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AN ALTERNATIVE APPROACH TO CHARACTERIZE THE COMMUTATIVITY OF ORTHOGONAL PROJECTORS

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Abstract

In an invited paper, Baksalary [Algebraic characterizations and statistical implications of the commutativity of orthogonal projectors. In: T. Pukkila, S. Puntanen (Eds.), Proceedings of the Second International Tampere Conference in Statistics, University of Tampere, Tampere, Finland, [2], pp. 113–142] presented 45 necessary and sufficient conditions for the commutativity of a pair of orthogonal projectors. Basing on these results, he discussed therein also statistical aspects of the commutativity with reference to problems concerned with canonical correlations and with comparisons between estimators and between sets of linearly sufficient statistics corresponding to different linear models. In the present paper, parts of this analysis are resumed in order to shed some additional light on the problem of commutativity. The approach utilized is different than the one used by Baksalary, and is based on representations of projectors in terms of partitioned matrices. The usefulness of such representations is demonstrated by reinvestigating some of Baksalary's statistical considerations.

Keywords: partitioned matrix, canonical correlations, ordinary least squares estimator, generalized least squares estimator, best linear unbiased estimator.

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1. Introduction

In an invited paper, published in the Proceedings of the Second International Tampere Conference in Statistics, Jerzy K. Baksalary [2] presented 45 necessary and sufficient conditions for the commutativity of a pair of orthogonal projectors. Basing on these results, he discussed statistical aspects of the commutativity with reference to problems concerned with canonical correlations and with comparisons between estimators and between sets of linearly sufficient statistics corresponding to different linear models. Subsequently, we resume parts of this analysis to shed some additional light on the problem of commutativity. In Section 2 we restate Theorem 1 in Baksalary [2] taking into account conditions which depend on orthogonal projectors. However, unlike Baksalary [2] who restricted his considerations to the real case only, our results are presented within the complex framework, what means that the projector matrices may contain complex entries. The next section is devoted to canonical correlations, whereas in Section 4 we follow Baksalary's traces to reinvestigate the equalities between ordinary least squares estimator (OLSE), generalized least squares estimator (GLSE), and best linear unbiased estimator (BLUE) in the general linear model. The paper is concluded with the section providing some further algebraic characterizations of the commutativity of orthogonal projectors.

Let $\mathbb{C}_{m,n}$ denote the set of $m \times n$ complex matrices and let $\mathbb{R}_{m,n}$ be its subset composed of real matrices. The symbols $\mathbf{F}^*$, $\mathbf{F}'$, $\mathcal{R}(\mathbf{F})$, $\mathcal{N}(\mathbf{F})$, and $\mathrm{rk}(\mathbf{F})$ will stand for the conjugate transpose, transpose, column space, null space, and rank of $\mathbf{F} \in \mathbb{C}_{m,n}$, respectively. Moreover, $\mathbf{I}_n$ will be the identity matrix of order n, and for given $\mathbf{F} \in \mathbb{C}_{n,n}$ we define $\overline{\mathbf{F}} = \mathbf{I}_n - \mathbf{F}$. The key role in the present paper is played by the notion of an orthogonal projector. Recall that a matrix $\mathbf{P} \in \mathbb{C}_{n,n}$ is called an orthogonal projector if $\mathbf{P}^2 = \mathbf{P} = \mathbf{P}^*$ ($\mathbf{P}^2 = \mathbf{P} = \mathbf{P}'$ in the real case), i.e., $\mathbf{P}$ is idempotent and Hermitian (idempotent and symmetric in the real case). An essential property of an orthogonal projector $\mathbf{P} \in \mathbb{C}_{n,n}$ is that it is expressible as $\mathbf{F}\mathbf{F}^\dagger$

for some $\mathbf{F} \in \mathbb{C}_{n,m}$, where $\mathbf{F}^\dagger \in \mathbb{C}_{m,n}$ is the Moore-Penrose inverse of $\mathbf{F}$, i.e., the unique solution to the equations

$$\mathbf{F}\mathbf{F}^\dagger\mathbf{F} = \mathbf{F},\ \mathbf{F}^\dagger\mathbf{F}\mathbf{F}^\dagger = \mathbf{F}^\dagger,\ (\mathbf{F}\mathbf{F}^\dagger)^* = \mathbf{F}\mathbf{F}^\dagger,\ (\mathbf{F}^\dagger\mathbf{F})^* = \mathbf{F}^\dagger\mathbf{F}.$$

Since $\mathcal{R}(\mathbf{F}\mathbf{F}^\dagger) = \mathcal{R}(\mathbf{F})$, we say that $\mathbf{P_F} = \mathbf{F}\mathbf{F}^\dagger$ is the orthogonal projector onto $\mathcal{R}(\mathbf{F})$ and, consequently, $\mathbf{Q_F} = \mathbf{I}_n - \mathbf{F}\mathbf{F}^\dagger$ is the orthogonal projector onto $\mathcal{R}(\mathbf{Q_F}) = \mathcal{R}^\perp(\mathbf{F})$, where $\mathcal{R}^\perp(\mathbf{F})$ denotes the orthogonal complement of $\mathcal{R}(\mathbf{F})$, i.e., the subspace consisting of all vectors orthogonal to $\mathcal{R}(\mathbf{F})$.

For matrix $\mathbf{F} \in \mathbb{C}_{n,n}$ of rank r, Hartwig and Spindelböck [5, Corollary 6] derived the following representation

$$\mathbf{F} = \mathbf{U}\begin{pmatrix} \mathbf{\Sigma K} & \mathbf{\Sigma L} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{1.1}$$

where $\mathbf{U} \in \mathbb{C}_{n,n}$ is unitary and the matrices $\mathbf{K}$, $\mathbf{L}$ satisfy $\mathbf{KK}^* + \mathbf{LL}^* = \mathbf{I}_r$, $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1\mathbf{I}_{r_1}, ..., \sigma_t\mathbf{I}_{r_t})$ is the diagonal matrix of singular values of $\mathbf{F}$, $\sigma_1 > \sigma_2 > ... > \sigma_t > 0$, $r_1+r_2+...+r_t = r = \mathrm{rk}(\mathbf{F})$. Using this representation and the fact that every orthogonal projector $\mathbf{P} \in \mathbb{C}_{n,n}$ is expressible as $\mathbf{F}\mathbf{F}^\dagger$, straightforward calculations show that $\mathbf{P}$ can be written as

$$\mathbf{P} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{1.2}$$

where $r = \mathrm{rk}(\mathbf{P})$.

Let now $\mathbf{Q} \in \mathbb{C}_{n,n}$ be another orthogonal projector. It is clear that, referring to (1.2), it can be expressed as a partitioned matrix of the form

$$\mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^*, \tag{1.3}$$

where $\mathbf{A} \in \mathbb{C}_{r,r}$ and $\mathbf{D} \in \mathbb{C}_{n-r,n-r}$ are both Hermitian.

The following 6 lemmas provide results which will be useful in the forthcoming sections. The first of them concerns relationships between submatrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$ involved in the matrix $\mathbf{Q}$ given in (1.3).

Lemma 1. *Let* $\mathbf{Q} \in \mathbb{C}_{n,n}$ *be the orthogonal projector represented as in* (1.3)*. Then:*

(i) $\mathbf{A} = \mathbf{A}^2 + \mathbf{B}\mathbf{B}^*$,

(ii) $\mathbf{B} = \mathbf{A}\mathbf{B} + \mathbf{B}\mathbf{D}$,

(iii) $\mathbf{D} = \mathbf{D}^2 + \mathbf{B}^*\mathbf{B}$,

(iv) $\mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\mathbf{A})$,

(v) $\mathcal{R}(\mathbf{B}) \subseteq \mathcal{R}(\overline{\mathbf{A}})$,

(vi) $\mathcal{R}(\mathbf{B}^*) \subseteq \mathcal{R}(\mathbf{D})$,

(vii) $\mathcal{R}(\mathbf{B}^*) \subseteq \mathcal{R}(\overline{\mathbf{D}})$,

(viii) $\mathrm{rk}(\mathbf{Q}) = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B}) + \mathrm{rk}(\mathbf{D})$.

Proof. Conditions (i)–(iii) are straightforward consequences of the idempotency of $\mathbf{Q}$.

Inclusion (iv) is established on account of condition (i) combined with $\mathbf{A} = \mathbf{A}^*$, by noting that

$$\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}\mathbf{A}^* + \mathbf{B}\mathbf{B}^*) = \mathcal{R}(\mathbf{A}\mathbf{A}^*) + \mathcal{R}(\mathbf{B}\mathbf{B}^*) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{B}),$$

where the second equality follows from the fact that $\mathbf{A}\mathbf{A}^*$ and $\mathbf{B}\mathbf{B}^*$ are both nonnegative definite. Clearly, the next three conditions are obtained in a similar way.

