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GENERALIZED *F* TESTS IN MODELS WITH RANDOM PERTURBATIONS: THE GAMMA CASE

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Abstract

Generalized F tests were introduced for linear models by Michalski and Zmyślony (1996, 1999). When the observations are taken in not perfectly standardized conditions the F tests have generalized F distributions with random non-centrality parameters, see Nunes and Mexia (2006). We now study the case of nearly normal perturbations leading to Gamma distributed non-centrality parameters.

Keywords: generalized F distributions; random non-centrality parameters; Gamma distribution.

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1. Introduction

The statistics of the generalized F tests are the quotients of linear combinations of independent chi-squares. These tests were introduced by Michalski and Zmyślony (1996, 1999), first for variance components and later for linear combinations of parameters in mixed linear models.

These tests derived when we have a quadratic unbiased estimator $\widetilde{\theta}$ for a parameter θ and we want to test

$$H_0 : \theta = 0$$

against

$$H_1 : \theta > 0.$$

If $\widetilde{\theta}^+$ and $\widetilde{\theta}^-$ are, respectively, the positive and the negative parts of $\widetilde{\theta}$, when H_0 $[H_1]$ holds we have $E(\widetilde{\theta}^+) = E(\widetilde{\theta}^-)$ $[E(\widetilde{\theta}^+) > E(\widetilde{\theta}^-)]$. Thus, we are led to use the test statistic

$$\Im = \frac{\widetilde{\theta}^+}{\widetilde{\theta}^-}.$$

The following example shows the importance of these tests. In a balanced variance components model in which a first factor crosses with the second that nests a third, the variance component associated with the second factor is not the difference between two ANOVA mean squares, see Khuri *et al.* (1998). Thus, an usual F test cannot be derived for the nullity of this variance component. This problem is solved using generalized F tests. We can find a solution for this case, with a practical application of interest, in Fonseca *et al.* (2003b).

An exact expression for the distribution of quotients of linear combinations of independent central chi-squares was obtained in Fonseca *et al.* (2002), when the chi-squares, in the numerator or in the denominator, have even degrees of freedom and all coefficients are non-negative. This result was extended to the non-central case in Nunes and Mexia (2006). On carrying out this extension there were used the Robbins (1948) and Robbins and Pitman (1949) mixtures method for fixed non-centrality parameters.

When the vector of observations is the sum of a vector corresponding to the theoretical model plus an independent perturbation vector, the distribution of the generalized F statistics has, see Nunes and Mexia (2006), random non-centrality parameters. This kind of model perturbation is worthwhile to study since it would cover situations in which the collection of the observations was made on non standardized conditions. If we assume that the fluctuations in the observation conditions are approximately normal the non-centrality parameters would tend to be Gamma distributed. So, we decided to study this case.

Our aim is essentially theoretical. An alternative for our treatment, if practical applications are the main goal, is given by Imhof (1961). We can also use the algorithm presented by Davies (1980). This way, the previous approaches, such as the ones given by Satterthwaite (1946) and Gaylor and Hopper (1969), may be improved.

This article is organized in the following way. In Section 2 the central generalized F distributions and some particular cases are presented. Section 3 presents the non-central case of these distributions. This section is divided in tree Subsections. 3.1 is devoted to the case of random non-centrality parameters. The expressions of the distributions where the non-centrality parameters have Gamma distribution for the non-generalized case are obtained in 3.2. Finally 3.3 deals with the results for the generalized case.

2. Generalized F and related distributions

Let a_1^r and a_2^s be the vectors with non-negative components and being at least one of them not null. Consider also the independent random variables $U_i \sim \chi^2_{g_{1,i}}$, $i = 1, ..., r$, and $V_j \sim \chi^2_{g_{2,j}}$, $j = 1, ..., s$, the distribution of

$$\frac{\sum_{i=1}^{r} a_{1,i} U_i}{\sum_{j=1}^{s} a_{2,j} V_j} \text{ will be } F^+(z|a_1^r, a_2^s, g_1^r, g_2^s).$$

Let consider some particular cases of these distributions. With $(v^m)^{-1}$ the vector whose components are the inverses of the components of v^m, the central generalized F distribution will be

$$F(z|g_1^r, g_2^s) = F^+(z|(g_1^r)^{-1}, (g_2^s)^{-1}, g_1^r, g_2^s).$$

Another interesting case of $F^+(z|a_1^r, a_2^s, g_1^r, g_2^s)$ will be

$$\overline{F}(z|g_1^r, g_2^s) = F^+(z|(1^r, 1^s, g_1^r, g_2^s).$$

If $r = s = 1$, in the first case one will have the usual central F distribution with g_1 and g_2 degrees of freedom, $F(z|g_1, g_2)$, while for the second case one will have the $\overline{F}$ distribution, defined for the quotient of independent central chi-squares with g_1 and g_2 degrees of freedom, $\overline{F}(z|g_1, g_2)$.

