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A GENERAL SCHEME OF EQUATIONS COVERING RECTILINEARLY DRAWN SHELL STRUCTURES

1. INTRODUCTION

The paper presents a general scheme of equations covering rectilinearly drawn shell structures constructed of homogeneous isotropic material and working in a moment state. The assumed mathematical model representing the stress in shell is based on the linear theory of shells admitting that the material medium of which shells are made is subject to the principle of Hooke. The assumed model leads to some linear equations with partial derivatives called the *equations of equilibrium* and to linear differential connections between the functions describing the state of strain of the shell and the coordinates of the displacement vector of the middle inner surface of the shell. These equations, supplemented by algebraic connections between stresses, moments and functions describing the strain of the shell resulting from the assumed model of the material medium, lead to a system of equations describing the static work of the shell.

The essential problem in the theory of shell structures is the description of the strained middle inner surface with reference to that surface before straining.

In this paper, the vector $\boldsymbol{u}$ of displacement, defining the positions of the individual points of the surface and connected with the first differential form, and the vector $\boldsymbol{d}$, connected with turn and with the second differential form, have been introduced to the description of the strained surface. This way of formulation allows a better insight into the character of the work of the shell as resulting from the moment state and of introducing on these grounds new conceptions of tensors of the moment state straining ϱ_{ij} connected with the second differential form and of the bending strain ϑ_{ij} connected with the third differential form. In view of what said above, the hitherto in the theory of shell structures employed conception of the strain tensor γ_{ij} is not precise and, therefore, in this paper it is called the *tensor of momentless strain*, since it is the carrier of the momentless work being connected with the first differential form. Geometrical connections, prepared in this way, allow of a simple representation of any layer parallel to the middle inner surface of the shell. It has been proved that concerning the physical relations connecting stresses with strains, the stress in any point of the shell can be examined within the limits of the linear theory as a sum consisting of stresses resulting from both the momentless and the moment work of the shell. Furthermore, integrals describing sectional forces and moments can be evaluated and represented also by means of adequate sums composed of effects of the momentless and moment works of the shell.

An important fragment of the general scheme of equations are equations of continuity of the shell. In the hitherto published work this problem is not presented uniformly and refers most often to some particular types of shell structures. A uniform formulation of this problem can be obtained if we begin with the condition of the necessity of fulfilling the equations of Gauss and Codazzi by the coefficients of the first and second differential forms of the strained surface.

The previously introduced concept of tensors ϱ_{ij} and ϑ_{ij} facilitate this and, as a solution, gives three equations called the *equations of continuity of the shell*. These equations hold not only for shell structures but, under the assumption that $H = 0$, are valid also for flat and upright plates.

It should be underlined that most of the considerations presented n this paper refer to all shell structures and not only to rectilinearly drawn.

2. GEOMETRICAL REPRESENTATION

The middle surfaces of rectilinearly drawn shells are made by straight lines called *rectilinear generators*. This means that through each point of the rectilinearly drawn surface a straight line can be passed fully lying on it. If the given curve $\boldsymbol{p}(u^2)$ is intersected at each of its points by rectilinear generators of directions defined by the unitary vector $\boldsymbol{l}(u^2)$, then the vector equation of the rectilinearly drawn surface is of the form

$$\boldsymbol{r} = \boldsymbol{p}(u^2) + u^1 \boldsymbol{l}(u^2), \tag{2.1}$$

where u^1 and u^2 are curvilinear coordinates on the surface; u^1 gives the position of the point on the generator, and u^2 indicates the generator on which the point lies. Differentiating equation (2.1) with respect to the parameters u^1 and u^2, we get the following vectors tangential to the rectilinearly drawn surface (Fig. 1):

$$\boldsymbol{r}_1 = \boldsymbol{l} \quad \text{and} \quad \boldsymbol{r}_2 = \boldsymbol{p}_2 + u^1 \boldsymbol{l}_2.$$

The symbols $\boldsymbol{r}_i$, $\boldsymbol{p}_i$ and $\boldsymbol{l}_i$ indicate the derivatives with respect to the variable u^i. Introducing the unitary vector $\boldsymbol{t}$, tangential to the line u^2, we can write the derivative $\boldsymbol{r}_2$ also in the form

$$\boldsymbol{r}_2 = \sqrt{g_{22}}\,\boldsymbol{t}, \tag{2.2}$$

where g_{22} is the coefficient of the first differential form.

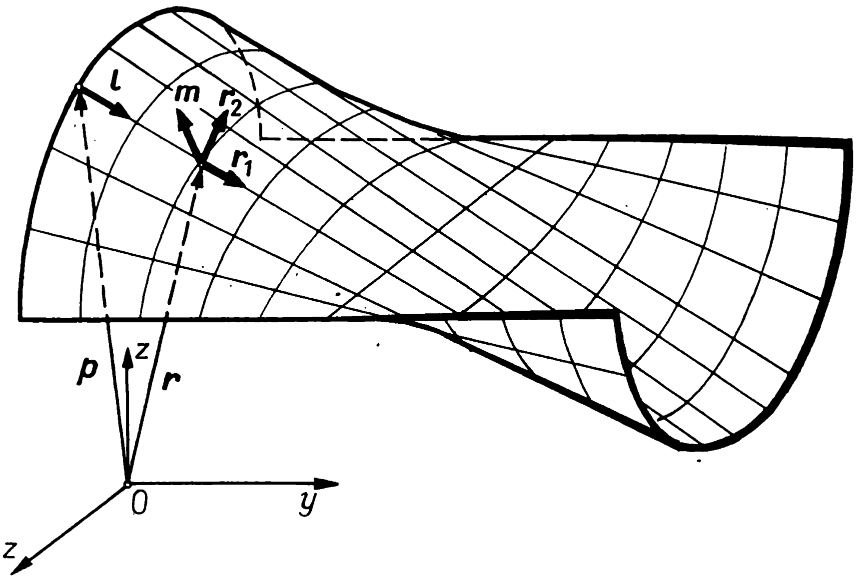

Fig. 1

The second derivatives of the vector $\boldsymbol{r}$ are the following:

$$\boldsymbol{r}_{11} = 0, \quad \boldsymbol{r}_{12} = \boldsymbol{r}_{21} = \boldsymbol{l}_2, \quad \boldsymbol{r}_{22} = \boldsymbol{p}_{22} + u^1\boldsymbol{l}_{22}.$$

