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GEOMETRICALLY STRICTLY SEMISTABLE LAWS AS THE LIMIT LAWS

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Abstract

A random variable X is geometrically infinitely divisible iff for every $p \in (0,1)$ there exists random variable X_p such that $X \stackrel{d}{=} \sum_{k=1}^{T(p)} X_{p,k}$, where $X_{p,k}$'s are i.i.d. copies of X_p, and random variable $T(p)$ independent of $\{X_{p,1}, X_{p,2}, \ldots\}$ has geometric distribution with the parameter p. In the paper we give some new characterization of geometrically infinitely divisible distribution. The main results concern geometrically strictly semistable distributions which form a subset of geometrically infinitely divisible distributions. We show that they are limit laws for random and deterministic sums of independent random variables.

Keywords: infinite divisibility, geometric infinite divisibility, geometric semistability, random sums, limit laws, characteristic function.

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1. Introduction

In many economic and physical phenomena we can often find a common feature, namely an observed quantity is a sum of very large amount of small summands which do not depend on each other. In such models an essential role play infinitely divisible, stable and semistable distributions.

In some problems we encounter with a situation that the number of summands is not deterministic, but rather random. Then a modeling with random sums is needed.

For example, consider a *System with Rapid Repair*, Gertsbach (1990). In this system, if the operating unit with a random lifetime X fails, it is immediately replaced by identical unit available with a probability $q = 1-p$ close to one. A lifetime of the system is then a random variable which is a random sum of the form

$$\sum_{k=1}^{T(p)} X_k,$$

where X_k's are i.i.d. copies of X, random variable $T(p)$ independent of X_k's has geometric distribution with a parameter p, i.e. $P(T(p) = n) = p(1-p)^{n-1}$, $n \in I\!N$.

These assumptions about summands X_k and random variable $T(p)$ we will assume throughout the paper.

If X has an exponential law we have that for every $p \in (0,1)$

$$\sum_{k=1}^{T(p)} X_k \stackrel{d}{=} \frac{1}{p} X,$$

where "$\stackrel{d}{=}$" denotes equality of distributions. So one can say that exponential distribution is invariant under geometric summation.

In this paper we consider these random variables X for which the following condition is satisfied

$$\exists p \in (0,1) \;\; \exists a > 0 \quad X \stackrel{d}{=} a \sum_{k=1}^{T(p)} X_k.$$

We will call them geometrically strictly semistable random variables.

Random variables X satisfying somewhat stronger condition, namely

$$\forall p \in (0,1) \;\; \exists a(p) > 0 \quad X \stackrel{d}{=} a(p) \sum_{k=1}^{T(p)} X_k,$$

are well characterized already, see Rachev and Samorodnitsky (1994). These random variables and their distributions we will call geometrically strictly stable. In Rachev and Samorodnitsky (1994) it is noticed that geometrically strictly stable distributions form a subset of so called geometrically infinitely divisible distributions. Indeed, since random variable X is geometrically infinitely divisible iff (see Rachev and Samorodnitsky, 1994)

$$\forall p \in (0,1) \;\; \exists X_p \quad X \stackrel{d}{=} \sum_{k=1}^{T(p)} X_{p,k},$$

the previous statement is obvious.

It is worth to indicate that geometrically infinitely divisible distributions are infinitely divisible in the classical sense (see Rachev and Samorodnitsky, 1994), so one can think about Lévy processes generated by such laws.

From now on we will use the following abbreviations and notations

r.v. – random variable,

ch.f. – characteristic function,

GID – geometrically infinitely divisible,

GSSe – geometrically strictly semistable,

ID – infinitely divisible,

SSe – strictly semistable,

Φ – a set of all characteristic functions,

$\mathbb{R}^+$ – a set $(0, +\infty)$,

$\mathbb{R}_0$ – a set $\mathbb{R} \setminus \{0\}$.