In Fonseca *et al.* (2002) the exact expressions of $F^+(z|a_1^r, a_2^s, g_1^r, g_2^s)$ are given when the degrees of freedom in the numerator or in the denominator are even. Moreover, the second case reduces to the first one since

$$F^+(z|a_1^r, a_2^s, g_1^r, 2m^s) = 1 - F^+(z^{-1}|a_2^s, a_1^r, 2m^s, g_1^r).$$

An example to show how these expressions may be used to check the precision of Monte-Carlo methods in tabling such distributions may be seen in Fonseca *et al.* (2002).

3. Non-central generalized F distributions

The exact expression of

$$F^+(z|1, a_2^s, g_1, g_2^s, \delta) = e^{-\delta/2} \sum_{\ell=0}^{+\infty} \frac{(\frac{\delta}{2})^\ell}{\ell!} F^+(z|1, a_2^s, g_1 + 2\ell, g_2^s),$$

which is the distribution of

$$\frac{\chi^2_{g_1,\delta}}{\sum_{i=2}^{s+1} a_i \chi^2_{g_i}},$$

was obtained in Nunes and Mexia (2006) when g_1 is even.

Distributions $\chi^2_{g,\delta}$ are a mixture of the distributions χ^2_{g+2j}, $j = 0, \ldots$ The coefficients in this mixture are the probabilities for non-negative integers of the Poisson distribution with parameter $\frac{\delta}{2}$, $P_{\delta/2}$. Thus, if $U \sim \chi^2_{g,\delta}$, it can be assumed that there is an indicator variable $J \sim P_{\delta/2}$ such that $U \sim \chi^2_{g+2\ell}$, when $J = \ell$, $\ell = 0, \ldots$

If the $U_i \sim \chi^2_{g_{1,i},\delta_{1,i}}$, $i = 1,...,r$, and $V_j \sim \chi^2_{g_{2,j},\delta_{2,j}}$, $j = 1,...,s$, are independent, their joint distribution

$$\chi^2_{g_1^r,g_2^s,\delta_1^r,\delta_2^s} = \prod_{i=1}^{r} \chi^2_{g_{1,i},\delta_{1,i}} \prod_{j=1}^{s} \chi^2_{g_{2,j},\delta_{2,j}}$$

will be a mixture with coefficients

$$c(\ell_1^r,\ell_2^s,\delta_1^r,\delta_2^s) = \prod_{i=1}^{r} e^{-\frac{\delta_{1,i}}{2}} \frac{(\frac{\delta_{1,i}}{2})^{\ell_{1,i}}}{\ell_{1,i}!} \prod_{j=1}^{s} e^{-\frac{\delta_{2,j}}{2}} \frac{(\frac{\delta_{2,j}}{2})^{\ell_{2,j}}}{\ell_{2,j}!} \tag{3.1}$$

of the

$$\chi^2_{g_1^r+2\ell_1^r,g_2^s+2\ell_2^s} = \prod_{i=1}^{r} \chi^2_{g_{1,i}+2\ell_{1,i}} \prod_{j=1}^{s} \chi^2_{g_{2,j}+2\ell_{2,j}}.$$

Using the mixtures method, see Robbins (1948) and Robbins and Pitman (1949), the distribution of

$$Z = \frac{\sum_{i=1}^{r} a_{1,i}U_i}{\sum_{j=1}^{s} a_{2,j}V_j}$$

will be

$$\begin{aligned} & F^+(z|a_1^r,a_2^s,g_1^r,g_2^s,\delta_1^r,\delta_2^s) \\ & = \sum_{\ell_{1,1}=0}^{+\infty} \cdots \sum_{\ell_{1,r}=0}^{+\infty} \sum_{\ell_{2,1}=0}^{+\infty} \cdots \sum_{\ell_{2,s}=0}^{+\infty} c(\ell_1^r,\ell_2^s,\delta_1^r,\delta_2^s) \\ & \quad F^+(z|a_1^r,a_2^s,g_1^r+2\ell_1^r,g_2^s+2\ell_2^s). \end{aligned} \tag{3.2}$$

