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INFERENCE FOR RANDOM EFFECTS IN PRIME BASIS FACTORIALS USING COMMUTATIVE JORDAN ALGEBRAS

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Abstract

Commutative Jordan algebras are used to drive an highly tractable framework for balanced factorial designs with a prime number p of levels for their factors. Both fixed effects and random effects models are treated. Sufficient complete statistics are obtained and used to derive UMVUE for the relevant parameters. Confidence regions are obtained and it is shown how to use duality for hypothesis testing.

Keywords: prime basis factorial, commutative Jordan algebras, complete sufficient statistics, UMVUE, confidence regions.

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1. Introduction

A factorial design where all the factors have the same number of levels is balanced factorial. In this work we use Commutative Jordan Algebras (CJA) to carry out the study of the balanced factorial designs where factors have a prime number of levels.

As we shall see the framework provided by CJA is highly treatable both for fixed effects and random effects models. In both cases we obtain complete sufficient statistics as well as UMVUE for the relevant parameters. Confidence regions are obtained and it is shown how, through duality, hypothesis testing may be carried out.

1.1. Commutative Jordan algebras

A CJA $\mathcal{A}$ is a vector space constituted by symmetric commuting matrices that contain the squares of their matrices. Seely (1971), showed that all the CJA have only one principal base constituted by orthogonal projections matrices, all of them mutually orthogonal.

Let $\mathbf{Q}_1, ..., \mathbf{Q}_w$ be the matrices from the principal basis of a CJA $\mathcal{A}$. If $\boldsymbol{Q_1} = \frac{1}{n}\mathbf{1}^n\mathbf{1}'^n = \frac{1}{n}\mathbf{J}_n$, where $\mathbf{1}^n$ is a matrix $n \times 1$ with elements equal to 1, the CJA will be regular. Moreover, if $\sum_{j=1}^{w} Q_j = \boldsymbol{I}_n$, where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix, the CJA will be complete.

In this work we only consider regular and complete CJA.

Let $\mathbf{Q}$ be an orthogonal projection matrix belonging to $\mathcal{A}$. We have

$$\mathbf{Q} = e\sum_{j=1}^{w} a_j \mathbf{Q}_j \tag{1}$$

with $a_j = 0$ or $a_j = 1$, $j = 1, \dots, w$.

Let $R(\mathbf{Q}_j) = \nabla_j$, $j = 1, ..., w$, be the range spaces of the matrices $\mathbf{Q}_j$, $j = 1, ..., w$. We have, see Mexia (1995), $\mathbf{Q}_j = \mathbf{A}'_j\mathbf{A}_j$, $j = 1, ..., w$, if the row vectors of $\mathbf{A}_j$ constitute an orthonormal basis for ∇_j, $j = 1, ..., w$. If $\mathcal{A}$ is a complete CJA, the line vectors of

$$\mathbf{P} = \left[\mathbf{A}'_1 ... \mathbf{A}'_n\right]' \tag{2}$$

constitute an orthonormal basis for R^n since we have

$$\mathbf{I}_n = \sum_{j=1}^{w} \mathbf{Q}_j = \sum_{j=1}^{w} \mathbf{A}'_j \mathbf{A}_j. \tag{3}$$

and $\mathbf{P}$ is an orthogonal matrix associated to $\mathcal{A}$.

Inversely, the matrices $\mathbf{Q}_j = \mathbf{A}'_j\mathbf{A}_j$, $j = 1, \dots, w$, are symmetric and idempotents and, as $\mathbf{Q}_j\mathbf{Q}_{j'} = 0_{n\times n}$, $j \neq j'$, are mutually orthogonal constituting the principal basis of a CJA associated to $\mathbf{P}$.

If $\mathcal{A}$ is a complete and regular CJA with principal basis $\aleph=\{\mathbf{Q}_1,\ldots,\mathbf{Q}_w\}$, we have $\mathbf{Q}_1 = \frac{1}{n}\mathbf{J}_n = \left(\frac{1}{\sqrt{n}}\mathbf{1}^n\right)\left(\frac{1}{\sqrt{n}}\mathbf{1}'^n\right)$, so $\mathbf{A}_1 = \frac{1}{\sqrt{n}}\mathbf{1}'^n$ is the first row of an orthogonal matrix $\mathbf{P}$ associated to $\mathcal{A}$, constituted by elements equals to $\frac{1}{\sqrt{n}}$. The matrix $\mathbf{P}$ will be, see Mexia (1988), orthogonal and standardized.

