ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 2 (1984), p. 337-342

ALGORITHM 90

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A PROCEDURE REALIZING A SECOND ORDER ONE-STEP METHOD FOR SOLVING A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

1. Procedure declaration. The procedure *diffsystbobkov2* solves the initial value problem of the form

$$y_k' = f_k\big(x, y_1(x), y_2(x), \ldots, y_n(x)\big), \tag{1}$$

$$y_k(x_0) = y_{0k} \quad (k = 1, 2, \ldots, n) \tag{2}$$

at the points $x_1, x_2, \ldots$

Data:

x — the value of x_0 in (2);

x1 — the value of the argument for which we solve the problem;

eps — the relative error (the given tolerance);

eta — the number which is used instead of zero when the obtained solution is zero or near to zero;

hmin — the least admissible absolute value of the step length;

n — the number of differential equations in (1) and (2);

y[*1* : *n*] — the values of the right-hand sides of (2).

Results:

x — the value of *x1*;

y[*1* : *n*] — the values of the approximate solution $y_k(x)$ $(k = 1, 2, \ldots, n)$.

Additional parameters:

steph — label outside of the body of the procedure *diffsystbobkov2* to which a jump is made if the absolute value of the step length is smaller than *hmin*; after the jump, *x* is equal to the value $\tilde{x}$ ($\tilde{x} < x1$) for which the approximate solution has a relative error equal to the given *eps*, and *y*[*1* : *n*] contains the value of this approximate solution;

f — identifier of the procedure which computes the values of the right-hand sides of (1) and puts them in *d*[*1* : *n*]; this procedure has the following heading: **procedure** *f*(*x*, *n*, *y*, *d*); **value** *x*, *n*; **real** *x*; **integer** *n*; **array** *y*, *d*.

```
procedure diffsystbobkov2(x,x1,eps,eta,hmin,n,y,steph,f);
 value x1,eps,eta,hmin,n;
 real x,x1,eps,eta,hmin;
 integer n;
 array y;
 label steph;
 procedure f;
 begin
  real h,hh,w,w1,w2,w3,w4;
  integer i;
  Boolean last;
  array d,df,dff,y1,y2[1:n];
  eps:=.166666666666/eps;
  h:=x1-x;
  last:=true;
  f(x,n,y,d);
conth:
  hh:=h×.5;
  w:=h×.375;
  for i:=1 step 1 until n do
   begin
    w1:=y[i];
    w2:=d[i];
    y1[i]:=w1+hh×w2;
    y2[i]:=w1+w×w2
   end i;
  f(x+hh,n,y1,df);
  f(x+w,n,y2,dff);
  hh:=h×.75;
  for i:=1 step 1 until n do
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 begin
  w1:=y[i];
  y1[i]:=w1+h×df[i];
  y2[i]:=w1+hh×dff[i]
 end i;
f(x+h,n,y1,df);
f(x+hh,n,y2,dff);
hh:=h×.5;
w2:=h×.333333333333;
w:=.0;
for i:=1 step 1 until n do
 begin
  w1:=d[i];
  w3:=w2×(w1+2.0×dff[i]);
  w1:=w3-hh×(w1+df[i]);
  w3:=y2[i]:=y[i]+w3+w1×.333333333333;
  w1:=abs(w1);
  w3:=abs(w3);
  if w3<eta
   then w3:=eta;
  w3:=w1/w3;
  if w3>w
   then w:=w3
 end i;
w:=if w=.0 then eta else 1.25×(eps×w)↑.333333333333;
hh:=h/w;
if w>1.25
 then
 begin
  if abs(hh)<hmin
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     then go to steph;
    last:=false
   end w>1.25
   else
   begin
    x:=x+h;
    for i:=1 step 1 until n do
     y[i]:=y2[i];
    if last
     then go to endp;
    f(x,n,y,d);
    w:=x1-x;
    if (w-hh)×h<0
     then
     begin
      hh:=w;
      last:=true
     end(w-hh)×h<0
   end w<1.25;
  h:=hh;
  go to conth;
endp:
 end diffsystbobkov2
```

