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ON MIESHALKIN-ROGOZIN THEOREM AND SOME PROPERTIES OF THE SECOND KIND BETA DISTRIBUTION*

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Abstract

The decomposition of the r.v. X with the beta second kind distribution in the form of finite (formula (9), Theorem 1) and infinity products (formula (17), Theorem 2 and form (21), Theorem 3) are presented. Next applying Mieshalkin – Rogozin theorem we receive the estimation of the difference of two c.d.f. $F(x)$ and $G(x)$ when $\sup|f(t)-g(t)|$ is known, improving the result of Gnedenko – Kolmogorov (formulae (23) and (24)).

Keywords: Mieshalkin – Rogozin theorem, result of Kolmogorov, Knar formula.

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1. Introduction

We consider a random variable X with the second kind beta distribution

$$(1)\qquad \beta_2\,(x|p\,,\,q) = \frac{x^{p-1}}{B\,(p\,,\,q)\;(x+1)^{p+q}}\,,\qquad 0\le x<\infty\,,\quad p,q>0.$$

The random variable $X = Z/\,(1-Z)$ has this distribution when Z has the first kind beta distribution

$$g\,(z) = B^{-1}\,(p,q)\,z^{p-1}\,(1-z)^{q-1}\,,\quad 0\le z\le 1.$$

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A quotient $X = Y_1/Y_2$ of two independent gamma random variables with a probability density function $f_{Y_j}(y_j|b_j) = \Gamma^{-1}(b_j)\, y_j^{b_j-1} \exp(-y_j)$, $b_j > 0$, $0 \le y_j < \infty$, $j = 1, 2$, has also the same distribution with $b_1 = p$, $b_2 = q$. In [16] Podolski has defined distributions of products and quotients of the powers of independent random variables with the second kind beta distribution using Meijer's G-function [3,9.3] and their generalization; the functions $H_{p,q}^{m,n}$ of Saksena and Mathai [18].

To represent a random variable X with the density function (1) as the finite product we shall use the Mellin transform $M_X(s)$ in the form of Zolotarev [20]

$$M_X(s) = EX^s,$$

where s is complex number, which gives

$$M_X(s) = \Gamma(p+s)\,\Gamma(q-s)\,\Gamma^{-1}(p)\,\Gamma^{-1}(q)\,, \quad -p < \mathrm{Re}\ \ s < q. \tag{2}$$

We can present the gamma functions in (2) as a finite product [3,8.335].

2. The case of finite product

Appyling the Knar formula [3,8.335]

$$\Gamma(nx) = (2\pi)^{\frac{1-n}{2}}\, n^{nx-0,5} \prod_{k=1}^{n} \Gamma\left(x + \frac{k-1}{n}\right)\,, \quad \mathrm{Re}\ \ x > 0\,, \quad n \ge 2, \tag{3}$$

to the gamma function in (2), we rewrite (2) for $n = n_1$ and $k = k_1$, after some simplifications, as

$$\begin{aligned} M_X(s) = &\prod_{k_1=1}^{n_1} \Gamma((p+k_1-1+s)/n_1)\,\Gamma((q+k_1-1-s)/n_1)\,\cdot \\ &\cdot \Gamma^{-1}((p+k_1-1)/n_1)\cdot\Gamma^{-1}((q+k_1-1)/n_1) = \\ = &\prod_{k_1=1}^{n_1} g_{k_1}(s), \quad -p < \mathrm{Re}\ \ s < q, \quad n_1 \ge 2. \end{aligned} \tag{4}$$

From the formula (2) it follows that the Mellin transform of random variable X_{k_1} with the second kind beta distribution with parameters $(p+k_1-1)/n_1$ and $(q+k_1-1)/n_1$ differs from the factor in the right hand side of the formula (4) only by a numerical coefficient at s. Because of $M_{X^{1/n}} = E\left(X^{1/n}\right)^s$, it follows that each factor $g_{k_1}(s)$ of the product (4) is the Mellin transform of the random variable $X_{k_1}^{1/n_1}$, where the random variables X_{k_1} have as densities the functions $\beta_2(x|p_{k_1}, q_{k_1})$, where

$$(5) \qquad p_{k_1} = (p+k_1-1)/n_1, \;\; q_{k_1} = (q+k_1-1)/n_1, \;\; k_1 = 1, 2, ..., n_1.$$