1.2. Prime basis factorials

We will consider the factorial design p^N, with N factors each one having a prime number p of levels. The p levels are numbered from 0 to $p-1$ and the treatments are represented by vectors $\mathbf{x} = (x_1,\ldots,x_N)$, with $x_j = 0,\ldots,p-1$, $j=1,\ldots,m$, whose components are the factors levels.

If in $G[p] = \{0,\ldots,p-1\}$ we define the addition and the multiplication module p, where the results from the usual operations are replaced by the rest of their division by p, we have the Galois field $G[p]$ with support $[p]$. So, the vectors $\mathbf{x}$ will belong to the vector space $G[p]^N$ whose vectors have N components belonging to $G[p]$.

The treatments may be identified by the vectors $\mathbf{x}^N \in G[p]^N$ with components $x_j \in G[p]$, $j=1,\ldots,N$. This treatments my be ordered by the indexes:

$$j = 1 + \sum_{i=1}^{N} x_i p^{i-1}. \tag{4}$$

Let $\mathbb{L}[p]^N$ be the family of the linear functions

$$\mathbb{L}(\mathbf{x}) = \sum_{j=1}^{N} a_j x_j, \qquad a_j = 1,\ldots,p-1. \tag{5}$$

The values of these functions are obtained using the module p arithmetic.

Let also $\mathbb{L}_r[p]^N$ be the family of reduced linear functions, i.e. the functions whose first non null coefficient is 1.

As $\mathbb{L}[p]^N$ is a vector space with dimension N and each one of the N coefficients can take p values, in $\mathbb{L}[p]^N$ will exist p^N functions of which p^N-1 are not null. There are $\frac{p^N-1}{p-1}$ reduced linear functions.

For more details about prime basis factorials see, for exemple, Day and Mukerjee (1999).

1.3. Incidence matrices

For each function $L \in \mathbb{L}_r[p]^N$ we can assign a matrix $\boldsymbol{C}(L) = [c_{ij}(L)]_{p \times p^N}$ where

$$c_{i,j}(L) = \begin{cases} 0; & L(\mathbf{x}_j) \neq i-1 \\ 1; & L(\mathbf{x}_j) = i-1, \end{cases} \tag{6}$$

with $\boldsymbol{x}_j$ the vector with index j.

Then, with $\boldsymbol{K}$ obtained deleting the first row equal to $\frac{1}{\sqrt{p}}\mathbf{1}'^p$ from a $p \times p$ standardized orthogonal matrix, we take

$$\boldsymbol{B}(L) = \frac{1}{q}\boldsymbol{K}\boldsymbol{C}(L) \tag{7}$$

with $q = p^{\frac{N-1}{2}}$.

In Mexia (1988) it was prooven that the matrix

$$\mathbf{P}\left(p^N\right) = \left[\frac{1}{p^{\frac{N}{2}}}\,\mathbf{1}_{p^N}, \mathbf{B}'(L_1), \ldots, \mathbf{B}'(L_w)\right]' \tag{8}$$

with $w = \frac{p^N - 1}{p-1}$, is orthogonal standardized. This matrix will be associated to the CJA $\mathcal{A}\left(p^N\right)$ with principal basis $\aleph(p^N) = \{\frac{1}{p^N}\mathbf{J}_{p^N},\ \mathbf{Q}(L_1), \ldots, \mathbf{Q}(L_w)\}$, $\mathbf{Q}(L_j) = \mathbf{B}'(L_j)\,\mathbf{B}(L_j)$, $j = 1, \ldots, w$.

2. Fixed effects model

We will use the CJA to construct the model and make the inference. Let us assume that we are working with the reduced linear functions $L_1, \ldots, L_w$ with $w = \frac{p^{N'} - 1}{p-1}$ for which we have the matrices $\mathbf{B}_j = \mathbf{B}(L_j)$, $j = 1, \ldots, w$, and the orthogonal matrix

$$\mathbf{P} = \left[\frac{1}{p^{\frac{N'}{2}}}\,\mathbf{1}_{p^{N'}}\ \mathbf{B}'_1 \ldots \mathbf{B}'_w\right]'$$

associated to the CJA relevant for the model.

2.1. Model

The fixed effects model will be

$$\mathbf{Y} = \mathbf{1}^n \mu + \sum_{j=1}^{w} \left(\mathbf{B}_j' \otimes \mathbf{1}^r\right) \beta_j + \mathbf{e}, \tag{9}$$

where $n = p^N r$, the vectors $\beta_1, \ldots, \beta_w$ are fixed with $p-1$ components and $\mathbf{e}$ is normal with zero mean vector and covariance matrix $\sigma^2 \mathbf{I}_n$.