The proof of identity (viii) is more involved. In view of Lemma 1(iv), applying Corollary 19.1 in Marsaglia and Styan [6] to matrix $\mathbf{Q}$ of the form (1.3) leads to

$$\mathrm{rank}(\mathbf{Q}) = \mathrm{rank}(\mathbf{A}) + \mathrm{rank}(\mathbf{D} - \mathbf{B}^*\mathbf{A}^\dagger\mathbf{B}). \tag{1.4}$$

As can be shown by referring to the relationships between matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{D}$, the difference occurring in the latter summand on the right-hand side of (1.4) satisfies $\mathbf{D} - \mathbf{B}^*\mathbf{A}^\dagger\mathbf{B} = \mathbf{Q}_{\overline{\mathbf{D}}}$. Hence,

$$\mathrm{rank}(\mathbf{Q}) = \mathrm{rank}(\mathbf{A}) + n - r - \mathrm{rank}(\overline{\mathbf{D}}). \tag{1.5}$$

Further, since formula (2.12) in Tian and Styan [8] entails

$$\operatorname{rank}(\mathbf{D}^2 - \mathbf{D}) = \operatorname{rank}(\mathbf{D}) + \operatorname{rank}(\overline{\mathbf{D}}) - n + r,$$

on account of Lemma 1(iii), we get

$$\operatorname{rank}(\overline{\mathbf{D}}) = n - r + \operatorname{rank}(\mathbf{B}) - \operatorname{rank}(\mathbf{D}).$$

Substituting this equality into (1.5) yields condition (viii). ∎

Another consequences of (1.2) and (1.3) are given in what follows.

Lemma 2. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be the orthogonal projectors represented as in* (1.2) *and* (1.3)*, respectively. Then:*

(i) $\mathbf{PQ} = \mathbf{QP}$ *if and only if* $\mathbf{B} = \mathbf{0}$,

(ii) $\mathbf{PQ} = \mathbf{Q}$ *if and only if* $\mathbf{D} = \mathbf{0}$.

Proof. Equivalence (i) is obtained straightforwardly, whereas to establish statement (ii) use was made of Lemma 1(iii), according to which $\mathbf{D} = \mathbf{0}$ implies $\mathbf{B} = \mathbf{0}$. ∎

In what follows we make repeatedly use of the two facts which can be directly shown. Namely, for suitable matrices $\mathbf{F}$ and $\mathbf{G}$,

$$\operatorname{rk}(\mathbf{F}^*\mathbf{G}) = \operatorname{rk}(\mathbf{P_F}\mathbf{P_G}), \tag{1.6}$$

$$\mathcal{R}(\mathbf{FG}) = \mathcal{R}(\mathbf{F}\mathbf{P_G}). \tag{1.7}$$

Important tools for considering the orthogonal projectors onto given column spaces are provided by the next lemma.

Lemma 3. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be the orthogonal projectors. Then:*

(i) $\mathbf{P} + \overline{\mathbf{P}}(\overline{\mathbf{P}}\mathbf{Q})^\dagger$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})$,

(ii) $\mathbf{P} - \mathbf{P}(\mathbf{P}\overline{\mathbf{Q}})^\dagger$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$.

Proof. Statements (i) and (ii) constitute equivalences (3.1) $\Leftrightarrow$ (3.6) and (4.1) $\Leftrightarrow$ (4.8) in Piziak *et al.* [7], respectively. ■

Using Lemma 3 we obtain the following representations of the orthogonal projectors onto sums and intersections of certain subspaces, including their dimensions.

Lemma 4. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be the orthogonal projectors represented as in* (1.2) *and* (1.3)*, respectively. Then:*

(i) $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*,$

where $\dim[\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] = r + \mathrm{rk}(\mathbf{D}),$

(ii) $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*,$

where $\dim[\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})] = n+\mathrm{rk}(\mathbf{B})-\mathrm{rk}(\mathbf{D}),$

(iii) $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*,$

where $\dim[\mathcal{N}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] = n - r + \mathrm{rk}(\mathbf{A}),$

(iv) $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{U}^*,$

where $\dim[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})] = n-\mathrm{rk}(\mathbf{A})+\mathrm{rk}(\mathbf{B}).$

Proof. The proof is limited to the observations that the matrices representing projectors given in the theorem are obtained on account of Lemma 3(i), whereas the expressions for the dimensions of the corresponding subspaces are consequences of the fact that there is one-to-one correspondence between an orthogonal projector and the subspace onto which it projects. ■

Lemma 5. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be the orthogonal projectors represented as in* (1.2) *and* (1.3), *respectively. Then:*

(i) $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{Q}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*$,

where $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$,

(ii) $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{Q}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*$,

where $\dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})] = r - \mathrm{rk}(\mathbf{A})$,

(iii) $\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}^*$,

where $\dim[\mathcal{N}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] = \mathrm{rk}(\mathbf{D}) - \mathrm{rk}(\mathbf{B})$,

(iv) $\mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{\mathbf{D}} \end{pmatrix}\mathbf{U}^*$,

where $\dim[\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})] = n - r - \mathrm{rk}(\mathbf{D})$.

Proof. The proof is based on the same observations as the proof of the preceding lemma, with the reference to Lemma 3(i) replaced with the one to Lemma 3(ii). ■

The next lemma concerns ranks of functions of $\mathbf{P}$ and $\mathbf{Q}$.

Lemma 6. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be the orthogonal projectors represented as in* (1.2) *and* (1.3). *Then:*

(i) $\mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{QP}) = \mathrm{rk}(\mathbf{A})$,

(ii) $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) = n - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$,

(iii) $\mathrm{rk}(\mathbf{P}\overline{\mathbf{Q}}) = r - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$,

(iv) $\mathrm{rk}(\mathbf{P}+\mathbf{Q}) = r + \mathrm{rk}(\mathbf{D})$,

(v) $\mathrm{rk}(\mathbf{P}-\mathbf{Q}) = r - \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B}) + \mathrm{rk}(\mathbf{D})$,

(vi) $\mathrm{rk}(\mathbf{PQ}+\mathbf{QP}) = \mathrm{rk}(\mathbf{A}) + \mathrm{rk}(\mathbf{B})$.

Proof. Conditions (i), (iv), and (vi) are established directly by finding the orthogonal projectors onto the column spaces of the functions of $\mathbf{P}$ and $\mathbf{Q}$ on the left-hand side of the equations constituting them. The remaining three conditions are obtained similarly, but here one needs a formula for $\mathrm{rk}(\overline{\mathbf{A}})$. It is derived on account of equality (2.12) in Tian and Styan [8], which, in view of Lemma 1(i), entails $\mathrm{rank}(\overline{\mathbf{A}}) = r + \mathrm{rank}(\mathbf{B}) - \mathrm{rank}(\mathbf{A})$. ■

Observe that the proofs delivered so far may be also applied to the orthogonal projectors represented by matrices of real entries. Then the matrix $\mathbf{U}$ occurring in the representations of $\mathbf{P}$ and $\mathbf{Q}$ is to be chosen orthogonal (instead of unitary) and the conjugate transpose is to be replaced with the ordinary transpose.

2. Algebraic characterizations of commutativity

Our first theorem is closely related to Theorem 1 in Baksalary [2]. The A-numbers given in the right-hand side column refer to this theorem.

Theorem 1. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ be the orthogonal projectors. Then the following statements are equivalent:*

(i) $\mathbf{PQ} = \mathbf{QP}$, (A1)

(ii) $\mathbf{PQ} = (\mathbf{PQ})^2$, (A7)

(iii) $\mathbf{QPQ} = \mathbf{QPQPQ}$, (A8)

(iv) $\mathbf{Q}\overline{\mathbf{P}}\mathbf{Q} = \mathbf{Q}\overline{\mathbf{P}}\mathbf{Q}\overline{\mathbf{P}}\mathbf{Q}$, (A9)

(v) $\overline{\mathbf{Q}}\mathbf{PQ} = \mathbf{0}$, (A10)

(vi) $\mathbf{PQ} = \mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$, (A11)

(vii) $\mathbf{PQ} = \mathbf{P}\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\mathbf{Q}$, (A12)

(viii) $\mathbf{P}\overline{\mathbf{Q}}$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$, (A13)

(ix) $\mathbf{P}\overline{\mathbf{Q}}$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})$, (A14)

(x) $\mathbf{P}\overline{\mathbf{Q}}$ *is the orthogonal projector onto* $[\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] \cap \mathcal{N}(\mathbf{Q})$, (A15)

(xi) $\mathbf{P} + \mathbf{Q} - \mathbf{PQ}$ *is the orthogonal projector onto* $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})$, (A16)

(xii) $\mathcal{R}(\mathbf{Q}_{\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})}\mathbf{P})$ *and* $\mathcal{R}(\mathbf{Q}_{\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})}\mathbf{Q})$ *are orthogonal,* (A19)

