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PERIODIC POINTS AND BIFURCATION OF ONE-DIMENSIONAL MAPS

1. This note concerns the problem of dependence of the behaviour of the sequence $\{x_n\}$ generated by the difference equation

(1) $$x_{n+1} = f_t(x_n)$$

on the value of a real parameter t if f_t is a one-dimensional function. This problem is important in applications, for example in theoretical biology (see [7] and References therein) or in the theory of weather prediction (see [6]). In situations important in applications the following conditions are satisfied.

For every t the function $f = f_t$ maps the interval $I = [0, 1]$ into itself,

(2) $$f(0) = f(1) = 0,$$
$$f \text{ has exactly one critical point } c = c_t \in (0, 1).$$

The classical situation where f_t is given by the formula

(3) $$f_t(x) = tx(1-x), \quad x \in I, \ t \in [0, 4],$$

has been studied extensively in many papers. It has been shown (see for example [5], [6], [7]) that f_t has for $1 < t$ exactly one non-zero fixed point p_t which for $t \leqslant 3$ attracts all points of I except of 0 and 1 (all definitions will be given later). For $3 < t$ the point p_t bifurcates into a 2-periodic orbit which for $t \leqslant 1 + \sqrt{6}$ attracts all points of I except of 0, 1, p_t and succesive inverse images of p_t under f_t. Similarly there appears a 4-periodic orbit, an 8-periodic orbit, etc. These orbits succesively attract all points of $I \setminus E$, where E is a countable set. However for $t > t_d$ (approx. $t_d = 3.570$) there appear n-periodic orbits with n different from 2^k for all k. For example there are for $t = 3.627$ a 6-periodic orbit, for $t = 3.75$ a 5-periodic orbit, for $t = 3.83$ a 3-periodic orbit. For $t = 3.76$ it is hard to decide, whether the attracting periodic orbit exists. For $t = 4$ there are no attracting periodic orbits.

Similar phenomena occur for other families of functions (see [4] and [7] where some numerical results are given). We give the qualitative description of the above phenomena in the case if f_t satisfies for every t the Singer condition

(4) $$S(f)(x) < 0,$$

where

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2, \quad \text{for} \quad x \neq c,$$

is the Schwarzian derivative of f (see [3], [11]). It is shown in [11] that the above condition is satisfied in most situations considered in applications. Our results generalize and complete known results obtained for the family (3) (see [5], [6]) and for families similar to (3) (see [7]). Some of these results one can probably obtain using kneading theory (see [1], [3], and References in [1]). However the methods presented here are more elementary. Our consideration can be repeated if f_t maps an interval $[0, A]$ into itself, $0 < A \leqslant \infty$. If $A = \infty$, the condition $f(A) = 0$ means that $\lim_{x \to \infty} f(x) = 0$.

2. Let $f\colon I \to I$ be a continuous function. Define the nth iteration of f as follows: $f^0 = \text{identity}$, $f^{n+1} = f \circ f^n$. For $x \in I$ the *orbit* $o(x)$ of x is the sequence $\{f^n(x)\}_0^\infty$. The *ω-limit set* $\omega(x)$ of x is the set of all limit points of $o(x)$. A point $x \in I$ is *n-periodic* if $o(x)$ contains exactly n points, *eventually n-periodic* if $f^k(x)$ is n-periodic for some k and *asymptotically n-periodic* if $\omega(x) = o(y)$ for some n-periodic point y. The 1-periodic point is said to be the *fixed point*. A set $A \subset I$ is said to be *n-periodically invariant* if $f^n(A) \subset A$ and $f^k(A) \not\subset A$ for $1 \leqslant k < n$. The 1-periodically invariant set is called the *invariant* set. If A is n-periodically invariant, then the set $o(A) = A \cup f(A) \cup \ldots \cup f^{n-1}(A)$ is said to be the *generalized n-periodic orbit*. An invariant set $A \subset I$ is *absorbing* in a set $J \subset I$ if for every $x \in J$ there exists k such that $f^k(x) \in A$. An invariant set $A \subset I$ is *attracting* in a set $J \subset I$ if $\omega(x) \subset A$ for every $x \in J$ and if J is a neighbourhood of A we call the set A an *attractor*.