We will also use the convention that if X is GID (GSSe, ID) r.v., then also its ch.f. and its distribution will be called GID (GSSe, ID).

The paper is organized as follows: in Section 2 we give some remarks on GID distributions and we formulate the new characterization of GID distributions; in Section 3 we consider a subclass of GID distributions, namely GSSe distributions. We show there that GSSe distributions are the limit laws of weighted random sums of i.i.d. r.v.'s; and in Section 4, GSSe distributions are presented as the limit laws of deterministic sums of independent, but not necessarily identically distributed r.v.'s.

2. A new characterization of geometrically infinitely divisible random variable

The GID r.v.'s appeared as the answer to the question of V. M. Zolotarev who asked about such r.v.'s X for which the following condition is satisfied:

$$\forall p \in (0,1) \ \exists X_p \quad X \stackrel{d}{=} \epsilon_p X + X_p, \tag{1}$$

where ϵ_p, X, X_p are independent r.v.'s, and ϵ_p has distribution:

$$P(\epsilon_p = 0) = p, \ \ P(\epsilon_p = 1) = 1 - p.$$

It can be check that the condition (1) is equivalent with the following one

$$\forall p \in (0,1) \ \exists X_p \quad X \stackrel{d}{=} \sum_{k=1}^{T(p)} X_{p,k}, \tag{2}$$

which become the commonly accepted condition defining GID r.v. The pioneering note on GID r.v. is the paper of Klebanov *et al.* (1984). Since then the GID distributions gained in popularity. Very important result in this area is the one to one correspondence between GID and ID distributions. Namely, for ch.f. φ we have (see Klebanov *et al.*, 1984)

$$\varphi \text{ is GID} \quad \text{iff} \quad \exp\{1 - 1/\varphi\} \text{ is ID ch.f.}, \tag{3}$$

or equivalently

$$\varphi \text{ is ID} \quad \text{iff} \quad \frac{1}{1-\ln\varphi} \text{ is GID ch.f.} \tag{4}$$

In this section we want to present some results concerning GID distribution.

Let φ, φ_p denote ch.f.'s of X and X_p respectively. Since for the ch.f. of the random sum $\sum_{k=1}^{T(p)} X_{p,k}$ we have

$$\begin{aligned}
\mathbb{E}\exp\left\{it\sum_{k=1}^{T(p)} X_{p,k}\right\} &= \mathbb{E}\mathbb{E}\left(\exp\left\{it\sum_{k=1}^{T(p)} X_{p,k}\right\}\middle| T(p)\right) \\
&= \sum_{n=1}^{\infty} p(1-p)^{n-1}\mathbb{E}\left(\exp\left\{it\sum_{k=1}^{n} X_{p,k}\right\}\right) \\
&= \sum_{n=1}^{\infty} p(1-p)^{n-1}\varphi_p(t)^n \\
&= \frac{p\varphi_p(t)}{1-(1-p)\varphi_p(t)},
\end{aligned}$$

then the condition (2) can be rewritten as

$$\forall p\in(0,1)\ \ \exists\varphi_p\in\Phi\ \ \forall t\in\mathbb{R}\ \ \varphi(t)=\frac{p\varphi_p(t)}{1-(1-p)\varphi_p(t)}. \tag{5}$$

From (5) we see that for GID ch.f. $\varphi(t)$ and every $p\in(0,1)$ the function

$$\frac{\varphi(t)}{p+(1-p)\varphi(t)}$$

is generally a ch.f. Moreover it is the ch.f. of the r.v. X_p appearing in the condition (2). It is possible to infer more about this ch.f.

Proposition 1. *Let X be GID r.v. with ch.f. φ.*

(i) *The function*

$$\psi_a = \frac{\varphi}{a + (1-a)\varphi} \tag{6}$$

is also the GID ch.f. for every $a \geq 0$.