(xiii) $\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$ *and* $\mathcal{R}(\mathbf{Q}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$ *are orthogonal,* (A20)

(xiv) $\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$ *and* $\mathcal{R}(\mathbf{Q})$ *are orthogonal,* (A21)

(xv) $\mathcal{R}(\mathbf{PQ}) \subseteq \mathcal{R}(\mathbf{Q})$, (A22)

(xvi) $\mathcal{R}(\mathbf{PQ}) \subseteq \mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$, (A23)

(xvii) $\mathcal{R}(\mathbf{PQ}) = \mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})$, (A24)

(xviii) $\mathcal{R}(\overline{\mathbf{Q}}\mathbf{P}) = \mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$, (A26)

(xix) $\mathcal{R}(\overline{\mathbf{Q}}\mathbf{P}) = \mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})$, (A27)

(xx) $[\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] \cap \mathcal{N}(\mathbf{Q}) = \mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]$, (A28)

(xxi) $[\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] \cap \mathcal{N}(\mathbf{Q}) = \mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})$, (A29)

(xxii) $\mathcal{R}(\mathbf{P}) = [\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] \oplus \{[\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})] \cap \mathcal{N}(\mathbf{Q})\}$, (A30)

(xxiii) $\mathcal{R}(\mathbf{P}) = [\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})] \oplus \mathcal{R}(\overline{\mathbf{Q}}\mathbf{P})$, (A31)

(xxiv) $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q}) = \mathcal{R}(\mathbf{Q}) \oplus [\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$, (A33)

(xxv) $\mathrm{rk}(\mathbf{PQ}) = \dim[\mathcal{R}(\mathbf{P}) \cap \mathcal{R}(\mathbf{Q})]$, (A34)

(xxvi) $\dim\{\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]\} = \mathrm{rk}(\mathbf{P}) - \mathrm{rk}(\mathbf{PQ})$, (A35)

(xxvii) $\dim\{[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]\cap\mathcal{N}(\mathbf{Q})\}=\mathrm{rk}(\mathbf{P})-\mathrm{rk}(\mathbf{PQ})$, (A36)

(xxviii) $\mathrm{rk}(\overline{\mathbf{Q}}\mathbf{P})=\mathrm{rk}(\mathbf{P})-\mathrm{rk}(\mathbf{PQ})$, (A37)

(xxix) $\mathrm{rk}(\mathbf{Q}+\mathbf{P})=\mathrm{rk}(\mathbf{P})+\mathrm{rk}(\mathbf{Q})-\mathrm{rk}(\mathbf{PQ})$, (A38)

(xxx) $\dim[\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]=\mathrm{rk}(\overline{\mathbf{Q}}\mathbf{P})$, (A39)

(xxxi) $\dim[\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]=\mathrm{rk}(\mathbf{P}+\mathbf{Q})-\mathrm{rk}(\mathbf{Q})$, (A40)

(xxxii) $\dim[\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]=\mathrm{rk}(\mathbf{P})-\dim[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]$, (A41)

(xxxiii) $\dim[\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]=\dim\{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\}$. (A42)

Proof. According to Lemma 2(i), condition (i) holds if and only if $\mathbf{B}=\mathbf{0}$. In the following we show that all other 32 conditions listed in the theorem are also equivalent to $\mathbf{B}=\mathbf{0}$.

First observe that condition $\mathbf{PQ}=(\mathbf{PQ})^2$ can be equivalently expressed as the conjunction $\mathbf{A}=\mathbf{A}^2$, $\mathbf{B}=\mathbf{AB}$, which by Lemma 1(i) is equivalent to $\mathbf{BB}^*=\mathbf{0}$, i.e., $\mathbf{B}=\mathbf{0}$.

Next, note that condition (iii) is equivalent to $\mathbf{A}^2=\mathbf{A}^3$, $\mathbf{AB}=\mathbf{A}^2\mathbf{B}$, and $\mathbf{B}^*\mathbf{B}=\mathbf{B}^*\mathbf{AB}$. Since $\mathbf{A}$ is Hermitian, $\mathbf{A}^2=\mathbf{A}^3$ implies $\mathbf{A}=\mathbf{A}^2$, and by Lemma 1(i) we have $\mathbf{B}=\mathbf{0}$. The reverse direction is trivial. Correspondingly, the equivalence (iv) $\Leftrightarrow$ (i) is obtained by replacing in condition (iii) the orthogonal projector $\mathbf{P}$ with the orthogonal projector $\overline{\mathbf{P}}$. Hence, it is seen that (iv) is equivalent to $\mathbf{Q}\overline{\mathbf{P}}=\overline{\mathbf{P}}\mathbf{Q}$, i.e., $\mathbf{PQ}=\mathbf{QP}$.

The proof referring to condition (v) is also obtained by direct calculations with the use of Lemma 1(i).

Further, by utilizing the formula for $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$ given in Lemma 5(i) it follows that $\mathbf{PQ}=\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$ holds if and only if $\mathbf{B}=\mathbf{0}$ and $\mathbf{Q}_{\overline{\mathbf{A}}}=\mathbf{A}$, with the latter of these conditions meaning that $\overline{\mathbf{A}}=\mathbf{P}_{\overline{\mathbf{A}}}$. However, in view of Lemma 1(i), the former of these conditions ensures that $\overline{\mathbf{A}}$ is idempotent, and, since $\overline{\mathbf{A}}$ is Hermitian, it is seen that $\overline{\mathbf{A}}^{\dagger}=\overline{\mathbf{A}}$, which in turn means that $\mathbf{B}=\mathbf{0}\Rightarrow\mathbf{Q}_{\overline{\mathbf{A}}}=\mathbf{A}$.

For the proof referring to condition (vii) note that

$$\mathbf{PP}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\mathbf{Q}=\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\mathbf{Q}=\mathbf{U}\begin{pmatrix}\mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{A} & \mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{B}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*.$$

Thus, condition (vii) can be expressed as $\mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{A} = \mathbf{A}$ and $\mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{B} = \mathbf{B}$. The former of these equations can be rewritten as $\mathbf{Q}_{\overline{\mathbf{A}}}(\mathbf{I}_r - \overline{\mathbf{A}}) = \mathbf{I}_r - \overline{\mathbf{A}}$, and hence further simplified to $\mathbf{P}_{\overline{\mathbf{A}}} = \overline{\mathbf{A}}$. Since $\overline{\mathbf{A}}$ is Hermitian, the following chain of equivalences holds $\mathbf{P}_{\overline{\mathbf{A}}} = \overline{\mathbf{A}} \Leftrightarrow \overline{\mathbf{A}}^2 = \overline{\mathbf{A}} \Leftrightarrow \mathbf{A}^2 = \mathbf{A} \Leftrightarrow \mathbf{B} = \mathbf{0}$.

Applying Lemma 3(ii) to $\mathbf{P}_{\mathcal{R}(\mathbf{P})} = \mathbf{P}$ and $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})}$ given in (1.2) and Lemma 4(iv), respectively, leads to

$$\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*. \tag{2.1}$$

Since

$$\mathbf{P}\overline{\mathbf{Q}} = \mathbf{U}\begin{pmatrix}\overline{\mathbf{A}} & -\mathbf{B}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*, \tag{2.2}$$

in view of Lemma 1(i), it is seen that condition (viii) is equivalent to $\mathbf{B} = \mathbf{0}$. The next equivalence, i.e., (ix) $\Leftrightarrow$ (i), is established in a similar way, with the use of the formula for $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})}$ given in Lemma 5(ii).

Since $\mathbf{P}_{\mathcal{N}(\mathbf{Q})} = \overline{\mathbf{Q}}$, on account of Lemma 1(vi), from Lemma 3(ii) we get

$$\mathbf{P}_{[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]\cap\mathcal{N}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix}\overline{\mathbf{A}} & -\mathbf{B}\\ -\mathbf{B}^* & \mathbf{P}_{\mathbf{D}} - \mathbf{D}\end{pmatrix}\mathbf{U}^*. \tag{2.3}$$

Comparing (2.2) with (2.3) leads to the conclusion that condition (x) holds if and only if $\mathbf{B} = \mathbf{0}$ and $\mathbf{P}_{\mathbf{D}} = \mathbf{D}$. However, in view of Lemma 1(iii), the former of these equalities ensures that $\mathbf{D}$ is idempotent, and, since $\mathbf{D}$ is Hermitian, it is seen that $\mathbf{B} = \mathbf{0}$ implies $\mathbf{P}_{\mathbf{D}} = \mathbf{D}$. This implication is utilized also in the next step of the proof, in which from (1.2), (1.3), and Lemma 4(i) we have

$$\mathbf{P} + \mathbf{Q} - \mathbf{P}\mathbf{Q} = \mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})} \;\Leftrightarrow\; \mathbf{U}\begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ \mathbf{B}^* & \mathbf{D}\end{pmatrix}\mathbf{U}^* = \mathbf{U}\begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_{\mathbf{D}}\end{pmatrix}\mathbf{U}^*.$$

In consequence, the equivalence (xi) $\Leftrightarrow$ (i) is satisfied if and only if $\mathbf{B} = \mathbf{0}$ and $\mathbf{P}_{\mathbf{D}} = \mathbf{D}$, what can be reduced to $\mathbf{B} = \mathbf{0}$ only.

