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RECURRENCE RELATIONS FOR SINGLE AND PRODUCT MOMENTS OF k-th LOWER RECORD VALUES FROM THE INVERSE DISTRIBUTIONS OF PARETO'S TYPE AND CHARACTERIZATIONS

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Abstract

We give recurrence relations for single and product moments of k-th lower record values from the inverse Pareto, inverse generalized Pareto and inverse Burr distributions. We present also characterization conditions for these distributions.

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1. Introduction

A random variable X is said to have an inverse Pareto distribution if its probability density function is of the form

$$(1.1) \qquad f(x) = \frac{\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{\alpha-1}, \; x \in (0, \sigma); \; \alpha, \; \sigma > 0.$$

Note that for the inverse Pareto distribution we have

$$(1.2) \qquad f(x)x = \alpha F(x),$$

where F denotes the distribution function of (1.1).

A random variable X is said to have an inverse generalized Pareto distribution if its probability density function is of the form

$$f(x) = \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}, \; x > 0; \; \tau, \; \theta > 0. \tag{1.3}$$

The inverse generalized Pareto distribution satisfies the condition

$$f(x)(x+\theta)x = \tau\theta F(x). \tag{1.4}$$

A random variable X is said to have an inverse Burr distribution if its pdf is of the form

$$f(x) = \frac{\tau\gamma(x/\theta)^{\gamma\tau}}{x[1+(x/\theta)^{\gamma}]^{\tau+1}}, \; x > 0; \; \theta, \; \tau, \; \gamma > 0. \tag{1.5}$$

For the inverse Burr distribution we have

$$f(x)x\left(\theta^{\gamma} + x^{\gamma}\right) = \tau\gamma\theta^{\gamma}F(x). \tag{1.6}$$

Remark. More details on inverse distributions and their applications one can find in [1].

Let $\{X_n, \; n \geq 1\}$ be a sequence of i.i.d. random variables with a cumulative distribution function F and a probability density function f. The j-th order statistic of a sample $(X_1, ..., X_n)$ is denoted by $X_{j:n}$. For a fixed $k \geq 1$ we define the sequence $\{L_k(n), \; n \geq 1\}$ of k-th lower record times of $\{X_n, \; n \geq 1\}$ as follows

$$L_k(1) = 1,$$
$$L_k(n+1) = \min\{j > L_k(n) : \; X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}.$$

For $k = 1$ we put $L(n) := L_1(n)$, $n \geq 1$, which are k-th lower record times of $\{X_n, \; n \geq 1\}$.

The sequence $\{Z_n^{(k)}, n \geq 1\}$ with $Z_n^{(k)} = X_{k:L_k(n)+k-1}$, $n = 1, 2, ...$, is called the sequence of k-th lower record values of $\{X_n, \; n \geq 1\}$. For convenience, we shall also take $Z_0^{(k)} = 0$. Note that for $k = 1$ we have

$Z_n^{(1)} = X_{L(n)}$, $n \geq 1$, i.e. record values of $\{X_n,\ n \geq 1\}$. Moreover, we see that $Z_1^{(k)} = \max(X_1, ..., X_k) := X_{k:k}$. It is know (cf. [3]) that the pdf of $Z_n^{(k)}$ and the joint pdf of $(Z_m^{(k)}, Z_n^{(k)})$ are given by:

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!}(-\ln F(x))^{n-1}(F(x))^{k-1}f(x),\ n \geq 1, \tag{1.7}$$

$$\begin{aligned} f_{Z_m^{(k)}, Z_n^{(k)}}(x,y) &= \frac{k^n}{(m-1)!(n-m-1)!}[\ln F(x) - \ln F(y)]^{n-m-1} \\ &\cdot[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}[F(y)]^{k-1}f(y),\ x > y;\ 1 \leq m < n,\ n \geq 2, \end{aligned} \tag{1.8}$$

respectively.