It is easy to see that $o(x)$ is an attracting periodic orbit with respect to f if and only if x is the attracting fixed point of f^n and $f^k(x) \neq x$ for $1 \leqslant k < n$. For an n-periodic point x the eigenvalue $\lambda(x)$ is the number $(f^n)'(x) = f'(f^{n-1}(x)) \cdot \ldots \cdot f'(x)$. For $|\lambda(x)| < 1$ the set $o(x)$ is an attractor, for $|\lambda(x)| > 1$ the set $o(x)$ is not an attractor (we say non-attractor). If the set A is absorbing with respect to f^n (in some set J) then the generalized n-periodic orbit is absorbing.

3. The Schwarzian derivative $S(f)$ of f defined for functions of class C^3 by formula (3) has been used by David Singer ([11]) to investigate the difference equation of the form (1). We quote some results from [8] and [11].

PROPOSITION 0. (i) *Let f and g be functions of class C^3. Then*

$$S(f\circ g)(x) = S(f)\big(g(x)\big)g'(x)^2 + S(g)(x).$$

Assume that the function f satisfies the condition (4).

(ii) *For every n the function f^n satisfies the condition* (4).

(iii) *The function f' cannot have a positive local minimum.*

(iv) *If f has finitely many critical points, then f has finitely many fixed points.*

(v) *If p is the fixed point of f with $|\lambda(p)| \leqslant 1$, then p is attracting in the interval J containing p and d, where d is a critical point or is the end of I. If additionally $\lambda(p) \neq 1$, then p is the attractor.*

(vi) *If f satisfies the condition* (2), *then there exists at most one attracting periodic orbit.*

4. We give two propositions from the bifurcation theory provided the Singer condition (4) holds. We start from a simple and not hard to prove lemma.

LEMMA 1. *Let $f_t\colon I\to I$ be a family of functions defined for t from an interval T. Let $A\subset I$ be a compact set, let the mapping $T\times A\ni(t, x)\to f_t(x)\in I$ be continuous. Put $R_t(A) = \{x\in A\colon f_t(x) = x\}$. Then for every $t_0\in T$, $\varepsilon > 0$ there exists $\delta > 0$ such that for $|t-t_0| < \delta$,*

$$R_t(A)\subset B\big(R_{t_0}(A), \varepsilon\big) = \{x\in A\colon |x-y| < \varepsilon \text{ for some } y\in R_{t_0}(A)\}.$$

Now, let $f_t\colon I\to I$ be a family of functions of class C^3 defined for t from an interval T, such that the mapping $(t, x)\to f_t^{(i)}(x)$, $i = 0, 1, 2$, is continuous ($f^{(i)}$ denotes the ith derivative of f). Assume that f_t satisfies for every $t\in T$ the condition (4).

PROPOSITION 1. *Let $s\in T$, $p\in(0, 1)$. Assume that $f_s(p) = p$, $f_s(x) > x$ ($f_s(x) < x$ respectively) for x from a neighbourhood V of p, $x\neq p$. Then there exist neighbourhoods S of s and U of p such that for $t\in S$ the equation $f_t(x) = x$ either has no solution in U, or has exactly one solution q in U and then $f_t'(q) = 1$, or has exactly two solutions $q_1 < q_2$ in U and then $0 < f_t'(q_1) < 1 < f_t'(q_2)$ ($0 < f_t'(q_2) < 1 < f_t'(q_1)$ respectively).*

Proof. Assume for instance $f_s(x) > x$. The condition (4) implies that $f_s''(p) > 0$. This condition and the Mean Value Theorem permit to choose neighbourhoods S and U such that for $t\in S$ the equation $f_t(x) = x$ has at most two solutions in U and $f_t(x) > x$ for $x\in V\setminus U$. One can apply once more the Mean Value Theorem to complete the proof.

PROPOSITION 2. *Let $s\in T$, $p\in(0, 1)$. Assume that $f_s(p) = p$, $f_s'(p) = -1$. Then there exist neighbourhoods S of s and U of p such that for $t\in S$ the* *equation* $f_t(x) = x$ *has exactly one solution* q *in* U, *and either* $f_t'(q) \geqslant -1$, *or there exist exactly two points* q_1, q_2 *in* U *such that* $q_1 < q < q_2$, $f_t^2(q_i) = q_i$, $i = 1, 2$, *and then* $0 < (f_t^2)'(q_i) < 1$.