(ii) *For $a > 0$ function ψ_a is ch.f. of the r.v. $\mathcal{X}_{\Gamma(a)}$, where $\{\mathcal{X}_s, s \geq 0\}$ is a Lévy process such that $\mathcal{X}_1$ has ch.f. $\exp\{1 - 1/\varphi\}$, and r.v. $\Gamma(a)$ independent of $\{\mathcal{X}_s, s \geq 0\}$ has exponential distribution with mean equal to a.*

Moreover,

$$\mathcal{X}_{\Gamma(a)} \stackrel{d}{=} \begin{cases} X & \textit{for} \quad a = 1, \\ X_a & \textit{for} \quad a \in (0,1), \\ \displaystyle\sum_{k=1}^{T(1/a)} X_k & \textit{for} \quad a > 1, \end{cases}$$

where X_a is given by (2).

Proof.

(i) From (4) we can write $\varphi(t) = 1/(1 - \ln \psi(t))$, where ψ is the ch.f. of some ID distribution. Hence

$$\psi_a(t) = \frac{1}{1 - \ln(\psi(t)^a)}.$$

Since for $a \geq 0$ the function ψ^a is ID ch.f. then again by (4) we infer that ψ_a is GID.

(ii) It is known from subordination of Lévy processes (see Sato, 1999, pp. 197–198) that if $\{\mathcal{X}_s, s \geq 0\}$ is the Lévy process with $\mathcal{X}_1$ having ch.f. of the form $\mathbb{E}\exp\{it\mathcal{X}_1\} = e^{f(t)}$, and $\{\mathcal{Y}_s, s \geq 0\}$ is a subordinator (an increasing Lévy process) with Laplace transform $\mathbb{E}\exp\{-u\mathcal{Y}_s\} = e^{sg(-u)}$, $u \geq 0$, then the subordinated process $\{\mathcal{Z}_s = \mathcal{X}_{\mathcal{Y}_s}, s \geq 0\}$ is a Lévy process with ch.f. $\mathbb{E}\exp\{it\mathcal{Z}_s\} = e^{sg(f(t))}$.

In our case $\{\mathcal{X}_s, s \geq 0\}$ is the Lévy process with ID ch.f. $\mathbb{E}\exp\{it\mathcal{X}_1\} = e^{1-1/\varphi(t)}$, $\{\mathcal{Y}_s, s \geq 0\}$ is a Gamma process with $\mathbb{E}\exp\{-u\mathcal{Y}_s\} = \left(\frac{1}{1+au}\right)^s$ (see Sato, 1999, p. 203). Thus $\mathcal{Z}_s$ has ch.f.

$$\mathbb{E}\exp\{it\mathcal{Z}_s\} = \left(\frac{1}{1-a(1-1/\varphi(t))}\right)^s = \left(\frac{\varphi(t)}{a+(1-a)\varphi(t)}\right)^s.$$

Hence

$$\mathcal{Z}_1 = \mathcal{X}_{\mathcal{Y}_1} \stackrel{d}{=} \mathcal{X}_{\Gamma(a)}$$

since $\mathcal{Y}_1 \stackrel{d}{=} \Gamma(a)$.

For the second part of (ii) notice that

- if $a = 1$ then of course $\psi_a = \varphi$;

- if $a \in (0,1)$ then $\psi_a = \frac{\varphi}{a+(1-a)\varphi}$ is by (5) the ch.f. of r.v. X_p, for $p = a$, which appears in the condition (2) from the definition of geometric infinite divisibility of X;

- if $a > 1$ then

$$\psi_a = \frac{(1/a)\varphi}{1-(1-1/a)\varphi} = (1/a)\sum_{k=1}^{\infty}(1-1/a)^{k-1}\varphi^k$$

is the ch.f. of the r.v. $\displaystyle\sum_{k=1}^{T(1/a)} X_k$.

■

It can be shown that for $a > 0$ the opposite implication to this from Propostion 1 (i) holds, but the following weaker remark is also true.