From (1.2), (1.3), and Lemma 5(i) it follows that

$$\mathbf{Q}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\mathbf{P} = \mathbf{U}\begin{pmatrix}\mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}^*$$

and

$$\mathbf{Q}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}}\mathbf{A} & \mathbf{P}_{\overline{\mathbf{A}}}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

whence it is seen that the column spaces of these matrices are orthogonal if and only if $\mathbf{P}_{\overline{\mathbf{A}}}\mathbf{A} = \mathbf{0}$ and $\mathbf{P}_{\overline{\mathbf{A}}}\mathbf{B} = \mathbf{0}$. The former of these conditions can be expressed as $\mathbf{P}_{\overline{\mathbf{A}}} = \overline{\mathbf{A}}$, whereas, on account of Lemma 1(v), the latter one is equivalent to $\mathbf{B} = \mathbf{0}$. Thus, the assertion is established by the same arguments as those utilized in the proof referring to condition (vi).

In view of Lemma 1(v), applying Lemma 3(ii) to $\mathbf{P}_{\mathcal{R}(\mathbf{Q})} = \mathbf{Q}$ and $\mathbf{P}_{\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})}$ given in (1.2) and Lemma 4(iv), respectively, leads to

$$\mathbf{P}_{\mathcal{R}(\mathbf{Q})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} - \overline{\mathbf{A}} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^*. \tag{2.4}$$

As easy to observe, on account of Lemma 1(v), the products of projectors (2.1) and (2.4) are equal to zero if and only if $\mathbf{B} = \mathbf{0}$. Similarly, the column spaces of projectors (2.1) and $\mathbf{Q}$ are orthogonal if and only if (i) holds.

Condition (xv) can be equivalently expressed as $\mathbf{QPQ} = \mathbf{PQ}$, which is known to be satisfied if and only if $\mathbf{P}$ and $\mathbf{Q}$ commute; see e.g., Theorem in Baksalary *et al.* [3]. Correspondingly, condition (xvi) can be expressed as $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}\mathbf{PQ} = \mathbf{PQ}$. From (1.2), (1.3), and Lemma 5(i) it follows that condition (xvi) holds if and only if $\mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{A} = \mathbf{A}$, $\mathbf{Q}_{\overline{\mathbf{A}}}\mathbf{B} = \mathbf{B}$. Since the same conjunction was obtained in the proof referring to condition (vii), the equivalence (xvi) $\Leftrightarrow$ (i) is obtained.

Direct calculations with the use of Lemma 1(i) and Lemma 1(iv) show that the Moore-Penrose inverse of $\mathbf{PQ}$ is of the form

$$(\mathbf{PQ})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{B}^*\mathbf{A}^\dagger & \mathbf{0} \end{pmatrix}\mathbf{U}^*. \tag{2.5}$$

Hence, we obtain

$$\mathbf{P}_{\mathcal{R}(\mathbf{PQ})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*, \tag{2.6}$$

and comparing this projector with $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$, given in Lemma 5(i), leads to the conclusion that condition (xvii) holds if and only if $\mathbf{P}_{\mathbf{A}} = \mathbf{Q}_{\overline{\mathbf{A}}}$, or, equivalently, $\mathcal{R}(\mathbf{A}) = \mathcal{N}(\overline{\mathbf{A}})$. This condition is equivalent to $\mathbf{A}^2 = \mathbf{A}$, which in turn means that $\mathbf{B} = \mathbf{0}$.

Conditions (xviii) and (xix) involve column space $\mathcal{R}(\overline{\mathbf{Q}}\mathbf{P})$. Since

$$\overline{\mathbf{Q}}\mathbf{P} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & \mathbf{0} \\ -\mathbf{B}^* & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

and, as can be verified by direct calculations with the use of Lemma 1(i) and Lemma 1(v),

$$(\overline{\mathbf{Q}}\mathbf{P})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & -\overline{\mathbf{A}}^\dagger\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

it follows that

$$\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\mathbf{P})} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{B} \\ -\mathbf{B}^* & \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B} \end{pmatrix}\mathbf{U}^*. \tag{2.7}$$

Hence, the corresponding assertions are established by comparing the projector (2.7) with $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]}$ given in (2.1) and $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})}$ given in Lemma 5(ii), respectively. Analogously, the proofs corresponding to the next two conditions are obtained by comparing (2.3) with (2.1) and $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})}$, respectively.

Applying Lemma 3(i), with the use of Lemma 1(v), to $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$ given in Lemma 5(i) and formulae (2.3) and (2.7) gives

$$\mathbf{P}_{[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]\oplus\{[\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]\cap\mathcal{N}(\mathbf{Q})\}} = \mathbf{U}\begin{pmatrix} \mathbf{Q}_{\overline{\mathbf{A}}}+\overline{\mathbf{A}} & -\mathbf{B} \\ -\mathbf{B}^* & \mathbf{P}_{\mathbf{D}}-\mathbf{D} \end{pmatrix}\mathbf{U}^*, \tag{2.8}$$

$$\mathbf{P}_{[\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})]\oplus\mathcal{R}(\overline{\mathbf{Q}}\mathbf{P})} = \mathbf{U}\begin{pmatrix} \mathbf{Q}_{\overline{\mathbf{A}}}+\overline{\mathbf{A}} & -\mathbf{B} \\ -\mathbf{B}^* & \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B} \end{pmatrix}\mathbf{U}^*, \tag{2.9}$$

respectively. Thus, equivalences (xxii) $\Leftrightarrow$ (i) and (xxiii) $\Leftrightarrow$ (i) are established directly by comparing (2.8) and (2.9) with $\mathbf{P}_{\mathcal{R}(\mathbf{P})} = \mathbf{P}$. (Parenthetically note that combining matrices (2.8) and (2.9) leads to the conclusion that $\mathcal{R}(\overline{\mathbf{Q}}\mathbf{P}) = [\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]\cap\mathcal{N}(\mathbf{Q})$. Actually, it can be shown that $\mathbf{P}_{\mathbf{D}}-\mathbf{D} = \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B}$.)

The next condition involves the column space $\mathcal{R}(\mathbf{Q}) \oplus [\mathcal{R}(\mathbf{P}) \cap \mathcal{N}(\mathbf{Q})]$, which is, on account of Lemma 1(iv), attributed to

$$\mathbf{P}_{\mathcal{R}(\mathbf{Q})\oplus[\mathcal{R}(\mathbf{P})\cap\mathcal{N}(\mathbf{Q})]} = \mathbf{U}\begin{pmatrix} \mathbf{Q_A} + \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{D} \end{pmatrix}\mathbf{U}^*.$$

Hence, by comparing this projector with $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})}$ given in Lemma 4(i) shows that also condition (xxiv) holds if and only if $\mathbf{B} = \mathbf{0}$.

The proof corresponding to condition (xxv) is obtained simply by comparing the latter part of Lemma 5(i) with Lemma 6(i).

The proof is concluded with the general observation that the equivalences between each of conditions (xxvi)–(xxxiii) and (i) are obtained on account of conditions: (viii) of Lemma 1; (i), (ii) of Lemma 5; (i), (iii), (iv) of Lemma 6, formulae (2.1), (2.3), and the properties of the rank of a matrix, by utilizing the fact that $\mathrm{rk}(\mathbf{P}) = r$. ∎

The theorem is followed by a comment that condition (A32) listed in Baksalary's [2] Theorem 1, i.e., $\mathcal{R}(\mathbf{P}) + \mathcal{R}(\mathbf{Q}) = \mathcal{R}(\mathbf{Q}) \oplus \{\mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})]\}$, does not yield a characterization of commutativity. This fact is based on the observation that $\mathbf{P}_{\mathcal{R}(\mathbf{P})+\mathcal{R}(\mathbf{Q})}$, given in Lemma 4(i), always coincides with $\mathbf{P}_{\mathcal{R}(\mathbf{Q})\oplus\{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\}}$. This can be shown by using Lemma 3(i) and formula (2.4).