Let

$$\begin{cases} \mathbf{B}_j^0 = \mathbf{B}_j \otimes \frac{1}{\sqrt{r}} \mathbf{1}'^r, \ j = 1, \ldots, w \\ \mathbf{B}^\perp = \mathbf{I}_n \otimes \mathbf{T}_r, \end{cases} \tag{10}$$

where $\mathbf{T}_r = \mathbf{I}_r - \frac{1}{r} \mathbf{J}_r$.

As the model $\mathbf{Y}$ has the mean vector and the variance-covariance matrix

$$\boldsymbol{\mu} = (\mathbf{1}^n \otimes \mathbf{1}^r)\mu + \sum_{j=1}^{w} (\boldsymbol{B}_j \otimes \mathbf{1}^r)\boldsymbol{\beta} \quad \text{and} \quad \boldsymbol{V} = \sigma^2 \boldsymbol{I}_n, \tag{11}$$

respectively, it can be shown, see Fonseca *et al.* (2006), that the density probability function is

$$n\left(\mathbf{Y}\right) = \frac{e^{-\frac{1}{2\sigma^2} \sum_{j=1}^{w} \left\| \widetilde{\beta}_j - \beta_j \right\|^2 - \frac{S^\perp}{2\sigma^2}}}{(2\pi)^{\frac{n}{2}} \sigma^n} \tag{12}$$

with the complete sufficient statistics

$$\begin{cases} S^\perp = \left\| \mathbf{B}^\perp \mathbf{Y} \right\|^2 \sim \sigma^2 \chi^2_{g^\perp}, \quad g^\perp = n\left(r-1\right) \\ \widetilde{\beta}_j = \frac{1}{\sqrt{r}} (\mathbf{B}_j \otimes \mathbf{1}'^r)\mathbf{Y} \sim \mathbf{N}\left(\beta_j, \sigma^2 \mathbf{I}_{p-1}\right), \quad j = 1, \ldots, w. \end{cases} \tag{13}$$

2.2. UMVUE

According to the Blackwell-Lehman-Scheffé theorem, we have the UMVUE

$$\left\{\begin{array}{ll} \widetilde{\sigma^2} = \frac{S^{\perp}}{g^{\perp}}; & g^{\perp} = n(r-1) \\ \widetilde{\boldsymbol{\Psi}}_j = C_j\widetilde{\beta}_j; & j = 1, \dots, w \end{array}\right. \tag{14}$$

for σ^2 and for $\boldsymbol{\Psi}_j = C_j\beta_j$, $j = 1, \dots, w$. If $\mathbf{C}_j = \mathbf{I}$ we have $\boldsymbol{\Psi}_j = \beta_j$ and $\widetilde{\boldsymbol{\Psi}}_j = \widetilde{\beta}_j$, $j = 1, \dots, w$.

2.3. Tests of hypothesis and confidence intervals for σ^2

We can use the statistic $S^{\perp} \sim \sigma^2\chi^2_{g^{\perp}}$ to construct confidence intervals for σ^2. And, through the duality we can also obtain tests of hypothesis for the null hypothesis

$$H_0 : \sigma^2 = \sigma_0^2. \tag{15}$$

This hypothesis is rejected at a significance level q, if and only if σ_0^2 is outside the $1-q$ level confidence interval.

2.4. Tests of hypothesis and confidence intervals for $\Psi_j = \mathbf{G}_j\beta_j$

The UMVUE for $\boldsymbol{\Psi}_j = \boldsymbol{G}_j\boldsymbol{\beta}_j$, $j = 1, \dots, w$, is $\widetilde{\boldsymbol{\Psi}}_j = \boldsymbol{G}_j\widetilde{\boldsymbol{\beta}}_j$, and we have

$$\widetilde{\boldsymbol{\Psi}}_j \sim N\left(\boldsymbol{\Psi}_j, \sigma^2\boldsymbol{G}_j\boldsymbol{G}_j'\right)$$

$$(i)$$

$$S^{\perp} \sim \sigma^2\chi^2_{g^{\perp}},$$

where $\boldsymbol{G}_j$, $j = 1, \dots, w$, is a matrix whose line vectors are linearly independent.