Since the finite product of the Mellin transform of independent random variables satisfies

$$(6) \qquad \prod_{k=1}^{n} M_{X_k^{1/n}}(s) = M_{\prod_1^n X_k^{1/n}}(s),$$

then by (2) and (4), we have

$$(7) \qquad M_X(s) = \prod_{k=1}^{n} M_{X_k^{1/n}}(s).$$

Thus from (6) and (7) it follows that

$$(8) \qquad M_X(s) = M_{\prod_1^n X_k^{1/n}}(s).$$

If we now apply the inverse transform of the Mellin transform to both the sides of the last relation we obtain the following stochastic equality

$$(9) \qquad X \stackrel{st}{=} \prod_{k=1}^{n} X_k^{1/n}.$$

The notation $X \stackrel{st}{=} Y$ means that X and Y have the same distribution function (d.f.).

The formula (9) means that

Theorem 1. *The second kind beta density $g(x|p,q)$ (formula* (1)*) of a r.v. X is equal to the density of the geometric mean of n_1 independent and nonnegative r.v.'s X_{k_1}, $k_1 = 1, ..., n_1$, with the second kind beta distribution with the respective parameters* (5).

Now we use the Mellin transform (4) to each density of the r.v. $X_{k_1}^{1/n_1}$ for $k_1 = 1, ..., n_1$, replacing the relevant parameters as follows

$$(10)\quad \begin{cases} (p+k_1-1)/n_1 \text{ by } p_{k_1,k_2} = [(p+k_1-1)/n_1 + k_2 - 1]/n_2 \\ \text{and} \\ (q+k_1-1)/n_1 \text{ by } q_{k_1,k_2} = [(q+k_1-1)/n_1 + k_2 - 1]/n_2. \end{cases}$$

Then we obtain that the random variables $X_{k_1}^{1/n_1}$, $k_1 = 1, 2, ..., n_1$, can be presented as

$$(11)\qquad X_{k_1}^{1/n_1} \stackrel{st}{=} \prod_{k_2=1}^{n_2} \left(X_{k_1,k_2}^{1/n_1} \right)^{1/n_2},$$

where the random variables X_{k_1,k_2} have as their density function $\beta_2 \left(x | p_{k_1,k_2},\ q_{k_1,k_2} \right)$.

Repeating r times the above change of parameters we get

$$(12)\qquad X \stackrel{st}{=} \prod_{k_1=1}^{n_1} \prod_{k_2=1}^{n_2} ... \prod_{k_r=1}^{n_r} X_{k_1,k_2,...,k_r}^{1/(n_1 n_2 ... n_r)},$$

where the random variables $X_{k_1,k_2,...,k_r}$ have densities functions $\beta_2(x|p_{k_1,k_2,...,k_r},\ q_{k_1,k_2,...,k_r})$.

3. The case of infinite product

Case 1.

Applying the Knar formula (Gradzstein, Ryzhik [3,8.324])

$$(13)\qquad \Gamma(x+1) = 4^x \prod_{k=1}^{\infty} \left[\Gamma\left(\frac{1}{2} + \frac{x}{2^k} \right) \Gamma^{-1}\left(\frac{1}{2} \right) \right], \quad \text{Re } x > -1,$$

to the gamma functions in (2), we can rewrite (2), after some simplifactions, as

$$
\begin{aligned}
M_X(s) = &\prod_{k=1}^{\infty} \left[\Gamma\left(1/2+(p-1+s)\big/2^k\right)\Gamma\left(1/2+(q-1-s)\big/2^k\right)\cdot\right. \\
(14)\qquad &\left.\cdot\,\Gamma^{-1}\left(1\big/2+(p-1)\big/2^k\right)\Gamma^{-1}\left(1/2+(q-1)\big/2^k\right)\right] \\
= &\prod_{k=1}^{\infty} h_k(s),
\end{aligned}
$$

where $-p < \text{Re }\, s < q$, for which we obtain that every factor $h_k(s)$ is the Mellin transform of r.v. $\left(X_{(k)}\right)^{\frac{1}{2^k}}$ with the density of $X_{(k)}$ given by

$$
(15)\qquad \beta_2\left(x\,\middle|\,\frac{1}{2}+\frac{p-1}{2^k}\,,\ \frac{1}{2}+\frac{q-1}{2^k}\right).
$$

Finally, we shall prove that $\prod_1^\infty h_k(s)$ is the Mellin transform of $\prod_1^\infty \left(X_{(k)}\right)^{1/2^k}$.