Likewise, if indicator variables are considered, the conditional distribution of Z, when $J_{1,i} = \ell_{1,i}$, $i = 1,...,r$ and $J_{2,j} = \ell_{2,j}$, $j = 1,...,s$, will be $F^+(z|a_1^r,a_2^s,g_1^r+2\ell_1^r,g_2^s+2\ell_2^s)$. Thus, the expression of $F^+(z|a_1^r,a_2^s,g_1^r,g_2^s,$ $\delta_1^r,\delta_2^s)$ can be obtained desconditioning in order to the indicator variables.

Let consider now monotonicity properties for these distributions. With $\delta_{1,p}$ the p-th component of δ_1^r, there will be

$$\begin{aligned}(3.3)\qquad &\frac{\partial F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)}{\partial \delta_{1,p}}\\ &= \frac{F^+(z|a_1^r, a_2^s, g_1^r + 2q_p^r, g_2^s, \delta_1^r, \delta_2^s) - F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)}{2} < 0,\end{aligned}$$

as well as

$$\begin{aligned}(3.4)\qquad &\frac{\partial F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)}{\partial \delta_{2,h}}\\ &= \frac{F^+(z|a_1^r, a_2^s, g_1^r, g_2^s + 2q_h^s, \delta_1^r, \delta_2^s) - F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \delta_1^r, \delta_2^s)}{2} > 0,\end{aligned}$$

where q_p^r has all components null, except the p-th that is equal to 1.

The non-generalized case will be used to justify (3.3) and (3.4). With the independent chi-squares χ_2^2, $\chi_{m,\delta}^2$ and $\chi_{n,\delta'}^2$, there will be

$$(3.5)\qquad pr\left(\frac{\chi_{m,\delta}^2}{\chi_{n,\delta'}^2 + \chi_2^2} < \frac{\chi_{m,\delta}^2}{\chi_{n,\delta'}^2} < \frac{\chi_{m,\delta}^2 + \chi_2^2}{\chi_{n,\delta'}^2}\right) = 1,$$

so

$$(3.6)\qquad \overline{F}(z|m+2, n, \delta, \delta') < \overline{F}(z|m, n, \delta, \delta') < \overline{F}(z|m, n+2, \delta, \delta'),$$

with

$$\begin{cases} \dfrac{\chi_{m,\delta}^2 + \chi_2^2}{\chi_{n,\delta'}^2} \sim \overline{F}(z|m+2, n, \delta, \delta') \\[2ex] \dfrac{\chi_{m,\delta}^2}{\chi_{n,\delta'}^2} \sim \overline{F}(z|m, n, \delta, \delta') \\[2ex] \dfrac{\chi_{m,\delta}^2}{\chi_{n,\delta'}^2 + \chi_2^2} \sim \overline{F}(z|m, n+2, \delta, \delta') \end{cases}.$$

3.1. Random non-centrality parameters

So far we have considered the indicator variables $J_{1,i}$, $i = 1, ..., r$, and $J_{2,j}$, $j = 1, ..., s$, to have Poisson distributions with fixed parameters. Let now assume these parameters to be random variables.

Remark. To understand the "appearance" of randomized non-centrality parameters we point out that if the error vector e^n has normal distribution with null mean vector and variance-covariance matrix $\sigma^2 I_g$, $e^n \sim N(0^n, \sigma^2 I_g)$, with I_g the $g \times g$ identity matrix, one will have $\|e^n\|^2 \sim \sigma^2 \chi^2_g$. With μ^n the mean vector of the observations vector, $\|e^n + \mu^n\|^2 \sim \sigma^2 \chi^2_{g,\delta}$, with the non-centrality parameter $\delta = \frac{1}{\sigma^2}\|\mu^n\|^2$. Let consider a random perturbation vector of the model, W^n, independent of e^n. The conditional distribution of $\|e^n + W^n\|^2$, given $W^n = w^n$, will be $\sigma^2 \chi^2_{g,\delta(w)}$, with $\delta(w) = \frac{1}{\sigma^2}\|w^n\|^2$. Then, desconditioning in order to W^n, we obtain a chi-square with g degrees of freedom and random non-centrality parameters. In mixed models, see for example Khuri *et al.* (1998), Fonseca *et al.* (2003a) and Nunes *et al.* (2006), the F and generalized F tests are quotients of squares of norms of vectors or of linear combinations of such squares. These squares may happen to have random non-centrality parameters when, in the expression, a random perturbation vector W^n occurs.