3. Canonical correlations

In the present section, inspired by Section 3.1 in Baksalary [2], it is assumed that all matrices under investigation have real entries. As in Baksalary [2, Section 3.1], consider the linear model of the form

$$\mathcal{M}_a = \{\mathbf{y}, \mathbf{W}\boldsymbol{\gamma} + \mathbf{Z}\boldsymbol{\delta}, \sigma^2\mathbf{I}_n\}, \tag{3.1}$$

in which $\mathbf{y} \in \mathbb{R}_{n,1}$ is an observable random vector with expectation $\mathsf{E}(\mathbf{y}) = \mathbf{W}\boldsymbol{\gamma} + \mathbf{Z}\boldsymbol{\delta}$ and with dispersion matrix $\mathsf{Cov}(\mathbf{y}) = \sigma^2\mathbf{I}_n$, where $\mathbf{W} \in \mathbb{R}_{n,w}$, $\mathbf{Z} \in \mathbb{R}_{n,z}$ are known, while $\boldsymbol{\gamma} \in \mathbb{R}_{w,1}$, $\boldsymbol{\delta} \in \mathbb{R}_{z,1}$, and $\sigma^2 > 0$ are unknown parameters. It was shown therein that the number of nonzero canonical correlations between $\mathbf{W}'\mathbf{y}$ and $\mathbf{Z}'\mathbf{y}$ is equal to $s = \mathrm{rk}(\mathbf{W}'\mathbf{Z})$, whereas the number of canonical correlations equal to one is $s_1 = \mathrm{rk}(\mathbf{W}) + \mathrm{rk}(\mathbf{Z}) - \mathrm{rk}(\mathbf{W} : \mathbf{Z})$. If now $\mathbf{P} = \mathbf{W}\mathbf{W}^\dagger = \mathbf{P}_\mathbf{W}$ and $\mathbf{Q} = \mathbf{Z}\mathbf{Z}^\dagger = \mathbf{P}_\mathbf{Z}$, with $\mathbf{P}$ and $\mathbf{Q}$ having representation as specified in the Introduction, then, using Lemma 6(i), we have $s = \mathrm{rk}(\mathbf{PQ}) = \mathrm{rk}(\mathbf{A})$. Furthermore, on account of Lemma 1(viii) and Lemma 6(iv), we get $s_1 = \mathrm{rk}(\mathbf{P}) + \mathrm{rk}(\mathbf{Q}) - \mathrm{rk}(\mathbf{P}+\mathbf{Q}) = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$. With the use of these observations, we can formulate the following theorem, which is closely related to Theorem 2 in Baksalary [2].

Theorem 2. *Let $\mathbf{P}, \mathbf{Q} \in \mathbb{R}_{n,n}$ be the orthogonal projectors and let $\mathbf{y}$, $\mathbf{Z}$, and $\mathbf{W}$ be as specified in the linear model* (3.1)*. Moreover, let s and s_1 be the numbers of zero and unit canonical correlations between $\mathbf{W}'\mathbf{y}$ and $\mathbf{Z}'\mathbf{y}$, respectively. Then the following conditions are equivalent:*

(i) $\mathbf{PQ} = \mathbf{QP}$,

(ii) $s = s_1$,

(iii) $\mathsf{Cov}(\mathbf{W}'\overline{\mathbf{Q}}\mathbf{y}, \mathbf{Z}'\overline{\mathbf{P}}\mathbf{y}) = \mathbf{0}$,

(iv) $\mathbf{W}'\overline{\mathbf{Q}}\mathbf{W} = \mathbf{W}'\overline{\mathbf{Q}}\mathbf{P}\overline{\mathbf{Q}}\mathbf{W}$.

Proof. As stated in Lemma 2(i), condition (i) is satisfied if and only if $\mathbf{B} = \mathbf{0}$. We show that also the three remaining conditions given in the theorem are equivalent to $\mathbf{B} = \mathbf{0}$.

First observe that condition (ii) can be expressed as $\mathrm{rk}(\mathbf{A}) = \mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})$. Since $\mathbf{B} = \mathbf{0} \Leftrightarrow \mathrm{rk}(\mathbf{B}) = 0$, the equivalence (ii) $\Leftrightarrow$ (i) is established.

For the proof referring to condition (iii), note that it is satisfied if and only if $\mathbf{W}'\overline{\mathbf{Q}}\,\overline{\mathbf{P}}\mathbf{Z} = \mathbf{0}$, or, equivalently, $\mathrm{rk}(\mathbf{W}'\overline{\mathbf{Q}}\,\overline{\mathbf{P}}\mathbf{Z}) = 0$. On account of (1.6), we have $\mathrm{rk}(\mathbf{W}'\overline{\mathbf{Q}}\,\overline{\mathbf{P}}\mathbf{Z}) = \mathrm{rk}(\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\mathbf{W})}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{P}}\mathbf{Z})})$, where $\mathrm{rk}(\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\mathbf{W})}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{P}}\mathbf{Z})}) = \mathrm{rk}(\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\mathbf{P})}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{P}}\mathbf{Q})})$. From (1.2) and (1.3) it is seen that

$$\overline{\mathbf{P}}\mathbf{Q} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}' & \mathbf{D} \end{pmatrix}\mathbf{U}',$$

and direct calculations with the use of Lemma 1(iii) and Lemma 1(vi) confirm that

$$(\overline{\mathbf{P}}\mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{B}\mathbf{D}^\dagger \\ \mathbf{0} & \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}'.$$

In consequence, on account of Lemma 1(iii),

$$\mathbf{P}_{\mathcal{R}(\overline{\mathbf{P}}\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{\mathbf{D}} \end{pmatrix}\mathbf{U}'. \tag{3.2}$$

Hence, utilizing Lemma 1(vi), from (2.7) and (3.2), we have

$$\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\mathbf{P})}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{P}}\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{0} & -\mathbf{B} \\ \mathbf{0} & \mathbf{B}'\overline{\mathbf{A}}^{\dagger}\mathbf{B} \end{pmatrix}\mathbf{U}'.$$

Thus, clearly $\mathrm{rk}(\mathbf{W}'\overline{\mathbf{Q}}\,\overline{\mathbf{P}}\mathbf{Z}) = 0$ if and only if $\mathbf{B} = \mathbf{0}$.

Finally, condition (iv) can be rewritten as $\mathbf{W}'\overline{\mathbf{Q}}\,\overline{\mathbf{P}}\,\overline{\mathbf{Q}}\mathbf{W} = \mathbf{0}$ which is satisfied if and only if $\mathbf{W}'\overline{\mathbf{Q}}\,\overline{\mathbf{P}} = \mathbf{0}$, or, equivalently, $\mathrm{rk}(\mathbf{P}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\,\overline{\mathbf{P}})}) = 0$. Since, as can be directly verified with the use of Lemma 1(iii) and Lemma 1(viii), the Moore-Penrose inverse of

$$\overline{\mathbf{Q}}\,\overline{\mathbf{P}} = \mathbf{U}\begin{pmatrix} \mathbf{0} & -\mathbf{B} \\ \mathbf{0} & \overline{\mathbf{D}} \end{pmatrix}\mathbf{U}'$$

is of the form

$$(\overline{\mathbf{Q}}\,\overline{\mathbf{P}})^{\dagger} = \mathbf{U}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -\overline{\mathbf{D}}^{\dagger}\mathbf{B}' & \mathbf{P}_{\overline{\mathbf{D}}} \end{pmatrix}\mathbf{U}',$$

on account of Lemma 1(vii), we obtain

$$\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\,\overline{\mathbf{P}})} = \mathbf{U}\begin{pmatrix} \mathbf{B}\overline{\mathbf{D}}^{\dagger}\mathbf{B}' & -\mathbf{B} \\ -\mathbf{B}' & \overline{\mathbf{D}} \end{pmatrix}\mathbf{U}'.$$

In consequence, it is seen that $\mathrm{rk}(\mathbf{P}\mathbf{P}_{\mathcal{R}(\overline{\mathbf{Q}}\,\overline{\mathbf{P}})}) = 0$ if and only if $\mathbf{B} = \mathbf{0}$. The proof is complete. ■

4. Comparison among three estimators

Similarly as in the previous section, also in the present one it is assumed that the matrices under investigation have real entries. Following Baksalary [2, Section 3.2], consider the general linear model of the form

$$\mathcal{M} = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta}, \sigma^2\mathbf{V}\}, \tag{4.1}$$

where $\mathbf{X} \in \mathbb{R}_{n,x}$, $\mathbf{V} \in \mathbb{R}_{n,n}$ are known (with possibly $\mathbf{X}$ not of full column rank and $\mathbf{V}$ singular), $\boldsymbol{\beta} \in \mathbb{R}_{x,1}$ is unknown, and $\mathbf{y}$, σ^2 are as defined in (3.1). We are interested in the equations

$$\text{OLSE}(\mathbf{X}\boldsymbol{\beta}) = \text{BLUE}(\mathbf{X}\boldsymbol{\beta}),$$

$$\text{GLSE}(\mathbf{X}\boldsymbol{\beta}) = \text{BLUE}(\mathbf{X}\boldsymbol{\beta}),$$

where

$$\text{OLSE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{P_X y},$$

$$\text{GLSE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}(\mathbf{X}'\mathbf{V}^\dagger\mathbf{X})^\dagger\mathbf{X}'\mathbf{V}^\dagger\mathbf{y},$$

and

$$\text{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{Ty},$$

with $\mathbf{T}$ being any solution to the matrix equations $\mathbf{T}(\mathbf{X} : \mathbf{VQ_X}) = (\mathbf{X} : \mathbf{0})$.