Remark 1. *Let φ be a ch.f. If for some $a > 0$ the function $\psi_a = \varphi/(a + (1-a)\varphi)$ is GID ch.f., then the ch.f. φ is also GID.*

Proof. The proof is simple and will be omitted. ■

In the rest of this section we characterize GID distributions as limit distributions. To get this we will need the following lemma, and from now on we will assume that if $T(p) = 1$ then $\sum_{k=1}^{T(p)-1} X_k \equiv 0$.

Lemma 1. *For the r.v.'s Y_p, $p \in (0,1)$ we have*

$$\sum_{k=1}^{T(p)} Y_{p,k} \xrightarrow{d} Y, \text{ when } p \to 0 \iff \sum_{k=1}^{T(p)-1} Y_{p,k} \xrightarrow{d} Y, \text{ when } p \to 0,$$

where Y is some r.v., and "$\xrightarrow{d}$" denotes the convergence in distribution.

Proof. Let φ_p, ψ denote the ch.f.'s of Y_p and Y respectively. If $\sum_{k=1}^{T(p)} Y_{p,k} \xrightarrow{d} Y$, when $p \to 0$, then we can write

$$\frac{p\varphi_p(t)}{1-(1-p)\varphi_p(t)} \overset{p\to 0}{\longrightarrow} \psi(t) \ \ \forall t \in I\!R.$$

Since $p\varphi_p(t) \overset{p\to 0}{\longrightarrow} 0$ then the denominator of this fraction also has to tend to zero, thus $(1-p)\varphi_p(t) \overset{p\to 0}{\longrightarrow} 1$, and consequently $\varphi_p(t) \overset{p\to 0}{\longrightarrow} 1$. Hence

$$\psi(t) = \lim_{p\to 0} \frac{p\varphi_p(t)}{1-(1-p)\varphi_p(t)} = \lim_{p\to 0} \frac{p}{1-(1-p)\varphi_p(t)}.$$

It can be checked that $p/(1-(1-p)\varphi_p(t))$ is a ch.f. of random sum $\sum_{k=1}^{T(p)-1} Y_{p,k}$. The proof of the second implication is similar and will be omitted. ■

The next theorem shows that for characterization of the GID r.v. X we can use some weaker conditions than (2) or the one which is formulated in Theorem 2.2. (v) of Rachev and Samorodnitsky (1994), i.e.

$$\forall p \in (0,1) \;\; \exists X_p \;\; \sum_{k=1}^{T(p)} X_{p,k} \xrightarrow{d} X, \text{ when } p \to 0.$$

Theorem 1. *For a r.v. X the following conditions are equivalent:*

(a) *X is GID;*

(b) *for every sequence $\{p_n\} \subset (0,1)$ there exist r.v.'s Y_n, $n \in \mathbb{N}$, such that*

$$\sum_{k=1}^{T(p_n)} Y_{n,k} \xrightarrow{d} X, \text{ when } n \to \infty; \tag{7}$$

(c) *there exist a sequence $\{p_n\} \subset (0,1)$, $p_n \overset{n\to\infty}{\longrightarrow} 0$, and the r.v.'s Y_n, $n \in \mathbb{N}$, such that the convergence (7) holds.*

Proof.

(a)$\Rightarrow$ (b). Since X is GID then by (2) for every $p \in (0,1)$ there exists r.v. X_p such that $\sum_{k=1}^{T(p)} X_{p,k} \overset{d}{=} X$. Now it is to enough to define the r.v. Y_n as having the same distribution as X_{p_n} for each p_n from any chosen sequence $\{p_n\} \subset (0,1)$ and we have

$$\sum_{k=1}^{T(p_n)} Y_{n,k} \overset{d}{=} X \xrightarrow{d} X, \text{ when } n \to \infty.$$

The implication (b)$\Rightarrow$ (c) is trivial.