Let consider now the random variables $L_{1,i}$, $i = 1, ..., r$ and $L_{2,j}$, $j = 1, ..., s$, with $\lambda_{L_1^r, L_2^s}(t_1^r, t_2^s)$ the joint moment generating function for these variables and

$$\lambda^{<\ell_1^r, \ell_2^s>}_{L_1^r, L_2^s}(t_1^r, t_2^s) = \frac{\partial^{\sum_{i=1}^r \ell_{1,i} + \sum_{j=1}^s \ell_{2,j}} \lambda_{L_1^r, L_2^s}(t_1^r, t_2^s)}{\prod_{i=1}^r \partial t_{1,i}^{\ell_{1,i}} \prod_{j=1}^s \partial t_{2,j}^{\ell_{2,j}}}. \tag{3.7}$$

Desconditioning

$$\begin{aligned} & F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, l_1^r, l_2^s) \\ = & \sum_{\ell_{1,1}=0}^{+\infty} \cdots \sum_{\ell_{1,r}=0}^{+\infty} \sum_{\ell_{2,1}=0}^{+\infty} \cdots \sum_{\ell_{2,s}=0}^{+\infty} c(\ell_1^r, \ell_2^s, l_1^r, l_2^s) \\ & F^+(z|a_1^r, a_2^s, g_1^r + 2\ell_1^r, g_2^s + 2\ell_2^s) \end{aligned} \tag{3.8}$$

in order to the random parameters vectors L_1^r and L_2^s, we will have

$$F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r,L_2^s}) = \sum_{\ell_{1,1}=0}^{+\infty} \dots \sum_{\ell_{1,r}=0}^{+\infty} \sum_{\ell_{2,1}=0}^{+\infty} \dots \sum_{\ell_{2,s}=0}^{+\infty}$$

$$\times \frac{\lambda_{L_1^r,L_2^s}^{<\ell_1^r,\ell_2^s>} \left(-\frac{1}{-} \qquad - \right.}{\prod_{i=1}^{r} \ell_{1,i}!2^{\ell_{1,i}} \prod_{j=1}^{s} \ell_{2,j}!2^{\ell_{2,j}}} F^+(z|a_1^r, a_2^s, g_1^r + 2\ell_1^r, g_2^s + 2\ell_2^s). \tag{3.9}$$

With q_i^r $[q_j^s]$ be the vector with all r $[s]$ components null, except the i-th [j-th] which is 1, all components of $\overline{L}_i^r = (1^r - q_i^r)L_1^r$ $[\overline{L}_j^s = (1^s - q_j^s)L_2^s]$ will be equal to the ones of L_1^r $[L_2^s]$ to exception of the i-th [j-th] that is null. From (3.3) and (3.4) it is easy to obtain

$$\begin{cases} F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{\overline{L}_i^r,L_2^s}) > F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r,L_2^s});\ i = 1, ..., r \\ \\ F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r,L_2^s}) > F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r,\overline{L}_j^s});\ j = 1, ..., s. \end{cases} \tag{3.10}$$

So, when one of the components of L_2^s $[L_1^r]$ is null, with probability 1, the values of $F^+(z|a_1^r, a_2^s, g_1^r, g_2^s, \lambda_{L_1^r,L_2^s})$ decrease [increase].

3.2. F distribution with non-centrality parameters with Gamma distribution

As it was previously seen, if $a_1^r = 1^r$ and $a_2^s = 1^s$, with $r = s = 1$ one will have the $\overline{F}$ distribution defined for the quotient of independent chi-squares with g_1 and g_2 degrees of freedom. So, (3.9) can be rewritten as

$$\overline{F}(z|g_1, g_2, \lambda_{L_1,L_2}) = \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} \frac{\lambda_{L_1,L_2}^{<i,j>} \left(-\frac{1}{-} \qquad - \right.}{2^{i+j} i! j!} \overline{F}(z|g_1 + 2i, g_2 + 2j). \tag{3.11}$$