$$\text{OLSE}(\mathbf{X}\boldsymbol{\beta}) = \text{BLUE}(\mathbf{X}\boldsymbol{\beta}) \;\Leftrightarrow\; \mathbf{P_X V} = \mathbf{VP_X}, \tag{4.2}$$

$$\text{GLSE}(\mathbf{X}\boldsymbol{\beta}) = \text{BLUE}(\mathbf{X}\boldsymbol{\beta}) \;\Leftrightarrow\; \mathbf{P_V P_X} = \mathbf{P_X}. \tag{4.3}$$

Following Hartwig and Spindelböck [5, Corollary 6], denoting $\text{rk}(\mathbf{V}) = r$, let us write

$$\mathbf{V} = \mathbf{U}\begin{pmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}',$$

where $\mathbf{G} \in \mathbb{R}_{r,r}$ is positive definite and $\mathbf{U} \in \mathbb{R}_{n,n}$ is orthogonal. In consequence,

$$\mathbf{P_V} = \mathbf{VV}^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}'.$$

If $\mathbf{P_X}$ takes the role of the projector $\mathbf{Q}$ defined in (1.3), we may write

$$\mathbf{P_X} = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{pmatrix}\mathbf{U}',$$

and straightforward calculations show that the right-hand sides conditions in (4.2) and (4.3) satisfy

$$\mathbf{P_X V} = \mathbf{VP_X} \;\Leftrightarrow\; \mathbf{B} = \mathbf{0},\; \mathbf{AG} = \mathbf{GA}, \tag{4.4}$$

$$\mathbf{P_V P_X} = \mathbf{P_X} \;\Leftrightarrow\; \mathbf{D} = \mathbf{0}. \tag{4.5}$$

In what follows we restate Theorem 3 in Baksalary [2], with its part (i) modified on account of (1.7).

Theorem 3. *Let $\mathcal{M}$ be the general linear model of the form* (4.1)*. Then:*

(i) $\text{OLSE}(\mathbf{X}\boldsymbol{\beta}) = \text{BLUE}(\mathbf{X}\boldsymbol{\beta}) \Leftrightarrow \mathbf{P_X P_V} = \mathbf{P_V P_X}$ *holds along with* $\mathcal{R}(\mathbf{VP}_{\mathcal{R}(\mathbf{X})\cap\mathcal{R}(\mathbf{V})}) \subseteq \mathcal{R}(\mathbf{X})$,

(ii) $\text{GLSE}(\mathbf{X}\boldsymbol{\beta}) = \text{BLUE}(\mathbf{X}\boldsymbol{\beta}) \Leftrightarrow \mathbf{P_X P_V} = \mathbf{P_V P_X}$ *holds along with* $\mathcal{R}(\mathbf{X}) \cap \mathcal{R}^{\perp}(\mathbf{V}) = \{\mathbf{0}\}$.

Proof. It is easily seen that $\mathbf{P_X P_V} = \mathbf{P_V P_X} \Leftrightarrow \mathbf{B} = \mathbf{0}$. Moreover, the latter condition on the right-hand side of statement (i) can be rewritten in the form

$$\mathbf{P_X V P}_{\mathcal{R}(\mathbf{X})\cap\mathcal{R}(\mathbf{V})} = \mathbf{VP}_{\mathcal{R}(\mathbf{X})\cap\mathcal{R}(\mathbf{V})}. \tag{4.6}$$

Note that $\mathbf{P}_{\mathcal{R}(\mathbf{X})\cap\mathcal{R}(\mathbf{V})}$ has the same form as $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$ given in Lemma 5(i). In consequence, equality (4.6) can be expressed as $\mathbf{AGQ}_{\overline{\mathbf{A}}} = \mathbf{GQ}_{\overline{\mathbf{A}}}$ and $\mathbf{B}'\mathbf{GQ}_{\overline{\mathbf{A}}} = \mathbf{0}$. Since, on account of Lemma 1(i), $\mathbf{B} = \mathbf{0}$ holds if and only if $\mathbf{Q}_{\overline{\mathbf{A}}} = \mathbf{A}$, the conjunction of $\mathbf{P_X P_V} = \mathbf{P_V P_X}$ and (4.6) is equivalent to $\mathbf{B} = \mathbf{0}$ and $\mathbf{AGA} = \mathbf{GA}$. In view of the fact that $\mathbf{A}$ and $\mathbf{G}$ are symmetric, condition $\mathbf{AGA} = \mathbf{GA}$ can be simplified to $\mathbf{AG} = \mathbf{GA}$, and thus, we have obtained the right-hand side of (4.4).

For the proof of equivalence (ii) first observe that condition $\mathcal{R}(\mathbf{X}) \cap \mathcal{R}^{\perp}(\mathbf{V}) = \{\mathbf{0}\}$ can be expressed as $\mathbf{P}_{\mathcal{R}(\mathbf{P_X})\cap\mathcal{R}(\mathbf{Q_V})} = \mathbf{0}$, where $\mathbf{P}_{\mathcal{R}(\mathbf{P_X})\cap\mathcal{R}(\mathbf{Q_V})} = \mathbf{P}_{\mathcal{R}(\mathbf{Q})\cap\mathcal{R}(\overline{\mathbf{P}})} = \mathbf{P}_{\mathcal{N}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$, with the last projector given in Lemma 5(iii). In consequence, condition $\mathcal{R}(\mathbf{X}) \cap \mathcal{R}^{\perp}(\mathbf{V}) = \{\mathbf{0}\}$ is satisfied if and only if $\mathbf{Q}_{\overline{\mathbf{D}}} = \mathbf{0}$. Combining $\mathbf{Q}_{\overline{\mathbf{D}}} = \mathbf{0}$, which means that $\overline{\mathbf{D}}$ is nonsingular, with $\mathbf{B} = \mathbf{0}$, which, on account of Lemma 1(iii) ensures that $\overline{\mathbf{D}}$ is idempotent, leads to the conclusion that

$$\mathbf{P_X P_V} = \mathbf{P_V P_X},\ \mathcal{R}(\mathbf{X}) \cap \mathcal{R}^{\perp}(\mathbf{V}) = \{\mathbf{0}\} \;\Leftrightarrow\; \mathbf{D} = \mathbf{0}.$$

Thus, in view of (4.5), the proof is completed. ∎

In part (a) $\Leftrightarrow$ (f) of his Theorem 4, Baksalary [2] has shown the equivalence of conditions

$$\mathsf{Cov}[\text{BLUE}(\mathbf{X}\boldsymbol{\beta})] = \mathsf{Cov}[\text{GLSE}(\mathbf{X}\boldsymbol{\beta})],$$

$$\text{BLUE}(\mathbf{X}\boldsymbol{\beta}) = \mathbf{X}(\mathbf{X}'\mathbf{V}^{\dagger}\mathbf{X})^{\dagger}\mathbf{X}'\mathbf{V}^{\dagger}\mathbf{y} + \mathbf{X}(\mathbf{X}'\mathbf{Q_V}\mathbf{X})^{\dagger}\mathbf{X}'\mathbf{Q_V}\mathbf{y}.$$

In what follows we demonstrate that the four other equivalent conditions given in Theorem 4 in Baksalary [2] can be expressed in terms of orthogonal projectors. For this purpose, denoting $\text{rk}(\mathbf{X}) = r$, let us write the model matrix $\mathbf{X}$ in its singular value decomposition

$$\mathbf{X} = \mathbf{U}\begin{pmatrix}\boldsymbol{\Omega} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}_1', \tag{4.7}$$

where $\mathbf{U} \in \mathbb{R}_{n,n}$, $\mathbf{U}_1 \in \mathbb{R}_{x,x}$ are orthogonal matrices, and $\boldsymbol{\Omega} \in \mathbb{R}_{r,r}$ is the diagonal matrix of singular values of $\mathbf{X}$; cf. Ben-Israel and Greville [4, p. 15]. From (4.7) it follows that

$$\mathbf{X}\mathbf{X}' = \mathbf{U}\begin{pmatrix}\boldsymbol{\Omega}^2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{U}', \tag{4.8}$$

with the elements of $\boldsymbol{\Omega}^2$ being the nonzero eigenvalues of $\mathbf{X}\mathbf{X}'$. Accordingly, we assume that

$$\mathbf{P_V} = \mathbf{U}\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}' & \mathbf{D}\end{pmatrix}\mathbf{U}'. \tag{4.9}$$

Now we get the following result, which involves conditions (b)–(e) of Baksalary's Theorem 4, with conditions occurring in points (b) and (e) restated on account of equality (1.7). Furthermore, point (d), originally being a conjunction of two conditions, is in the theorem below replaced with a single condition only.