2. Relations for single moments

We start our study with the inverse Burr distribution. Using (1.6) and (1.7) we obtain the following relations for single moments of k-th lower record values from the inverse Burr distribution in (1.5).

Theorem 2.1 *Fix a positive integer k. For $n \geq 2$, $r = 0, 1, 2, ...$,*

$$E\left(Z_n^{(k)}\right)^r = \frac{k\tau\gamma}{k\tau\gamma + r}E\left(Z_{n-1}^{(k)}\right)^r - \frac{r}{\theta^\gamma(r + k\tau\gamma)}E\left(Z_n^{(k)}\right)^{r+\gamma} \tag{2.1}$$

and, consequently, for $0 \leq m \leq n-1$

$$\begin{aligned} E\left(Z_n^{(k)}\right)^r &= \left[\frac{k\tau\gamma}{k\tau\gamma + r}\right]^{n-s} E\left(Z_s^{(k)}\right)^r \\ &- \left[\frac{r}{\theta^\gamma(r + k\tau\gamma)}\right] \sum_{p=s+1}^{n} \left[\frac{k\tau\gamma}{k\tau\gamma + r}\right]^{n-p} E\left(Z_p^{(k)}\right)^{r+\gamma}. \end{aligned} \tag{2.2}$$

Proof. By (1.7) and (1.6) we have for $1 \leq r \leq n$

$$\theta^{\gamma} E\left(Z_n^{(k)}\right)^r + E\left(Z_n^{(k)}\right)^{r+\gamma} = \frac{k^n \tau\gamma\theta^{\gamma}}{(n-1)!}\int x^{r-1}[F(x)]^k[-\ln F(x)]^{n-1}dx.$$

Integrating by parts treating x^{r-1} as the part for integration we get

$$\theta^{\gamma} E\left(Z_n^{(k)}\right)^r + E\left(Z_n^{(k)}\right)^{r+\gamma} = \frac{k\tau\gamma\theta^{\gamma}}{r}\left[E\left(Z_{n-1}^{(k)}\right)^r - E\left(Z_n^{(k)}\right)^r\right],$$

which gives (2.1). Then (2.2) follows by induction.

Corollary 2.1. *The recurrence relations for single moments of lower record values from the inverse Burr distribution have the form*

$$E\left(X_{L(n)}\right)^r = \frac{\tau\gamma}{r+\tau\gamma}E\left(X_{L(n-1)}\right)^r - \frac{r}{\theta^{\gamma}(r+\tau\gamma)}E\left(X_{L(n)}\right)^{r+\gamma}.$$

Corollary 2.2. *Under the assumptions of Theorem* 2.1 *with* $\gamma = 1$ *we have the corresponding relations for the inverse generalized Pareto distribution* (1.3)

$$E\left(Z_n^{(k)}\right)^r = \frac{k\tau}{k\tau + r}E\left(Z_{n-1}^{(k)}\right)^r - \frac{r}{\theta(r+k\tau)}E\left(Z_n^{(k)}\right)^{r+1}.$$

Remark. For $k = 1$ we have the relations for moments of lower record values from the inverse generalized Pareto distribution of the form

$$E\left(X_{L(n)}\right)^r = \frac{\tau}{\tau + r}E\left(X_{L(n-1)}\right)^r - \frac{r}{\theta(r+\tau)}E\left(X_{L(n)}\right)^{r+1}.$$

Using (1.2) and (1.7) and using an argument similar to the one used in the proof of Theorem 2.1 we get recurrence relations for single moments of k-th lower record values from the inverse Pareto distribution.