For the implication (c)⇒ (a) notice that from Lemma 1 we have

$$S_n = \sum_{k=1}^{T(p_n)-1} Y_{n,k} \xrightarrow{d} X, \text{ when } n \to \infty.$$

Let φ_n denotes the ch.f. of Y_n. Then

$$\psi_n = \frac{p_n}{1-(1-p_n)\varphi_n(t)}$$

is a ch.f. of S_n. Notice that for every $s \in (0,1)$

$$\gamma_s := \frac{\psi_n}{s+(1-s)\psi_n} = \frac{p_n/(p_n+s(1-p_n))}{1-(1-p_n/(p_n+s(1-p_n)))\varphi_n}$$

and it is a ch.f. of the r.v. $\sum_{k=1}^{T(r_n)-1} Y_{n,k}$, where $r_n = p_n/(p_n+s(1-p_n))$.

Since $\psi_n = s\gamma_s/(1-(1-s)\gamma_s)$, then ψ_n and S_n are GID. Now applying Theorem 2.2. (ii) of Rachev and Samorodnitsky (1994), which states that the set of GID r.v.'s is closed under convergence in distribution, we infer that X is GID. ■

3. The geometrically strictly semistable laws

In the paper of Lin (1994) one can find the informations on characterizing some distributions connected with geometric compound, i.e. with distribution of random sums, where the number of summands is geometrically distributed r.v. More precisely, we find there considerations on r.v.'s X for which the following condition is satisfied

(8) $$X \stackrel{d}{=} a \sum_{k=1}^{T(p)} X_k \quad \text{for some } p \in (0,1) \text{ and some real } a.$$

The general result on this problem states (see Lin, 1994, Theorem 3) that the r.v.'s X satisfying (8) have ch.f. φ of the form

$$\varphi(t) \equiv 1 \quad \text{or} \quad \varphi(t) = \begin{cases} (1 + |t|^\alpha h(t))^{-1} & \text{for} \quad t \in \mathbb{R}_0, \\ 1 & \text{for} \quad t = 0, \end{cases}$$

where $|a|^\alpha = p$, and h is complex-valued function such that $h(at) = h(t)$ for every $t \in \mathbb{R}_0$. Moreover, it was proved that if condition (8) holds with a such that $|a| \geq 1$, then $X = 0$ almost everywhere. It is mentioned that relation (8) practically means the invariance of a rarefaction of renewal process $\{S_n, n \in \mathbb{N}\}$, where $S_n = \sum_{k=1}^{n} X_k$.

In this paper we will interested in r.v.'s X for which

$$\exists p \in (0,1) \;\; \exists a \in (0,1) \quad X \stackrel{d}{=} a \sum_{k=1}^{T(p)} X_k, \tag{9}$$

and we will call them GSSe r.v.'s, although in the paper of Mohan *et al.* (1993) one can find these r.v.'s under the name of geometrically-right-semistable. In some places we write GSSe(p, a) as we want to indicate the numbers p and a from (9).

In view of Theorem 3.1 of Mohan *et al.* (1993), which states that ch.f. φ is geometrically-right-semistable iff $\exp\{1 - 1/\varphi\}$ is ch.f. of right-semistable distribution, we infer that

$$1/(1 + |t|^\alpha h(t)) \text{ is GSSe ch.f. iff } \exp\{-|t|^\alpha h(t)\} \text{ is SSe ch.f.}$$

The changes of the distribution names are done due to present state of semistable distribution theory (see Sato, 1999, Maejima, 2001). However, remembering that the first results on semistable distributions belong to P. Lévy we find out that $\alpha \in (0, 2]$, see Lévy (1937).

Until now the GSSe r.v.'s were considered only as r.v.'s satysfying the stability condition (9) (Mohan *et al.,* 1993, Lin, 1994). Our aim is to prove that GSSe distributions are limit laws.