Proof. Since $M_X(it) = \varphi_{\ln X}(t)$, where $\varphi_X(t)$ is the characteristic function of X, we have

$$
\begin{aligned}
M_X(it) &= \varphi_{\ln X}(t) = \lim_{n\to\infty}\prod_{k=1}^{n}\varphi_{\ln X_{(k)}^{1/2^k}}(t) = \lim_{n\to\infty}\varphi_{\sum_1^n \ln X_{(k)}^{1/2^k}}(t) \\
(16)\qquad &= \lim_{n\to\infty}\varphi_{\ln\prod_1^n X_{(k)}^{1/2^k}}(t) = \varphi_{\ln\prod_1^\infty X_{(k)}^{1/2^k}}(t).
\end{aligned}
$$

Hence we get

$$
(17)\qquad X \overset{st}{=} \prod_{k=1}^{\infty}\left(X_{(k)}\right)^{2^{-k}}.
$$

We have proved

Theorem 2. *The second kind beta density* $g\left(x|p,q\right)$ *(formula* (1)*) is equal to the density of the infinite product of independent and non negative r.v.'s* $\left(X_{(k)}\right)^{2^{-k}}$ *with the densities of* $X_{(k)}$ *determined by* (15).

Case 2. Let us now apply the generalization of Knar formula [6]

$$\begin{aligned}
&\Gamma\left(z+1\right)= \\
(18)\qquad &\qquad = R^{Rz/(R-1)}\prod_{k=1}^{\infty}\prod_{n=1}^{R-1}\left[\Gamma\left(z\big/R^k+n/R\right)\Gamma^{-1}\left(n/R\right)\right]
\end{aligned}$$

to the four gamma functions in the Mellin transform in (2); $R>1$ is positive integer and Re$z>-1$. After some simplication we rewrite

$$\begin{aligned}
&M_X\left(s\right)= \\
&\quad=\prod_{k=1}^{\infty}\prod_{n=1}^{R-1}\left\{\Gamma\left[(p-1+s)\big/R^k+n/R\right]\Gamma\left[(q-1-s)\big/R^k+n/R\right]\cdot\right. \\
(19)\qquad &\quad\left.\cdot\,\Gamma^{-1}\left[(p-1)\big/R^k+n/R\right]\Gamma^{-1}\left[(q-1)\big/R^k+n/R\right]\right\}= \\
&\quad=\prod_{k=1}^{\infty}\prod_{n=1}^{R-1}h_{k,n}(s).
\end{aligned}$$

Let us note that this Mellin transform differs from the product (2) by the change p,q by $p_{k,n},\ q_{k,n}$ determined by

$$(20)\qquad p_{k,n}=(p-1)\big/R^k+n\big/R,\quad q_{k,n}=(q-1)\big/R^k+n/R$$

and by coefficent at s.

Therefore every factor in right hand side of (19) is the Mellin transform of r.v. which we denote by $h_{k,n}(s)$ and we put

$$\prod_{n=1}^{R-1} X^{R-k}_{p_{k,n},\, q_{k,n}} \overset{st}{=} X_{p_k, q_k}. \tag{21}$$

So we have proved

Theorem 3. *For each $R = 2, 3, ...,$ the density of the product of $R-1$ r.v.'s $X^{R-k}_{p_{k,n},q_{k,n}}$ with the densities $g\left(x|p_{k,n}, q_{k,n}\right)$, where $p_{k,n}, q_{k,n}$ are determined by (20), is equal to the density of r.v. X_{p_k,q_k}.*

Finally, we obtain

$$X \overset{st}{=} \prod_{k=1}^{\infty} \prod_{n=1}^{R-1} X^{R-k}_{p_{k,n},q_{k,n}}. \tag{22}$$