Moreover we have, see Mexia (1995),

$$U_j^0 = (\boldsymbol{\Psi}_j - \widetilde{\boldsymbol{\Psi}}_j)'(\boldsymbol{G}_j\boldsymbol{G}_j')^+(\boldsymbol{\Psi}_j - \widetilde{\boldsymbol{\Psi}}_j) \sim \sigma^2\chi^2_{c_j}$$

$$(i)$$

$$S^{\perp} \sim \sigma^2\chi^2_{g^{\perp}},$$

with $c_j = car(\boldsymbol{G}_j) = car(\boldsymbol{G}_j\boldsymbol{G}_j')$, $j = 1, \dots, w$.

So we have the pivot variable

$$\mathcal{F}_j^0 = \frac{g^\perp}{c_j}\frac{U_j^0}{S^\perp} \sim F_{c_j,g^\perp}, \quad j = 1, \dots, w. \tag{16}$$

Thus, the $1-q$ confidence ellipsoid for $\mathbf{\Psi}_j$ is

$$(\mathbf{\Psi}_j - \widetilde{\mathbf{\Psi}}_j)'(\boldsymbol{G}_j\boldsymbol{G}_j')^{-1}(\mathbf{\Psi}_j - \widetilde{\mathbf{\Psi}}_j) \le c_j f_{1-q,c_j,g^\perp}\frac{S^\perp}{g^\perp}, \quad j = 1, \dots, w. \tag{17}$$

where $f_{1-q,c_j,g^\perp}$ is the quantil with probability $1-q$ from the $\mathcal{F}$ distribution with c_j and $g^\perp$ degrees of freedom.

If we have to perform a test of hypothesis for the null hypothesis

$$H_{0,j} : \mathbf{\Psi}_j = \boldsymbol{b}_j, \quad j = 1, \dots, w, \tag{18}$$

we may use the duality property of $\mathcal{F}$ tests. So this hypothesis is rejected at a significance level q, if and only if $\boldsymbol{b}_j$ is outside the $1-q$ level confidence ellipsoid.

3. RANDOM EFFECTS MODEL

3.1. Model

The random effects model is

$$\mathbf{Y} = 1^n Y_o + \sum_{j=1}^{w} \mathbf{B}_j' \beta_j, \tag{19}$$

with $n = p^N$. The vectors β_j, $j = 1, \dots, w$ have normal distribution with null mean vector and covariance matrix $\sigma_1^2 \mathbf{I}_{p-1}, \dots, \sigma_w^2 \mathbf{I}_{p-1}$ and are independents. Y_o the general mean with mean vector μ and variance σ_o^2.

As $\mathbf{1}^n \boldsymbol{B}_j' = \mathbf{0}$, $j = 1, \dots, w$,

$$\begin{cases} \frac{1}{p^{N/2}} \mathbf{1}'^{p^N} \boldsymbol{Y} = p^{N/2} \boldsymbol{\mu} \\ \boldsymbol{B}\boldsymbol{Y} = \boldsymbol{\beta}, \quad \boldsymbol{B} = [\boldsymbol{B}_1, \dots, \boldsymbol{B}_w]', \end{cases} \tag{20}$$

where $\boldsymbol{\beta} = \left[\boldsymbol{\beta}_1', \dots, \boldsymbol{\beta}_w'\right]'$.

From the first equation on (20), $\boldsymbol{\mu}$ is known and $\mathbf{Y}$ depends only of β that have normal distribution with null mean vector and covariance matrix diagonal by blocks, $\mathbf{D}\left(\sigma_1^2\mathbf{I}_{p-1},\ldots,\sigma_w^2\mathbf{I}_{p-1}\right)$.

Then, the density probability function for β is

$$n(\boldsymbol{z}) = \frac{\exp\left\{-\left(\frac{(Y_o-\mu)^2}{2\sigma_o^2}+\frac{1}{2}\sum_{j=1}^{w}\frac{S_j}{\sigma_j^2}\right)\right\}}{(2\pi)^{(n-1)/2}\prod_{j=1}^{w}\sigma_j^{p-1}}, \tag{21}$$

where

$$S_j = ||\boldsymbol{B}_j\boldsymbol{z}||^2 = ||\boldsymbol{\beta}_j||^2 \sim \sigma_j^2\chi_{p-1}^2, \quad j=1,\ldots,w, \tag{22}$$

since

$$\mathbf{z}'\mathbf{D}\left(\sigma^2\mathbf{I}_{p-1},\ldots,\sigma_w^2\mathbf{I}_{p-1}\right)^{-1}\mathbf{z} = \sum_{j=1}^{w}\frac{1}{\sigma_j^2}\,||B_j\mathbf{z}||^2 = \sum_{j=1}^{w}\frac{\|\beta_j\|^2}{\sigma_j^2}. \tag{23}$$

According to the factorization theorem the $S_1,\ldots,S_w$ will be sufficient statistics. Moreover since the normal density belongs to the exponential family and the parameter space contains an open set, the statistics S_j, $j=1,\ldots,w$, and Y_o will be complete.