Theorem 2. *For a r.v. X the following conditions are equivalent:*

(a) *X is GSSe;*

(b) *there exist $p \in (0,1)$ and $\{a_n\} \subset \mathbb{R}^+$ such that*

$$a_n \sum_{k=1}^{T(p^n)} X_k \xrightarrow{d} X, \quad \text{when } n \longrightarrow \infty; \tag{10}$$

(c) *there exist $p \in (0,1)$, $\{a_n\} \subset \mathbb{R}^+$ and a r.v. Y such that*

$$a_n \sum_{k=1}^{T(p^n)} Y_k \xrightarrow{d} X, \quad \text{when } n \longrightarrow \infty;$$

(d) *there exist a sequence $\{p_n\} \subset (0,1)$, $p_n \overset{n\to\infty}{\longrightarrow} 0$, $p_{n+1}/p_n \overset{n\to\infty}{\longrightarrow} p \in (0,1]$, a sequence $\{a_n\} \subset \mathbb{R}^+$ and a r.v. Y such that*

$$a_n \sum_{k=1}^{T(p_n)} Y_k \xrightarrow{d} X, \quad \text{when } n \longrightarrow \infty.$$

Moreover, if X is GSSe(p,a) then the constants a_n in (10) can be repleaced by

$$p^{n/\alpha}(1 + o(1)),$$

where $\alpha = \ln p / \ln a$.

Proof. Let φ, ψ denote the ch.f.'s of X and Y respectively.

(a)$\Rightarrow$(b). From the definition of GSSe r.v. X we note that for its ch.f. φ the following condition is satisfied

$$\varphi(t) = \frac{p\varphi(at)}{1-(1-p)\varphi(at)}, \quad t \in \mathbb{R},$$

for some $p, a \in (0,1)$. Consequently by the mathematical induction we obtain that

$$\varphi(t) = \frac{p^n\varphi(a^n t)}{1-(1-p^n)\varphi(a^n t)} \quad \text{for every } n \in \mathbb{N}.$$

This proves (10) with $a_n = a^n$.

The implications (b)⇒(c) and (c)⇒(d) are trivial.

(d)⇒(a). Since (d) holds then

$$\frac{p_n\psi(a_n t)}{1-(1-p_n)\psi(a_n t)} \stackrel{n\to\infty}{\longrightarrow} \varphi(t) \qquad \text{for every } t \in \mathbb{R}.$$

From Theorem 1 we infer that φ is GID and consequently ID ch.f., so it never takes the value zero. Hence

$$1 - \frac{1-(1-p_n)\psi(a_n t)}{p_n\psi(a_n t)} = \frac{\psi(a_n t)-1}{p_n\psi(a_n t)} \stackrel{n\to\infty}{\longrightarrow} 1 - \frac{1}{\varphi(t)}.$$

Since $p_n\psi(a_n t) \stackrel{n\to\infty}{\longrightarrow} 0$ then, in view of the previous convergence, $\psi(a_n t) \stackrel{n\to\infty}{\longrightarrow} 1$. Thus

$$p_n^{-1}(\psi(a_n t)-1) \stackrel{n\to\infty}{\longrightarrow} 1 - 1/\varphi(t)$$

and

$$[p_n^{-1}](\psi(a_n t) - 1) \stackrel{n\to\infty}{\longrightarrow} 1 - 1/\varphi(t),$$

where $[x]$ denotes the greatest integer number not greater than x.