Theorem 4. *Let $\mathcal{M}$ be the general linear model of the form* (4.1). *Then the following conditions are equivalent:*

(i) $\mathcal{R}(\mathbf{X}\mathbf{X}'\mathbf{P_V}) \subseteq \mathcal{R}(\mathbf{P_V})$,

(ii) $\mathbf{X}\mathbf{X}'\mathbf{P_V} = \mathbf{P_V}\mathbf{X}\mathbf{X}'$,

(iii) $\mathcal{R}(\mathbf{X}'\mathbf{X}\mathbf{X}'\mathbf{P_V}) \subseteq \mathcal{R}(\mathbf{X}'\mathbf{P_V})$,

(iv) $\mathbf{P_X P_V} = \mathbf{P_V P_X}$ *and* $\mathcal{R}(\mathbf{X}\mathbf{X}'\mathbf{P}_{\mathcal{R}(\mathbf{X})\cap\mathcal{R}(\mathbf{V})}) \subseteq \mathcal{R}(\mathbf{P_V})$.

Proof. First observe that from (4.8) and (4.9) it follows that

$$\mathbf{X}\mathbf{X}'\mathbf{P_V} = \mathbf{U}\begin{pmatrix} \mathbf{\Omega}^2\mathbf{A} & \mathbf{\Omega}^2\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}', \tag{4.10}$$

$$\mathbf{P_V}\mathbf{X}\mathbf{X}' = \mathbf{U}\begin{pmatrix} \mathbf{A}\mathbf{\Omega}^2 & \mathbf{0} \\ \mathbf{B}'\mathbf{\Omega}^2 & \mathbf{0} \end{pmatrix}\mathbf{U}'. \tag{4.11}$$

Since $\mathbf{\Omega}$ is nonsingular, it is thus seen that condition (ii) holds if and only if $\mathbf{B} = \mathbf{0}$ and $\mathbf{\Omega}^2\mathbf{A} = \mathbf{A}\mathbf{\Omega}^2$. Clearly, by taking the square roots, the latter condition can be simplified to $\mathbf{\Omega}\mathbf{A} = \mathbf{A}\mathbf{\Omega}$.

On the other hand, condition (i) can be equivalently expressed as $\mathbf{P_V}\mathbf{X}\mathbf{X}'\mathbf{P_V} = \mathbf{X}\mathbf{X}'\mathbf{P_V}$, where

$$\mathbf{P_V}\mathbf{X}\mathbf{X}'\mathbf{P_V} = \mathbf{U}\begin{pmatrix} \mathbf{A}\mathbf{\Omega}^2\mathbf{A} & \mathbf{A}\mathbf{\Omega}^2\mathbf{B} \\ \mathbf{B}'\mathbf{\Omega}^2\mathbf{A} & \mathbf{B}'\mathbf{\Omega}^2\mathbf{B} \end{pmatrix}\mathbf{U}'. \tag{4.12}$$

In consequence, from (4.10) and (4.12) it is seen that condition (i) is satisfied if and only if $\mathbf{B} = \mathbf{0}$ and $\mathbf{A}\mathbf{\Omega}^2\mathbf{A} = \mathbf{\Omega}^2\mathbf{A}$. Taking the conjugate transposes on both sides of the latter condition gives $\mathbf{A}\mathbf{\Omega}^2\mathbf{A} = \mathbf{A}\mathbf{\Omega}^2$. Thus, $\mathbf{\Omega}^2\mathbf{A} = \mathbf{A}\mathbf{\Omega}^2$, i.e., $\mathbf{\Omega}\mathbf{A} = \mathbf{A}\mathbf{\Omega}$. Since, by Lemma 1(i), $\mathbf{B} = \mathbf{0}$ yields the idempotency of $\mathbf{A}$, the reverse implication is easily seen, and thus the equivalence (ii) $\Leftrightarrow$ (i) is established.

Further, condition (iii) can be expressed as $\mathbf{X}'\mathcal{R}(\mathbf{X}\mathbf{X}'\mathbf{P_V}) = \mathbf{X}'\mathcal{R}(\mathbf{P_V})$, or, alternatively,

$$\mathbf{X}'\mathbf{P}_{\mathcal{R}(\mathbf{X}\mathbf{X}'\mathbf{P_V})} = \mathbf{X}'\mathbf{P_V}. \tag{4.13}$$

In view of Lemma 1(i), applying formula $\mathbf{M}^\dagger = \mathbf{M}'(\mathbf{M}\mathbf{M}')^\dagger$ for any real matrix $\mathbf{M}$ (see e.g., Example 18 in Ben-Israel and Greville [4, Chapter 1]) to (4.10), gives

$$(\mathbf{X}\mathbf{X}'\mathbf{P_V})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{A}\mathbf{\Omega}^2(\mathbf{\Omega}^2\mathbf{A}\mathbf{\Omega}^2)^\dagger & \mathbf{0} \\ \mathbf{B}'\mathbf{\Omega}^2(\mathbf{\Omega}^2\mathbf{A}\mathbf{\Omega}^2)^\dagger & \mathbf{0} \end{pmatrix}\mathbf{U}'.$$

Hence, again referring to Lemma 1(i), we get

$$\mathbf{P}_{\mathcal{R}(\mathbf{XX'P_V})} = \mathbf{U}\begin{pmatrix} \mathbf{\Omega}^2\mathbf{A}\mathbf{\Omega}^2(\mathbf{\Omega}^2\mathbf{A}\mathbf{\Omega}^2)^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}'. \tag{4.14}$$

In consequence, using (4.7), (4.9), and (4.14), we can rewrite condition (4.13) in the form

$$\mathbf{U}_1\begin{pmatrix} \mathbf{\Omega}^3\mathbf{A}\mathbf{\Omega}^2(\mathbf{\Omega}^2\mathbf{A}\mathbf{\Omega}^2)^\dagger & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}' = \mathbf{U}_1\begin{pmatrix} \mathbf{\Omega}\mathbf{A} & \mathbf{\Omega}\mathbf{B} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}'. \tag{4.15}$$

It is seen that (4.15) is satisfied if and only if $\mathbf{B} = \mathbf{0}$ and $\mathbf{\Omega}^2\mathbf{A}\mathbf{\Omega}^2(\mathbf{\Omega}^2\mathbf{A}\mathbf{\Omega}^2)^\dagger = \mathbf{A}$. Since the latter of these conditions means that $\mathbf{A}$ is the orthogonal projector onto $\mathcal{R}(\mathbf{\Omega}^2\mathbf{A})$, it follows that $\mathbf{A}\mathbf{\Omega}^2\mathbf{A} = \mathbf{\Omega}^2\mathbf{A}$. Hence, we arrive at $\mathbf{B} = \mathbf{0}$ and $\mathbf{\Omega}\mathbf{A} = \mathbf{A}\mathbf{\Omega}$. To establish the converse implication, first note that on account of $\mathbf{B} = \mathbf{0}$, Lemma 1(i) combined with $\mathbf{A}' = \mathbf{A}$ ensures that $\mathbf{A}^\dagger = \mathbf{A}$. Hence, from Example 22 in Ben-Israel and Greville [4, Chapter 4] it follows that $\mathbf{\Omega}\mathbf{A} = \mathbf{A}\mathbf{\Omega}$ ensures $(\mathbf{\Omega}^2\mathbf{A}\mathbf{\Omega}^2)^\dagger = \mathbf{\Omega}^{-2}\mathbf{A}\mathbf{\Omega}^{-2}$, and, thus, identity (4.15) is clearly fulfilled.