Theorem 2.2. *Fix a positive integer* k. *For* $n \geq 2$ *and* $r = 0, 1, 2, \ldots,$

$$E\left(Z_n^{(k)}\right)^r = \frac{\alpha k}{\alpha k + r}E\left(Z_{n-1}^{(k)}\right)^r$$

and, consequently

$$E\left(Z_n^{(k)}\right)^r = \left(\frac{\alpha k}{\alpha k + r}\right)^{n-1} E\left(X_{k:k}\right)^r.$$

Corollary 2.3. *For $k = 1$ in the Theorem 2.2 we get*

$$E\left(X_{L(n)}\right)^r = \left(\frac{\alpha}{\alpha + r}\right)^{n-1} EX^r.$$

3. Relations for product moments

Using (1.6) and (1.8) we get the following recurrence relations for the inverse Burr distribution

Theorem 3.1. *Fix a positive integer k. For $m \geq 1$, $r, s = 0, 1, 2, ...,$*

$$E\left[\left(Z_m^{(k)}\right)^r \left(Z_{m+1}^{(k)}\right)^s\right] =$$

$$= \frac{k\tau\gamma}{s + k\tau\gamma} E\left(Z_m^{(k)}\right)^{r+s} - \frac{\gamma s}{\theta^\gamma (s + k\tau\gamma)} E\left[\left(Z_m^{(k)}\right)^r \left(Z_{m+1}^{(k)}\right)^{s+\gamma}\right],$$

and for $1 \leq m \leq n-2$

$$E\left[\left(Z_m^{(k)}\right)^r \left(Z_n^{(k)}\right)^s\right] =$$

$$= \frac{k\tau\gamma}{s + k\tau\gamma} E\left[\left(Z_m^{(k)}\right)^r \left(Z_{n-1}^{(k)}\right)^s\right] - \frac{s}{\theta^\gamma (s + k\tau\gamma)} E\left[\left(Z_m^{(k)}\right)^r \left(Z_n^{(k)}\right)^{s+\gamma}\right].$$

Corollary 3.1. *Under the assumptions of Theorem 3.1 with $\gamma = 1$ we have the relations for inverse generalized Pareto distribution*

$$E\left[\left(Z_m^{(k)}\right)^r \left(Z_{m+1}^{(k)}\right)^s\right] =$$

$$= \frac{k\tau}{s + k\tau} E\left(Z_m^{(k)}\right)^{r+s} - \frac{s}{\theta (s + k\tau)} E\left[\left(Z_m^{(k)}\right)^r \left(Z_{m+1}^{(k)}\right)^{s+1}\right],$$

and for $1 \le m \le n-2$

$$E\left[\left(Z_m^{(k)}\right)^r\left(Z_n^{(k)}\right)^s\right]=$$

$$=\frac{k\tau}{s+k\tau}E\left[\left(Z_m^{(k)}\right)^r\left(Z_{n-1}^{(k)}\right)^s\right]-\frac{s}{\theta(s+k\tau)}E\left[\left(Z_m^{(k)}\right)^r\left(Z_n^{(k)}\right)^{s+1}\right].$$

Using (1.2) and (1.8) we get relations for the inverse Pareto distribution.

Theorem 3.2. *Fix a positive integer* k*. For* $m \ge 1$, $r, s = 0, 1, 2, \ldots$,

$$E\left[\left(Z_m^{(k)}\right)^r\left(Z_{m+1}^{(k)}\right)^s\right]=\frac{k}{k+s}E\left(Z_m^{(k)}\right)^{r+s},$$

and for $1 \le m \le n-2$

$$E\left[\left(Z_m^{(k)}\right)^r\left(Z_n^{(k)}\right)^s\right]=\frac{k}{k+s}E\left[\left(Z_m^{(k)}\right)^r\left(Z_{n-1}^{(k)}\right)^s\right].$$

Remark. Recurrence relations for single and product moments of k-th record values from Pareto, generalized Pareto and Burr distributions were presented in [4].

4. Characterizations

This section contains characterizations of inverse distributions by moment relations for k-th lower record values.