Notice that

$$[p_n^{-1}] \ln \psi(a_n t) = [p_n^{-1}](\psi(a_n t) - 1)\left(1 + \frac{o(\psi(a_n t) - 1)}{\psi(a_n t) - 1}\right).$$

Hence

$$[p_n^{-1}] \ln \psi(a_n t) \stackrel{n\to\infty}{\longrightarrow} 1 - 1/\varphi(t),$$

and consequently

$$\psi(a_n t)^{[p_n^{-1}]} \stackrel{n\to\infty}{\longrightarrow} \exp\{1 - 1/\varphi(t)\}.$$

Because the function $\exp\{1 - 1/\varphi(t)\}$ is continuous at $t = 0$ and it is a limit of ch.f's sequence, then by Lévy–Cramer continuity theorem it is ch.f. of some distribution. Since $[p_n^{-1}]/[p_{n+1}^{-1}] \stackrel{n\to\infty}{\longrightarrow} p$ then limit function is ch.f. of some SSe distribution (see Maejima and Samorodnitsky, 1999). Now applying Theorem 3.1 of Mohan *et al.* (1993) we infer that φ is GSSe.

For the last statement notice that since X is GSSe(p, a), then $X \stackrel{d}{=} a\sum_{k=1}^{T(p)} X_k$ and its ch.f. φ is of the form $\varphi(t) = 1/(1 + |t|^\alpha h(t))$, where $\alpha = \ln p/\ln a$ and $h(t)$ is some complex-valued function with a property that $h(at) = h(t)$, $t \in I\!R_0$.

Notice that for the function h we have

$$h(p^{n/\alpha}(1 + o(1))t) = h(a^n(1 + o(1))t) = h((1 + o(1))t) \stackrel{n\to\infty}{\longrightarrow} h(t)$$

for every $t \in I\!R_0$.

Hence for the ch.f. γ_n of the sum $p^{n/\alpha}(1 + o(1))\sum_{k=1}^{T(p^n)} X_k$ we have

$$\begin{aligned}\gamma_n(t) &= \frac{p^n/\left(1+|p^{n/\alpha}(1+o(1))t|^\alpha h(p^{n/\alpha}(1+o(1))t)\right)}{1-(1-p^n)/\left(1+|p^{n/\alpha}(1+o(1))t|^\alpha h(p^{n/\alpha}(1+o(1))t)\right)} \\ &= \frac{1}{1+(1+o(1))^\alpha|t|^\alpha h(p^{n/\alpha}(1+o(1))t)} \overset{n\to\infty}{\longrightarrow} \frac{1}{1+|t|^\alpha h(t)}\end{aligned}$$

for every $t \in I\!R_0$. The convergence at the point $t = 0$ is obvious. ■

Theorem 3. *R.v. X is GSSe iff there exist a sequence $\{p_n\} \subset (0,1)$, $p_n \overset{n\to\infty}{\longrightarrow} 0$, sequence $\{a_n\} \subset I\!R^+$, $a_n \overset{n\to\infty}{\longrightarrow} 0$, $a_{n+1}/a_n \overset{n\to\infty}{\longrightarrow} a \in (0,1)$ and a r.v. Y such that*

$$a_n \sum_{k=1}^{T(p_n)} Y_k \overset{d}{\longrightarrow} X \text{ when } n \longrightarrow \infty. \tag{11}$$

Proof. If X is GSSe then (9) holds. Going similar as in the proof of implication (a)⇒(b) of previous theorem, we see that (11) holds with $p_n = p^n$, $a_n = a^n$ and Y such that $Y \overset{d}{=} X$.

For the opposite implication note that, going similar as in the proof of implication (d)⇒(a) of previous theorem, we have

$$\psi(a_n t)^{[p_n^{-1}]} \overset{n\to\infty}{\longrightarrow} \exp\{1 - 1/\varphi(t)\},$$

where φ, ψ are ch.f.'s of r.v.'s X and Y. This convergence implies that $\exp\{1 - 1/\varphi(t)\}$ is ID ch.f.