Proof. The first part follows from the Implicit Function Theorem. We prove the second part. Note that $f_s^2(p) = p$, $(f_s^2)'(p) = 1$, $(f_s^2)''(p) = 0$, and by the condition (4) $(f_s^2)'''(p) < 0$. By the Taylor Theorem, we have $f_s^2(x) > x$ for $x < p$ and $f_s^2(x) < x$ for $x > p$. One can choose neighbourhoods S and U such that for $t \in S$ $(f_t^2)'(x) > 0$ for $x \in U$ and $f_t^2(x) > x$ for $x < q$, $x \notin U$, $f_t^2(x) < x$ for $x > q$, $x \notin U$. Let $t \in S$ and let $f_t'(q) < -1$. Then $(f_t^2)'(q) > 1$ what means that the equation $f_t^2(x) = x$ has solutions $q_1 < q$ and $q_2 > q$ in U. The Mean Value Theorem and Proposition 0 (iii) imply that the above equation has no solutions except q, q_1, q_2 in U. Hence $q_2 = f_t(q_1)$, $q_1 = f_t(q_2)$ and $(f_t^2)'(q_1) = (f_t^2)'(q_2) < 1$.

5. Assume in this section that the C^3 mapping $f\colon I \to I$ satisfies conditions (2), (4) and

$$f'(0) > 1. \tag{5}$$

Lemma 2. *There exists exactly one non-zero fixed point of* f.

Proof. Conditions (2) and (5) imply that the fixed point exists. If two non-zero fixed points $q_1 < q_2$ exist, then $q_1 < c$. The Mean Value Theorem implies that $f'(a) = 1$ for some $a < q_1$. Proposition 0 (iii) implies that the function $f(x) - x$ decreases in the interval $(a, 1)$ which contradicts $f(q_2) = q_2$, $q_1 < q_2$.

In the sequel the unique non-zero point of f is denoted by p.

Lemma 3. *If the fixed point* p *is a non-attractor, then there exists a unique 2-periodic orbit.*

Proof. We show that $f^2(a) \leqslant a$ for some $a < p$, which together with the inequality $(f^2)'(0) > 1$ imply the existence of a 2-periodic orbit $\{p_1, p_2\}$, $p_1 < p < p_2$. Assume the contrary, i.e. that $f^2(x) > x$ for every $x < p$. By the uniqueness of p we have $c < p$. For $c \leqslant x < p$ we successively obtain: $p < f(x)$, $x < f^2(x) < p$, $p < f^3(x) < f(x)$, $x < f^2(x) < f^4(x) < p$, etc. The sequence $\{f^{2n}(x)\}$ has a limit $q \leqslant p$ and $f^2(q) = q$ what means that $q = p$, but this contradicts the non-attractivity of p.

Let $\{q_1, q_2\}$ be a 2-periodic orbit. It is clear that $q_1 < p < q_2$ for $q_1 < q_2$. Similarly as in the proof of Lemma 2 one can show that $q_2 = p_2$ and hence $q_1 = p_1$.

Proposition 3. *The following conditions are equivalent*:

(i) *The point* p *is the attractor*,

(ii) $f'(p) \geqslant -1$,

(iii) *There are no* n*-periodic orbits for* $n > 1$,

(iv) *The point* p *is attracting in* $(0, 1)$.

Proof. Obviously (iv) implies (iii). (iii) implies (i) by Lemma 3. (i) and (ii) are equivalent by Proposition 0 (v) and Lemma 2. It suffices to prove that (ii) implies (iv). Let $c < p$ (for $p \leqslant c$ the proof is obvious). By Proposition 0 (v), p is attracting in $[c, p]$ and hence $[c, f(c)]$. For every $x \in (0, c)$ there exists an integer k such that $c \leqslant f^k(x)$, otherwise the sequence $\{f^n(x)\}$ has a limit $q \leqslant c$ and $f(q) = q < p$. Since $f^k(x) \leqslant f(c)$, the sequence $\{f^n(x)\}$ tends to p. Since for $x \in (f(c), 1)$ we have $f(x) \in (0, f(c))$, the point p is attracting in $(0, 1)$.