2. Method used. In the construction of the procedure *diffsystbobkov2* we use two one-step methods of second order. The methods are of the form

$$\eta_{n+1} = \eta_n + \frac{h}{2}(f_n + f_{n+1}) \tag{3}$$

with the additional formulas

$$\eta_{n+1/2} = \eta_n + \frac{h}{2} f_n, \qquad \eta_{n+1} = \eta_n + h f_{n+1/2},$$

and

$$(4) \qquad \eta_{n+1} = \eta_n + \frac{h}{3}(f_n + 2f_{n+3/4})$$

with the additional formulas

$$\eta_{n+3/8} = \eta_n + \frac{3}{8}hf_n, \qquad \eta_{n+3/4} = \eta_n + \frac{3}{4}hf_{n+3/8},$$

where η_{n+a} is an approximate solution obtained at the point $x_n + ah$, and

$$f_{n+ah} = f(x_n + ah, \eta_{n+a}).$$

Using the results due to Bobkov [2], p. 38-42, we may treat the solution obtained by (4) as the solution calculated by method (3) twice with step $h/2$. In this way, we may use the solutions obtained by methods (3) and (4) to vary the step of integration.

In paper [3] the method (3) is used once with step h and twice with step $h/2$ for the control of the step size.

This step size control mechanism is described in [1]. The numerical results for the procedure *diffsystbobkov2* are better than those of the procedure *diffsystheun* (see [3]) which realizes the method (3).

3. Certification. The procedure *diffsystbobkov2* has been verified on the ODRA 1204 computer for many examples of the initial value problem. Some of them are presented here.

Examples.

$$\text{(A)} \qquad y_1' = 1/y_2, \quad y_1(0) = 1, \quad y_2' = -1/y_1, \quad y_2(0) = 1$$

with the exact solution $y_1 = e^x$, $y_2 = e^{-x}$.

$$\text{(B)} \qquad y_1' = 10\cos 10x, \quad y_1(0) = 0,$$

with the exact solution $y_1 = \sin 10x$.

$$\text{(C)} \qquad \begin{cases} y_1' = 10\,\mathrm{sgn}(\sin 20x)y_2, & y_1(0) = 0, \\ y_2' = -10\,\mathrm{sgn}(\sin 20x)y_1, & y_2(0) = 1, \end{cases}$$

with the exact solution $y_1 = |\sin 10x|$, $y_2 = |\cos 10x|$.

In the sequel we present the obtained relative error and the number of evaluations of the function f ($[f]$).

The results were obtained for *eps* = *eta* and *hmin* = $_{10}-15$ at the points .5, 1.0, 1.5, and 10.0. Here we give only the results for $x = 1.5$ and $x = 10.0$.

The results obtained for the problem (A)

x	$eps = {}_{10}-3$	$[f]$	$eps = {}_{10}-6$	$[f]$	$eps = {}_{10}-9$	$[f]$
1.5	$1.5_{10}-5$	14	$1.2_{10}-7$	74	$4.4_{10}-10$	689
	$2.4_{10}-4$		$3.6_{10}-7$		$3.2_{10}-11$	
10.0	$-2.8_{10}-2$	128	$-3.2_{10}-5$	1173	$-3.2_{10}-8$	11613
	$3.8_{10}-2$		$4.1_{10}-5$		$4.1_{10}-8$	

The results obtained for the problem (B)

x	$eps = {}_{10}-3$	$[f]$	$eps = {}_{10}-6$	$[f]$	$eps = {}_{10}-9$	$[f]$
1.5	$-7.4_{10}-4$	117	$5.1_{10}-7$	728	$4.1_{10}-9$	6942
10.0	$-5.0_{10}-3$	2181	$-3.7_{10}-6$	14217	$9.5_{10}-8$	134643

The results obtained for the problem (C)

x	$eps = {}_{10}-3$	$[f]$
1.5	$-2.9_{10}-3$	941
	$-2.9_{10}-3$	
10.0	$-5.0_{10}-2$	15558
	$-5.0_{10}-2$	

References

[1] J. S. Chomicz, A. Olejniczak and M. Szyszkowicz, *A method for finding the step size of integration of a system of ordinary differential equations*, Zastos. Mat. 17 (1983), p. 645-654.

[2] V. I. Krylov, V. V. Bobkov and P. I. Monastyrnyĭ (В. И. Крылов, В. В. Бобков и П. И. Монастырный), *Вычислительные методы*, т. 2, Москва 1977.

[3] M. Szyszkowicz, *Two algorithms for solving the initial value problem by using Runge-Kutta-Heun's method*, Raport Nr N-53, Instytut Informatyki Uniwersytetu Wrocławskiego, Wrocław 1978.

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Received on 16. 2. 1981