Denote $\gamma(t) = \exp\{1 - 1/\varphi(t)\}$. Since

$$\left(\psi\left(\frac{a_{n+1}}{a_n}\cdot a_n t\right)^{[p_n^{-1}]}\right)^{[p_{n+1}^{-1}]/[p_n^{-1}]} \stackrel{n\to\infty}{\longrightarrow} \gamma(t)$$

and

$$\psi\left(\frac{a_{n+1}}{a_n}\cdot a_n t\right)^{[p_n^{-1}]} \stackrel{n\to\infty}{\longrightarrow} \gamma(at)$$

we infer that $[p_{n+1}^{-1}]/[p_n^{-1}] \stackrel{n\to\infty}{\longrightarrow} p^{-1} > 0$ and

$$\gamma(at) = \gamma(t)^p. \tag{12}$$

Assuming that $p \geq 1$ we obtain

$$\gamma(t) = \gamma(a^n t)^{1/p^n} \stackrel{n\to\infty}{\longrightarrow} 1 \quad \text{for every } t \in I\!R,$$

it means $\gamma(t) \equiv 1$ and therefore $\varphi(t) \equiv 1$. Hence, for nontrivial case, we have $p \in (0,1)$ and (12) means that γ is SSe ch.f. Now by Theorem 3.1 of Mohan *et al.* (1993) we state that φ is GSSe ch.f. ∎

From the Theorem 1 and Theorem 2 one can see that GSSe distributions are GID. Consequently every GSSe distribution is ID.

4. Decomposability of GSSe laws

In this section, with a help of decomposability concept, we shall notice that GSSe r.v.'s are limits (in the sense of convergence in distribution) of not random, but deterministic sums of independent r.v.'s.

Let us remind, a r.v. X, its ch.f. and its distribution are decomposable, see Loève (1945), iff

(13) $$X \stackrel{d}{=} cX + X_c \quad \text{for some } c \in (0,1) \text{ and some r.v. } X_c,$$

X, X_c are independent. R.v. X is called then c-decomposable.

Proposition 2. *The GSSe(p,a) r.v.'s are a-decomposable.*

Proof. It is enough to notice that for r.v. X which is GSSe(p,a) we can write

$$X \stackrel{d}{=} aX + a \sum_{k=1}^{T(p)-1} X_k.$$

■

Loève (1945) in Theorem 4 stated that for $0 < c < 1$ the r.v. X is c-decomposable if and only if there exists r.v. Y such that

(14) $$X \stackrel{d}{=} \sum_{k=0}^{\infty} c^k Y_k,$$

He noticed also that Y has the same distribution as X_c in (13).

From Theorem 1 and Theorem 2 of Loève (1945) we find out that the class of c-decomposable laws coincides with the class of limit distributions for the sums

$$a_n \sum_{k=1}^{n} Z_k,$$

where $Z_1, Z_2, \ldots, Z_n$ are independent, but not necessarily identically distributed r.v.'s, and $\{a_n\}$ is a sequence of positive numbers such that

$$a_n \stackrel{n\to\infty}{\longrightarrow} 0, \qquad a_{n+1}/a_n \stackrel{n\to\infty}{\longrightarrow} c \in (0,1).$$

Moreover, Loève (1945) obtained the following characterization for the r.v. Y appearing in (14)

$$a_n Z_n \stackrel{d}{\longrightarrow} Y, \text{ when } n \to \infty.$$

Thus we have the following statement which this time characterizes GSSe r.v.'s as limits of nonrandom sums of independent, but not necessarily identically distributed r.v.'s.

Proposition 3. *Let* $p, a \in (0,1)$. *A r.v.* X *is* $GSSe(p,a)$ *iff there exist a sequence* $\{a_n\} \subset \mathbb{R}^+$, $a_n \stackrel{n\to\infty}{\longrightarrow} 0$, $a_{n+1}/a_n \stackrel{n\to\infty}{\longrightarrow} a$, *and independent r.v.'s* $Z_1, Z_2, \ldots$ *such that*

$$a_n \sum_{k=1}^{n} Z_k \xrightarrow{d} X, \text{ when } n \to \infty,$$

and

$$a_n Z_n \xrightarrow{d} a \sum_{k=1}^{T(p)-1} X_k, \text{ when } n \to \infty.$$

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