For $x \in [0, f(c)]$ denote by x_l and x_r the uniquely defined points of I satisfying $f(x_l) = f(x_r) = x$ and $x_l \leqslant c \leqslant x_r$. Similarly $x_{ll} = (x_l)_l$, $x_{lr} = (x_l)_r$, etc.

PROPOSITION 4. *Let the point p be the non-attractor and let*

$$p \leqslant f^3(c). \tag{6}$$

Then:

(i) *The interval $P = [p_l, p_{lr}]$ is absorbing in* $(0, 1)$,

(ii) *The intervals $L = [p_l, p]$ and $R = [p, p_{lr}]$ have the properties*: $f(L) \subset R$, $f(R) = L$, $f^2(L) \subset L$, $f^2(R) \subset R$,

(iii) *The restrictions $f^2|_L$, $f^2|_R$ are topologically conjugate with a mapping f_1: $I \to I$ satisfying conditions* (2), (4), (5),

(iv) *f has a $2n$-periodic orbit* (*attracting $2n$-periodic orbit resp.*), *if and only if f_1 has an n-periodic orbit* (*attracting n-periodic orbit resp.*),

(v) *f has no $(2n+1)$-periodic orbits for all $n \geqslant 1$*,

(vi) *If f_1 has a generalized n-periodic orbit* (*absorbing resp.*), *then f has a generalized $2n$-periodic orbit* (*absorbing resp.*),

(vii) *$f^3(c) = p$ if and only if $f_1(c_1) = 1$, where c_1 is the critical point of f_1.*

Proof. It is easy to see that $c < p$, and therefore

$$c_l < p_l < c < p < c_r < p_{lr}.$$

The mapping f^2 has three critical points: at c_l and c_r it has its maximum, and at c it has a local minimum. The condition (6) implies the inequalities $f^2(c) \geqslant p_l$, $f(c) = f^2(c_r) \leqslant p_{lr}$. Hence the interval P is invariant and (ii) is true. Similarly as in the proof of Proposition 3 one can show that for every $x \in (0, 1)$ there exists k such that $f^k(x) \in P$, what proves (i). Note that $g \circ f^2|_R = f^2|_L \circ g$, where $g = f|_R$ is the homeomorphism of R onto L. Define the homeomorphism h: $I \to R$ by $h(x) = (p_{lr} - p)x + p$. The mapping $f_1 = h^{-1} \circ f^2|_R \circ h$ with $c_1 = h^{-1}(c_r)$ satisfies conditions (2), (4), (5) (Proposition 0 (i) and (ii)), Proposition 3 (iii)). Statements (iv)–(vii) are simple consequences of hitherto proved (i)–(iii).

PROPOSITION 5. *Let the point p be the non-attractor and let*

$$f^3(c) < p. \tag{7}$$

Then there exists an $(2n+1)$*-periodic orbit for some* $n > 1$. *By* [10] *there exist also* $(2m+1)$*-periodic orbits for* $m > n$.

Proof. It suffices to prove that for some $n > 1$ holds $f^{2n+1}(c) \leqslant c$. First note that the condition (7) implies $f^2(c) < c$ and hence $q_1 < c$, where q_1 is the unique 2-periodic point less than p (Lemma 3). In particular $f^2(x) < x$ for $c \leqslant x < p$. Assume now that for every $n > 0$ holds $c < f^{2n+1}(c)$. Then the following inequalities hold:

$$c < f^3(c) < p,$$
$$c < f^5(c) = f^2(f^3(c)) < f^3(c) < p,$$

and generally

$$c < f^{2n+1}(c) = f^2(f^{2n-1}(c)) < f^{2n-1}(c) < \ldots < p.$$

The limit q of the sequence $\{f^{2n-1}(c)\}$ exists and satisfies the inequalities $c \leqslant q < p$ and $f^2(q) = q$, what contradicts the uniqueness of q_1.

We give the main result.

