}_1^{\mathbf{V}} = \mathbf{I} - \mathbf{P}_1^{\mathbf{V}}$ and $\mathbf{Q}_1^{\mathbf{W}} = \mathbf{I} - \mathbf{P}_1^{\mathbf{W}}$, are projectors. Note, however, that according to Theorem 1, if both projectors are orthogonal with respect to the same inner product, then idempotency of each of these products separately implies the commutativity of $\mathbf{P}_1^{\mathbf{V}}$ and $\mathbf{P}_2^{\mathbf{V}}$.

Summerizing, we can say, in view of Theorems 1 and 2, that the equivalence between the commutativity condition (2.1) and the idempotency condition (2.2) is not related with the Hermitianness of the projectors involved, but with their common $\mathbf{V}$-orthogonality.

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Received 14 December 2007