Consider now L_1 with Gamma distribution with parameters n_1 and α_1, $L_1 \sim G(n_1, \alpha_1)$,

$$\lambda_{L_1}(t_1) = \left(\frac{\alpha_1}{\alpha_1 - t_1}\right)^{n_1}, \; t_1 < \alpha_1$$

and consequently

$$\lambda_{L_1}^{<i>}(t_1) = \frac{(n_1 + i - 1)!\alpha_1^{n_1}(\alpha_1 - t_1)^{-n_1-i}}{(n_1 - 1)!}. \tag{3.12}$$

If L_1 is independent of L_2, with $L_2 \sim G(n_2, \alpha_2)$, one will have

$$\begin{aligned} \lambda_{L_1,L_2}^{<i,j>}(t_1, t_2) &= \lambda_{L_1}^{<i>}(t_1)\lambda_{L_2}^{<j>}(t_2) \\ &= \frac{(n_1 + i - 1)!\alpha_1^{n_1}(\alpha_1 - t_1)^{-n_1-i}}{(n_1 - 1)!} \\ &\quad \frac{(n_2 + j - 1)!\alpha_2^{n_2}(\alpha_2 - t_2)^{-n_2-j}}{(n_2 - 1)!}, \end{aligned} \tag{3.13}$$

and (3.11) will be

$$\begin{aligned} &\overline{F}(z|g_1, g_2, \lambda_{L_1,L_2}) \\ &= \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \frac{\binom{n_1 + i - 1}{i}\binom{n_2 + j - 1}{j}\alpha_1^{n_1}\alpha_2^{n_2}}{2^{i+j}(\alpha_1 + \frac{1}{\;} \quad -} \\ &\quad \overline{F}(z|g_1 + 2i, g_2 + 2j). \end{aligned} \tag{3.14}$$

$)^{n_1+i}(\alpha_2 + \frac{1}{\;}$

Let consider a particular case of Gamma distribution. If $L_1 \sim \chi^2_{n_1}$ and $L_2 \sim \chi^2_{n_2}$ then $L_1 \sim G(\frac{n_1}{2}, \frac{1}{\ }$ $\frac{\ }{2}, \frac{1}{\ }$

$$\frac{\frac{\ }{2} + i - 1)!(\frac{n_2}{2} + j - 1)!\frac{1}{\ } \quad \frac{\ }{2} \quad \left(\frac{1}{\ } \quad {}^{\frac{\ }{2}-i} \left(\frac{1}{\ } \quad {}^{\frac{\ }{2}-j}}{(\frac{n_1}{2} - 1)!(\frac{n_2}{2} - 1)!}$$

and

$$\overline{F}(z|g_1, g_2, \lambda_{L_1,L_2}) = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} \frac{\binom{\frac{n_1}{2}+i-1}{i}\binom{\frac{n_2}{2}+j-1}{j}}{2^{\frac{n_1}{2}+i+\frac{n_2}{2}+j}} \overline{F}(z|g_1 + 2i, g_2 + 2j), \tag{3.16}$$

if L_1 and L_2 are independent.

3.3. Generalized F distribution with non-centrality parameters with Gamma distribution

Consider the generalized case and the independent random variables $L_1^r \sim G(n_1^r, \alpha_1^r)$, with $n_{1,1}, ..., n_{1,r}$ $[\alpha_{1,1}, ..., \alpha_{1,r}]$ the components of n_1^r $[\alpha_1^r]$ and $L_2^s \sim G(n_2^s, \alpha_2^s)$, with $n_{2,1}, ..., n_{2,s}$ $[\alpha_{2,1}, ..., \alpha_{2,s}]$ the components of n_2^s $[\alpha_2^s]$,

$$\lambda_{L_1^r}(t_1^r) = \prod_{i=1}^{r} \lambda_{L_{1,i}(t_i)} = \prod_{i=1}^{r}\left(\frac{\alpha_{1,i}}{\alpha_{1,i} - t_i}\right)^{n_{1,i}}, \; t_i < \alpha_{1,i}, \; i = 1, ..., r.$$