THEOREM. *Let* γ *be a* 2^n*-periodic orbit for the fixed integer* n. *The following conditions are equivalent*:

(i) *Every point* $x \in I$ *is asymptotically* m*-periodic with* $m \leqslant 2^n$,

(ii) *Every point* $x \in I$ *is asymptotically* 2^k*-periodic with* $k \leqslant n$,

(iii) *The orbit* γ *is the attractor, it is attracting in* $I \setminus E_n$, *where* $E_n = \{x \in I$: *x is eventually a* 2^k*-periodic point,* $k < n\}$ *is the countable set*,

(iv) *For every* $k \leqslant n$ *there exists exactly one* 2^k*-periodic orbit. There are no other periodic orbits*,

(v) *One can define the sequence* $\{f_k\}$ *for* $k \leqslant n$, *where* $f_k\colon I \to I$ *satisfy conditions* (2), (4), (5), (6), $f_0 = f$, $f_{k+1} = (f_k)_1$ *as in Proposition* 4 (iii), *the non-zero fixed point* p_n *of* f_n *is attracting in* (0, 1).

Proof. For $n = 0$ the theorem follows from Proposition 3. Let $n \geqslant 1$, and assume that one of the conditions (i)–(iv) holds, we denote it by C. By Propositions 3 and 5, assumptions of Proposition 4 are satisfied, hence there is a mapping $f_1\colon I \to I$ satisfying the condition C for $n-1$. From the inductive assumption f_1 satisfies all other conditions, and again by Proposition 4, f satisfies them as well.

6. We consider families of functions similar to the family (3). Let $f_t\colon I \to I$ be functions of class C^3 defined for t from an interval T. The family $\{f_t\}$ has property (B) on T, if:

(8) $T \times I \ni (t, x) \to f_t^{(i)}(x)$ is continuous, $i = 0, 1, 2$,

(9) f_t satisfies conditions (2), (4), (5), for every $t \in T$,

(10) $T = [t_0, t_e]$ and the fixed point p is the attractor for $t = t_0$, $f(c) = 1$ for $t = t_e$, $f(c) < 1$ for $t < t_e$.

We write in many cases f instead of f_t, c instead of c_t, etc. The family defined by (3) has the property (B) on $[t_0, 4]$, for $t_0 \in (1, 3]$.

Let D denote a certain condition imposed on f_t. A point s in T is said to be the *minimal parameter* (*maximal* parameter) with respect to D provided $s = \inf\{t \in T\colon f_t \text{ satisfies D}\}$ ($s = \sup\{t \in T\colon f_t \text{ satisfies D}\}$).

Assume that $\{f_t\}$ has property (B) on T. The results of Sections 4 and 5 allow us to make some remarks about the properties of the family $\{f_t\}$.

For t close to t_0 there exists the unique fixed point p which is attracting in (0, 1). For $t = t_e$ point p is the non-attractor because $\omega(c) = \{0\}$. Let s_1 be the maximal parameter with respect to the condition $f'(p) = -1$. Proposition 2 implies that for t close to s_1 the point p bifurcates in a 2-periodic orbit which is attracting in $I \setminus E_1$ (the definition of E_n was given in the Theorem (iii)). Let $T_1 = [t_1, t^1]$ where t_1 is close enough to s_1 and $t^1 > t_1$ is the minimal parameter with respect to the condition $f^3(c) = p$. The family of functions $f_{1t}\colon I \to I$ defined as $f_{1t} = (f_t)_{I'}$ has the property (B) on T_1. By Proposition 4 (i) the interval P is absorbing in (0, 1). In a similar way families $\{f_{nt}\}$ exist and have the property (B) on suitable intervals $T_n = [t_n, t^n]$, $n = 2, 3, 4, \ldots$ For $t \in T_n$ close to t_n the 2^n-periodic orbit is attracting in $I \setminus E_n$. For $t \in T_n$ there exists a generalized 2^{n-1}-periodic orbit which is absorbing in $I \setminus E_{n-1}$. It is not known (even for the family (3)) whether the maximal period increases, however it tends to infinity as t tends to the infimum of $\{t^n\}$.