Let $L(a,b)$ stand for the space of all integrable functions on (a,b). A sequence $(h_n) \subset L(a,b)$ is called complete on $L(a,b)$ if for all functions $g \in L(a,b)$ the condition

$$\int_a^b g(x)f_n(x)dx = 0,\ n \in \mathbb{N},$$

implies $g(x) = 0$ a.e. on (a,b). We start with the following result of Lin [2].

Proposition. *Let n_0 be any fixed non-negative integer, $-\infty \leq a < b \leq \infty$ and $g(x) \geq 0$ an absolutely continuous function with $g'(x) \neq 0$ a.e. on (a, b). Then the sequence of functions $\{(g(x))^n e^{-g(x)},\ n \geq n_0\}$ is complete in $L(a, b)$ iff $g(x)$ is strictly monotone on (a, b).*

Using the above Proposition we get a stronger version of Theorem 2.1 and 2.2.

Theorem 4.1. *Fix a positive integer $k \geq 1$ and let r be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1.5) is that*

$$E\left(Z_n^{(k)}\right)^r = \frac{k\tau\gamma}{k\tau\gamma + r} E\left(Z_{n-1}^{(k)}\right)^r - \frac{r}{\theta^\gamma (r + k\tau\gamma)} E\left(Z_n^{(k)}\right)^{r+\gamma},\ n = 2, 3, \dots .$$

Proof. The necessary part follows immediately from (2.1). On the other hand, if the recurrence relation (2.1) is satisfied, then

$$\frac{k^n \theta^\gamma (r + k\tau\gamma)}{(n-1)!} \int x^r [-\ln F(x)]^{n-1} [F(x)]^{k-1} f(x) dx$$

$$-\frac{k^n \theta^\gamma \tau\gamma}{(n-2)!} \int x^r [-\ln F(x)]^{n-2} [F(x)]^{k-1} f(x) dx$$

$$+\frac{r k^n}{(n-1)!} \int x^{r+\gamma} [-\ln F(x)]^{n-1} [F(x)]^{r-1} f(x) dx = 0.$$

Integrating the second integral by parts we have

$$\int x^{r-1} [-\ln F(x)]^{n-1} [F(x)]^{k-1} \left[\theta^\gamma (r + k\tau\gamma) x f(x) - r\theta^\gamma \tau\gamma F(x)\right.$$

$$\left. - k\theta^\gamma \tau\gamma x f(x) + r x^{\gamma+1} f(x)\right] dx = 0$$

which reduces to

$$\int x^{r-1}[-\ln F(x)]^{n-1}[F(x)]^{k-1}\left[f(x)x(\theta^\gamma + x^\gamma) - \theta^\gamma \tau\gamma F(x)\right] dx = 0.$$

It now follows from Proposition with $g(x) = [-\ln F(x)]$ that

$$f(x)x(\theta^\gamma + x^\gamma) = \theta^\gamma \tau\gamma F(x),$$

which proves that $f(x)$ has the form (1.5).

Corollary 4.1. *Under the assumptions of Theorem* 4.1 *with* $\gamma = 1$ *the following equations*

$$E\left(Z_n^{(k)}\right)^r = \frac{k\tau}{k\tau + r} E\left(Z_{n-1}^{(k)}\right)^r - \frac{r}{\theta(r + k\tau)} E\left(Z_n^{(k)}\right)^{r+1}, \ n = 2, 3, ...,$$

characterize the inverse generalized Pareto distribution (1.3)*, i.e. only the distribution* (1.3) *satisfies that condition.*

Using an argument similar to the one in the proof of Theorem 2.1 we get the following theorem.

Theorem 4.2. *Fix a positive integer* $k \geq 1$ *and let* r *be a non-negative integer. A necessary and sufficient condition for a random variable* X *to be distributed with pdf given by* (1.1) *is that*

$$E\left(Z_n^{(k)}\right)^r = \frac{\alpha k}{\alpha k + r} E\left(Z_{n-1}^{(k)}\right)^r$$

for $n = 2, 3, ...$.

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