4. The modified Mieshalkin-Rogozin theorem

Our further aim is to determine an estimation of supremum of a difference between the two c.d.f. $F(x)$ and $G(x)$ if the $\sup|f(t) - g(t)|$ is known, where $f(t)$ and $g(t)$ are the characteristic functions corresponding to $F(x)$ and $G(x)$. We shall assume that $G(x)$ is the c.d.f. with the density determined by formula (1) and $F(x)$ is unknown c.d.f. which we want to estimate (see below). This problem was treated first by Gnedenko and Kolmogorov [2], where they give an estimation of such a difference using the integral $\int_{-T}^{T} |f(t) - g(t)|/tdt$. Next Dyson [1] showed that is not possible to determine for any $\sigma > 0$, such $\varepsilon > 0$ being dependent on σ only, that $\sup_x |F(x) - G(x)| < \sigma$ results from $\sup_x |f(t) - g(t)| < \varepsilon$. A full solution to this problem was given by Rogozin [17].

Five years later Mieshalkin and Rogozin published a paper [14] extending the results of [17]. We shall use here the Theorem 1 of the paper [14] in a somewhat modified form, concerning c.d.f's supported on $[0, \infty)$.

Let us assume that c.d.f. $F(x)$ and a function of bounded variation (f.b.v.) $G(x)$ fulfil the following conditions

(C1) $F(0) = G(0) = 0$,

(C2) $G'(x)$ exists for any $x > 0$ and $|G'(x)| < A < \infty$,

(C3) $|f(t) - g(t)| < \varepsilon$ for $|t| < T$, where $f(t)$ and $g(t)$ are the characteristic functions of $F(x)$ and $G(x)$, respectively.

There exists a constant C such that for $A, T, \varepsilon > 0$ and $L > 2T$ the following inequality holds

$$\sup_x |F(x) - G(x)| \leq C\left[\varepsilon \ln(LT) + A/T + \gamma(L)\right], \tag{23}$$

where

$$\gamma(L) = \operatorname*{var}_{0<x<\infty} G(x) - \sup_x \operatorname*{var}_{x\leq y\leq x+L} G(y)\,. \tag{24}$$

It follows from the proof of the Mieshalkin-Rogozin Theorem [14, p. 50] that

$$C \geq 16\left[\ln(LT) + 2^{-3}\pi^{-1} + 1\right]^{-1}. \tag{25}$$

The function $G(x)$ is c.d.f. so evidently var $G(x) = 1$ and

$$\sup_x \operatorname*{var}_{x\leq y\leq x+L} G(x) = \sup_x \int_x^{x+L} g\left(y|p,q\right) dy = \sup_x \left[G\left(x+L\right) - G(x)\right].$$

Let us compute the derivative

$$\begin{aligned} &d\beta_2(x|p,q)/(dx) = \\ &= \left[1/B\left(p,q\right)\right]\left(x+1\right)^{-2(p+q)} x^{p-2}\left(x+1\right)^{p+q-1}\left[-\left(q+1\right)x + p - 1\right]. \end{aligned} \tag{26}$$

It vanishes for

$$x_0 = (p-1)/(q+1) > 0 \text{ for } p > 1, q > 0. \tag{27}$$

Therefore

$$\sup_x \operatorname*{var}_{x \le y \le x+L<\infty} G(y) = [1/B(p,q)] \sup_x \int_x^{x+L} y^{p-1} \Big/ (1+y)^{p+q} dy \tag{28}$$

and for the derivative of the right hand side we have also

$$\begin{aligned} &d \operatorname*{var}_{x \le y \le x+L} G(y) \Big/ dx = \\ &= [1/B(p,q)] \left[(x+L)^{p-1} (1+x+L)^{-p-q} - x^{p-1} (1+x)^{-p-q} \right]. \end{aligned} \tag{29}$$

For $x < x_0$ the derivative is positive. This means that initial successive increments of the c.d.f. $G(x)$ on the segment of the lenghts L increase to the size of the interval $[x_0 - L, x_0)$, where x_0 is a point in which the derivative vanishes. For $x > x_0$ the derivative is negative and vanishes when $x \to \infty$. Therefore

$$\sup_x \operatorname*{var}_{x \le y \le x+L} G(y) = G(x_0) - G(x_0 - L) \tag{30}$$

and in (24) $\gamma(L)$ should be substituted by

$$\gamma(L) = 1 - G(x_0) + G(x_0 - L). \tag{31}$$

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