Let $t_c \geqslant t^1$ be the maximal parameter with respect to the condition $f^3(c) = p$. (For the family (3) $t^1 = t_c$ (approx. = 3.6786), see [9]). Proposition 5 implies that for $t > t_c$ there exist periodic orbits with odd periods. These periods decrease as periodic orbits appear. Proposition 1 implies that there exists a couple of $(2n+1)$-periodic orbits, one being the attractor and the other one the non-attractor. We show that this attractor bifurcates into $2(2n+1)$-periodic orbits, the next into $4(2n+1)$-periodic orbits etc. Roughly speaking the family $\{f_t^{2n+1}\}$ has local property (B). We show that similar phenomena occur in the case if $f^4(c) < p$ (and $f^2(c) < f^3(c) < c$), namely there appear periodic orbits with periods $\ldots, 10, 8, 6, 4$. Similarly there appear periodic orbits with periods $\ldots, 11, 9, 7, 5, \ldots, 12, 10, 8, 6$, etc.

PROPOSITION 6. *Let $N \geqslant 2$ and let*

(11) $$f^2(c) < f^3(c) < \ldots < f^N(c) < c.$$

Denote by P_N the interval $[g, g_r]$, where $g = p_{l\ldots l}$ ($N-1$ times). If

(12) $$p \leqslant f^{N-1}(c)$$

then P_N *is absorbing in* (0, 1). *For* $x \in (0, g)$ $f^n(x) > x$ *for all n. If* (12) *does not hold, i.e.*

$$f^{N-1}(c) < p, \tag{13}$$

then there exists an integer k such that $f^{N+1+2k}(c) \leqslant c$, *what means that there exists an* $(N+1+2k)$*-periodic orbit* $o(q)$ *and* $q < g$.

Proof. If conditions (11) and (12) hold, then $f(c) \leqslant g_r$, what proves the first part. The proof of the second part is similar to the proof of Proposition 5.

The family $\{f_t\}$ has property (LB) on the interval T, if for every $t \in T$ there exists a closed interval $I_t \subset I$ such that the restriction $f_t|_{I_t}$ is topologically conjugate with a function $g_t\colon I \to I$ and the family $\{g_t\}$ has property (B) on T.

As an example one can take the family $\{f_t^{2^n}\}$ on the interval T_n, $n = 0, 1, 2, \ldots$

Let $N \geqslant 2$, $k \geqslant 0$ be fixed. Put $n = N+1+2k$. Let $\{f_t\}$ has property (B) on T. Let t_{Nk} be the maximal parameter with respect to conditions (11) and (13), and let $f^n(c) = c$. It is easy to see ([9]) that the function $t \to c_t$ is continuous. Since $f^n(c) = 0$ for $t = t_e$ the number t_{Nk} is well-defined. Assume for a moment that $t = t_{Nk}$. The orbit $o(c)$ is n-periodic and the attractor. Moreover, $o(c) \cap (0, g) = \{q\}$, where $q = f^2(c)$. Conditions $f^n(c) = q$, $(f^n)'(q) = 0$ and $f^n(g) = p > g$ imply that there exists a second fixed point q_0 of f^n, $q < q_0 < g$. The definition of t_{Nk} implies that for $t \geqslant t_{Nk}$ there are two n-periodic points q_t, q_{0t} in the interval $(0, g)$ continuously depending on t and for $t = t_{Nk}$ equal to q and q_0, respectively. There is a unique critical point d_t in (q_t, q_{0t}) of f_t^n. Let $t^{Nk} > t_{Nk}$ be the minimal parameter with respect to the condition $f^{2n}(d) = q_0$, and put $T_{Nk} = [t_{Nk}, t^{Nk}]$. Proposition 0 (iii) implies that the family $\{f_t^n\}$ has the property (LB) on T_{Nk} (set $I_t = [u_t, q_{0t}]$, where u_t is the maximal point with respect to the properties $f^n(u) = q_0$, $u < q_0$). Note that $t_0 < \ldots < T_{22} < T_{21} < T_{20} < \ldots < T_{32} < T_{31} < T_{30} < \cdots < T_{42} < T_{41} < T_{40} < \ldots < t_e$ ($A < B$ means that for $a \in A$, $b \in B$ we have $a < b$), and $\lim_{N\to\infty} t_{Nk} = \lim_{N\to\infty} t^{Nk} = t_e$.

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