According now to the Blackwell-Lehman-Scheffé theorem, the estimators

$$\widetilde{\sigma}_j^2 = \frac{S_j}{p-1}, \quad j=1,\ldots,w, \tag{24}$$

are UMVUE for the σ_j^2, $j=1,\ldots,w$.

3.2. Confidence intervals and tests of hypotheses for σ_j^2

We can use the statistics S_j, $j=1,\ldots,w$, as pivot variables to obtain $1-q$ confidence intervals for σ_j^2. So we have

$$\left[\frac{S_j}{\chi_{p-1,1-q/2}};\frac{S_j}{\chi_{p-1,q/2}}\right], \tag{25}$$

where $\chi_{p-1,q}^2$ is the quantile for probability q of the chi-square distribution with $p-1$ degrees of freedmen.

If we have to perform a test of hypothesis for the null hypothesis

$$H_0 : \sigma^2 = \sigma_0^2, \tag{26}$$

we may use the duality property of χ^2 tests. Thus this hypothesis is rejected, at a significance level q, if and only if σ_0^2 is outside the $1-q$ level confidence interval.

3.4. Confidence intervals for $\theta_{j,l} = \frac{\sigma_j^2}{\sigma_l^2}$, $j \neq l$

Now we consider quotients of variance components.

With

$$\theta_{j,l} = \frac{\sigma_j^2}{\sigma_l^2}, j \neq l, \tag{27}$$

the test statistic for $H_0 : \theta_{j,l} = b$ is

$$\mathfrak{F}_{j,l} = \frac{S_j}{S_l} \sim \theta_{j,l} F_{p-1,p-1}, \quad j \neq l. \tag{28}$$

If $p > 3$,

$$E\left(\mathfrak{F}_{\mathrm{j,l}}\right) = \theta_{j,l} \frac{p-1}{p-3}, j \neq l. \tag{29}$$

And, according to the Blackwell-Lehman-Scheffé theorem, the estimators

$$\widetilde{\theta}_{j,l} = \frac{p-3}{p-1} \mathfrak{F}_{j,l} \tag{30}$$

are UMVUE for the $\theta_{j,l}$, $j \neq l$.

Let $F\left(\cdot \,|k,\ h\right)$ be the central $\mathfrak{F}$ distribution with k and h degrees of freedmen and $\mathfrak{f}_{k,h}\left(q\right)$ the quantile with probability q from $F\left(\cdot \,|k,\ h\right)$. Since

$$Pr\left(f_{p-1,p-1}(q') \leq \frac{\mathfrak{F}_{j,l}}{\theta_{j,l}} \leq f_{p-1,p-1}(q'')\right) = q'' - q', \tag{31}$$

we obtain the bounds for the two-side confidence interval for $\theta_{j,l}$, $j \neq l$, given by

$$(32) \qquad Pr\left(\frac{\mathfrak{F}_{j,l}}{f_{p-1,p-1}(q'')} \leq \theta_{j,l} \leq \frac{\mathfrak{F}_{j,l}}{f_{p-1,p-1}(q')}\right) = q'' - q'.$$

Similarly, we have the one-side confidence intervals

$$(33) \qquad \begin{cases} Pr\left(\theta_{j,l} \leq \frac{\mathfrak{F}_{j,l}}{f_{p-1,p-1}(q)}\right) = 1 - q \\ \\ Pr\left(\theta_{j,l} \geq \frac{\mathfrak{F}_{j,l}}{f_{p-1,p-1}(q)}\right) = q. \end{cases}$$

These confidence intervals can be used to perform, through duality, two-sided and one-sided tests of hypothesis for

$$(34) \qquad H_0 : \theta_{j,l} = \theta^0_{j,l}.$$

So these tests reject H_0 for a significant level q when $\theta^0_{j,l}$ is not covered by the corresponding $1 - q$ confidence interval.

4. Final comments

As the considered tests (chi-square tests and $\mathcal{F}$ tests) have the property of duality, it able us to unify the presentation for confidence intervals and the rejection regions for the tests of hypotheses.

We intend to extend this treatment to mixed models thus completing the study of prime basis factorial which up to now have been restricted to the fixed effects models.

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