Consequently

$$\lambda_{L_1^r}^{<\ell_1^r>}(t_1^r) = \prod_{i=1}^{r} \frac{(\alpha_{1,i})^{n_{1,i}}(n_{1,i} + \ell_{1,i} - 1)!}{(n_{1,i} - 1)!(\alpha_{1,i} - t_i)^{n_{1,i}+\ell_{1,i}}} \tag{3.17}$$

and

$$\lambda_{L_1^r,L_2^s}^{<\ell_1^r,\ell_2^s>}(t_1^r,t_2^s) = \lambda_{L_1^r}^{<\ell_1^r>}(t_2^s)\lambda_{L_2^s}^{<\ell_2^s>}(t_2^s)$$

$$= \prod_{i=1}^{r} \frac{(\alpha_{1,i})^{n_{1,i}}(n_{1,i}+\ell_{1,i}-1)!}{(n_{1,i}-1)!(\alpha_{1,i}-t_i)^{n_{1,i}+\ell_{1,i}}} \tag{3.18}$$

$$\prod_{j=1}^{s} \frac{(\alpha_{2,j})^{n_{2,j}}(n_{2,j}+\ell_{2,j}-1)!}{(n_{2,j}-1)!(\alpha_{2,j}-t_j)^{n_{2,j}+\ell_{2,j}}}.$$

This way, (3.9) can be rewritten as

$$F^+\big(z|a_1^r,a_2^s,g_1^r,g_2^s,\lambda_{L_1^r,L_2^s}\big) = \sum_{\ell_{1,1}=0}^{+\infty} \dots \sum_{\ell_{1,r}=0}^{+\infty}\sum_{\ell_{2,1}=0}^{+\infty}\sum_{\ell_{2,s}=0}^{+\infty}$$

$$\times \frac{\prod_{i=1}^{r}\begin{pmatrix} n_{1,i}+\ell_{1,i}-1 \\ \ell_{1,i}\end{pmatrix}(\alpha_{1,i})^{n_{1,i}}\prod_{j=1}^{s}\begin{pmatrix} n_{2,j}+\ell_{2,j}-1 \\ \ell_{2,j}\end{pmatrix}(\alpha_{2,j})^{n_{2,j}}}{\prod_{i=1}^{r}2^{\ell_{1,i}}\Big(\alpha_{1,i}+\frac{1}{-} \qquad -} \tag{3.19}$$

$$\overline{\quad 2}, \frac{1^r}{2}\Big), L_2^s \sim G\Big(\frac{n_2^s 1^s}{2}, \frac{1^s}{2}\Big)$$

and there will be

$$)^{n_{2,j}+\ell_{2,j}} \prod^{s} \ell \Big(\quad 1$$

$$\lambda_{L_1^r,L_2^s}^{<\ell_1^r,\ell_2^s>}\left(t_1^r,t_2^s\right)$$

$$=\prod_{i=1}^{r}\frac{\left(\frac{1}{\ }\ \overline{\ \ }^{2}\right)\left(\frac{n_{1,i}}{2}+\ell_{1,i}-1\right)!}{\left(\frac{n_{1,i}}{2}-1\right)!\left(\frac{1}{\ }\ \ \overline{\ \ }_{2}+\ell_{1,i}\right.}\prod_{j=1}^{s}\frac{\left(\frac{1}{\ }\ \overline{\ \ }^{2}\right)\left(\frac{n_{2,j}}{2}+\ell_{2,j}-1\right)!}{\left(\frac{n_{2,j}}{2}-1\right)!\left(\frac{1}{\ }\ \ \overline{\ \ }_{2}+\ell_{2,j}\right.},$$

and

$$F^{+}(z|a_1^r,a_2^s,g_1^r,g_2^s,\lambda_{L_1^r,L_2^s})$$

$$=\sum_{\ell_{1,1}=0}^{+\infty}\cdots\sum_{\ell_{1,r}=0}^{+\infty}\sum_{\ell_{2,1}=0}^{+\infty}\sum_{\ell_{2,s}=0}^{+\infty}$$

$$\times\frac{\prod_{i=1}^{r}\binom{\frac{n_{1,i}}{2}+\ell_{1,i}-1}{\ell_{1,i}}\prod_{j=1}^{s}\binom{\frac{n_{2,j}}{2}+\ell_{2,j}-1}{\ell_{2,j}}}{\prod_{i=1}^{r}2^{\frac{n_{1,i}}{2}+\ell_{1,i}}\prod_{j=1}^{s}2^{\frac{n_{2,j}}{2}+\ell_{2,j}}} \tag{3.21}$$

$$\times F^{+}(z|a_1^r,a_2^s,g_1^r+2\ell_1^r,g_2^s+2\ell_2^s).$$

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$n_{2,j}$ $n_{2,ij}$

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