For the proof referring to condition (iv), observe that from (4.7) it follows that

$$\mathbf{P}_{\mathcal{R}(\mathbf{X})} = \mathbf{U}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}'.$$

Hence, on the one hand, $\mathbf{P_X P_V} = \mathbf{P_V P_X} \Leftrightarrow \mathbf{B} = \mathbf{0}$, and, on the other hand, it is seen that $\mathbf{P}_{\mathcal{R}(\mathbf{X})\cap\mathcal{R}(\mathbf{V})}$ is of the same form as $\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap\mathcal{R}(\mathbf{Q})}$ given in Lemma 5(i). In consequence, condition (iv), being satisfied if and only if $\mathbf{P_X P_V} = \mathbf{P_V P_X}$ and $\mathbf{P_V XX'P}_{\mathcal{R}(\mathbf{X})\cap\mathcal{R}(\mathbf{V})} = \mathbf{XX'P}_{\mathcal{R}(\mathbf{X})\cap\mathcal{R}(\mathbf{V})}$, can be equivalently expressed as $\mathbf{B} = \mathbf{0}$ and $\mathbf{A}\mathbf{\Omega}^2\mathbf{Q}_{\overline{\mathbf{A}}} = \mathbf{\Omega}^2\mathbf{Q}_{\overline{\mathbf{A}}}$. In view of Lemma 1(i), the former of these conditions ensures that the latter can be simplified to $\mathbf{A}\mathbf{\Omega}^2\mathbf{A} = \mathbf{\Omega}^2\mathbf{A}$, so it is seen that (iv) is equivalent to $\mathbf{B} = \mathbf{0}$ and $\mathbf{\Omega}\mathbf{A} = \mathbf{A}\mathbf{\Omega}$. The proof is complete. ■

It is worth emphasizing that condition (d) in Theorem 4 of Baksalary [2] constitute the conjunction $\mathbf{P_X P_V} = \mathbf{P_V P_X}$ and $\mathcal{R}(\mathbf{X'XX'V}) = \mathcal{R}(\mathbf{X'V})$, whereas the corresponding condition (iii) in Theorem 4 above is void of the commutativity condition.

The next theorem corresponds to part (A1) $\Leftrightarrow$ (S4) of Theorem 5 in Baksalary [2].

Theorem 5. *Let $\mathcal{M}$ be the general linear model of the form* (4.1). *Then the following statements are equivalent:*

(i) $\mathbf{P_X P_V} = \mathbf{P_V P_X}$,

(ii) *there are no unit canonical correlations between* $\mathbf{P_X y}$ *and* $\mathbf{Q_X y}$.

Proof. It was mentioned in Baksalary [2, p. 123] that $t_1 = \mathrm{rk}(\mathbf{VX}) - \dim[\mathcal{R}(\mathbf{X}) \cap \mathcal{R}(\mathbf{V})]$ is the number of the unit canonical correlations between $\mathbf{P_X y}$ and $\mathbf{Q_X y}$. Hence, using (1.6), it follows that $t_1 = \mathrm{rk}(\mathbf{P_V P_X}) - \dim[\mathcal{R}(\mathbf{P_X}) \cap \mathcal{R}(\mathbf{P_V})]$. Setting $\mathbf{P_V} = \mathbf{P}$ and $\mathbf{P_X} = \mathbf{Q}$, with matrices on the right-hand sides being of the forms (1.2) and (1.3), respectively, by Lemma 5(i) and Lemma 6(i) we get $t_1 = \mathrm{rk}(\mathbf{A}) - [\mathrm{rk}(\mathbf{A}) - \mathrm{rk}(\mathbf{B})]$. Thus, $t_1 = 0$ is equivalent to $\mathrm{rk}(\mathbf{B}) = 0$, i.e., $\mathbf{P_X P_V} = \mathbf{P_V P_X}$. ■

5. Further characterizations of commutativity

There are many further, alternative, characterizations of the commutativity of a pair of orthogonal projectors. In the subsequent two theorems involving orthogonal projectors $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$, we provide conditions equivalent to $\mathbf{PQ} = \mathbf{QP}$ expressed, respectively, in terms of the ranks and ranges.

Theorem 6. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be the orthogonal projectors. Then the following conditions are equivalent:*

(i) $\mathbf{PQ} = \mathbf{QP}$,

(ii) $\mathrm{rk}(\mathbf{I}_n - \mathbf{PQ}) + \mathrm{rk}(\mathbf{PQ}) = n$,

(iii) $\mathrm{rk}(\mathbf{P\overline{Q}}) = \mathrm{rk}(\mathbf{P}) - \mathrm{rk}(\mathbf{PQ})$,

(iv) $\mathrm{rk}(\mathbf{P} - \mathbf{Q}) = \mathrm{rk}(\mathbf{P} + \mathbf{Q}) - \mathrm{rk}(\mathbf{PQ})$,

(v) $\mathrm{rk}(\mathbf{PQ} + \mathbf{QP}) = \mathrm{rk}(\mathbf{PQ})$.

Proof. The theorem is derived straightforwardly from Lemma 6. ■

The last theorem provides some results involving the ranges of functions of $\mathbf{P}$ and $\mathbf{Q}$.

Theorem 7. *Let* $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ *be the orthogonal projectors. Then the following conditions are equivalent:*

(i) $\mathbf{PQ} = \mathbf{QP}$,

(ii) $\mathcal{R}(\mathbf{PQ}) \cap \mathcal{R}(\mathbf{P} - \mathbf{Q}) = \{\mathbf{0}\}$,

(iii) $\mathcal{R}(\mathbf{PQ}) \subseteq \mathcal{R}(\mathbf{QP})$,

(iv) $\mathcal{N}(\mathbf{PQ}) \subseteq \mathcal{N}(\mathbf{QP})$,

(v) $[\mathcal{R}(\mathbf{P}) + \mathcal{N}(\mathbf{Q})] \cap \mathcal{R}(\mathbf{Q}) = \mathcal{R}(\mathbf{P}) \cap [\mathcal{N}(\mathbf{P}) + \mathcal{R}(\mathbf{Q})]$,

(vi) $\mathcal{R}(\mathbf{P}\overline{\mathbf{Q}})$ *and* $\mathcal{R}(\mathbf{Q}\overline{\mathbf{P}})$ *are orthogonal.*

Proof. We show that each of conditions (ii)–(vi) is equivalent to $\mathbf{B} = \mathbf{0}$. First note that direct calculations with the use of conditions (i), (v), (vi) of Lemma 1, and already mentioned identity $\mathbf{P_D} = \mathbf{B}^*\overline{\mathbf{A}}^\dagger\mathbf{B} + \mathbf{D}$, lead to the conclusion that the Moore-Penrose inverse of

$$\mathbf{P} - \mathbf{Q} = \mathbf{U}\begin{pmatrix} \overline{\mathbf{A}} & -\mathbf{B} \\ -\mathbf{B}^* & -\mathbf{D} \end{pmatrix}\mathbf{U}^*$$

is of the form

$$(\mathbf{P} - \mathbf{Q})^\dagger = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & -\overline{\mathbf{A}}^\dagger\mathbf{B} \\ -\mathbf{B}^*\overline{\mathbf{A}}^\dagger & -\mathbf{P_D} \end{pmatrix}\mathbf{U}^*.$$

Hence,

$$\mathbf{P}_{\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\overline{\mathbf{A}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{P_D} \end{pmatrix}\mathbf{U}^*, \tag{5.1}$$

and applying Lemma 3(ii) to (2.6) and (5.1) gives, after some rearrangements,

$$\mathbf{P}_{\mathcal{R}(\mathbf{PQ})\cap\mathcal{R}(\mathbf{P}-\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{P_B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*.$$

Thus, the equivalence (ii) $\Leftrightarrow$ (i) is seen.

The proofs corresponding to the next two conditions are based on the facts that (iii) is equivalent to $\mathbf{QP}(\mathbf{QP})^{\dagger}\mathbf{PQ} = \mathbf{PQ}$, whereas (iv) to $\mathbf{QP}(\mathbf{PQ})^{\dagger}\mathbf{PQ} = \mathbf{QP}$. Hence, the assertions follows straightforwardly with the use of (1.2), (1.3), (2.5), Lemma 1(i), and Lemma 1(iv).

Next, using Lemma 4(ii) and Lemma 4(iii), from Lemma 3(ii) we get

$$\mathbf{P}_{[\mathcal{R}(\mathbf{P})+\mathcal{N}(\mathbf{Q})]\cap\mathcal{R}(\mathbf{Q})} = \mathbf{U}\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{P}_{\overline{\mathbf{D}}} - \overline{\mathbf{D}} \end{pmatrix}\mathbf{U}^*,$$

$$\mathbf{P}_{\mathcal{R}(\mathbf{P})\cap[\mathcal{N}(\mathbf{P})+\mathcal{R}(\mathbf{Q})]} = \mathbf{U}\begin{pmatrix} \mathbf{P}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^*,$$

from where part (v) $\Leftrightarrow$ (i) follows.

The proof is concluded with the observation that the equivalence (vi) $\Leftrightarrow$ (i) is established by direct calculations. ∎

Many further necessary and sufficient conditions for the commutativity of the orthogonal projectors $\mathbf{P}, \mathbf{Q} \in \mathbb{C}_{n,n}$ can be formulated, including the ones referring to the eigenvalues. For instance, it is known that all eigenvalues of $\mathbf{PQ}$ belong to the set $[0,1]$; see e.g., Lemma 2 in Anderson *et al.* [1]. However, the projectors $\mathbf{P}$ and $\mathbf{Q}$ commute if and only if $\mathbf{PQ}$ has no eigenvalues belonging to the set $(0,1)$. Related results of this type will be reported in a forthcoming paper.

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