Then the derivative evaluated from (2.2) is

$$\boldsymbol{r}_{22} = \frac{\partial\sqrt{g_{22}}}{\partial u^2}\,\boldsymbol{t} + g_{22}\varkappa\boldsymbol{n},$$

where $\varkappa$ is the curvature of the line u^2, and $\boldsymbol{n}$ the vector of the normal principal curve u^2.

Knowing the first and second derivatives of the vector $\boldsymbol{r}$, we can evaluate the coefficients of the first differential form g_{ij} and of the second differential form b_{ij} as well as their discriminants g and b. We obtain

$$\begin{gathered} g_{11} = 1, \quad g_{12} = g_{21} = \boldsymbol{l}\boldsymbol{p}_2 = \sqrt{g_{22}}\,\boldsymbol{t}\boldsymbol{l}, \\ g_{22} = |\boldsymbol{p}_2 + u^1\boldsymbol{l}_2|^2, \quad g = g_{22}[1-(\boldsymbol{l}\boldsymbol{t})^2], \\ b_{11} = 0, \quad b_{12} = b_{21} = \boldsymbol{l}_2\boldsymbol{m}, \\ b_{22} = g_{22}\varkappa\boldsymbol{n}\boldsymbol{m}, \quad b = -(\boldsymbol{l}_2\boldsymbol{m})^2. \end{gathered} \tag{2.3}$$

The unitary vector $\boldsymbol{m}$ occurring in (2.3) is normal to the middle inner surface of the shell. This vector is perpendicular to vectors $\boldsymbol{l}$ and $\boldsymbol{t}$ and, therefore, can be represented by the vector product

$$\boldsymbol{m} = \frac{\boldsymbol{l}\times\boldsymbol{t}}{|\boldsymbol{l}\times\boldsymbol{t}|}.$$

The Christoffel symbols of the second kind for rectilinearly drawn surfaces in the assumed system of coordinates are given by the formulas

$$(2.4)\quad \begin{cases} \Gamma^1_{11} = 0, \quad \Gamma^2_{11} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = -\dfrac{g_{12}}{2g}\dfrac{\partial g_{22}}{\partial u^1}, \\ \Gamma^2_{12} = \Gamma^2_{21} = \dfrac{1}{2g}\dfrac{\partial g_{22}}{\partial u^1}, \quad \Gamma^1_{22} = -\dfrac{g_{22}}{2g}\left[\dfrac{\partial g_{22}}{\partial u^1} - 2\sqrt{g_{22}}\,\dfrac{\partial \boldsymbol{l}\boldsymbol{t}}{\partial u^2}\right], \\ \Gamma^2_{22} = \dfrac{1}{2g}\left[g_{12}\dfrac{\partial g_{22}}{\partial u^1} + \dfrac{\partial g}{\partial u^2}\right]. \end{cases}$$

In an orthogonal system of coordinates, the Christoffel symbols (2.4) are considerably simplified:

$$(2.5)\quad \begin{gathered} \Gamma^1_{11} = 0, \quad \Gamma^2_{11} = 0, \quad \Gamma^1_{12} = \Gamma^1_{21} = 0, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2g}\frac{\partial g}{\partial u^1}, \\ \Gamma^1_{22} = -\frac{1}{2}\frac{\partial g}{\partial u^1}, \quad \Gamma^2_{22} = \frac{1}{2g}\frac{\partial g}{\partial u^2}. \end{gathered}$$

The rectilinearly drawn surfaces with regard to their construction can be devided into two groups: evolving surfaces and all others. With regard to evaluation we devide rectilinear surfaces into three groups: I. evolving surfaces, II. helical surfaces and III. all other rectilinearly drawn surfaces.

If K is the Gauss curvature and H the mean curvature, the individual groups have the following characteristics.

Group I:

$$b_{11} = 0, \quad b_{12} = b_{21} = 0, \quad b_{22} = g_{22}\varkappa\boldsymbol{n}\boldsymbol{m}, \quad K = 0, \quad H \neq 0.$$

Group II:

$$b_{11} = 0, \quad b_{12} = b_{21} = \boldsymbol{l}_2\boldsymbol{m}, \quad b_{22} = 0, \quad K \neq 0, \quad H = 0.$$

Group III:

$$b_{11} = 0, \quad b_{12} = b_{21} = \boldsymbol{l}_2\boldsymbol{m}, \quad b_{22} = g_{22}\varkappa\boldsymbol{n}\boldsymbol{m}, \quad K \neq 0, \quad H \neq 0.$$

3. GEOMETRICAL RELATIONS OF THE SHELL

We can precisely define the position of a surface in the space by its first and second differential forms. It follows that to define any strained surface it is necessary to know its first and second differential forms. Let the inner middle surface of the shell $\boldsymbol{r} = \boldsymbol{r}(u^1, u^2)$ turn after the straining

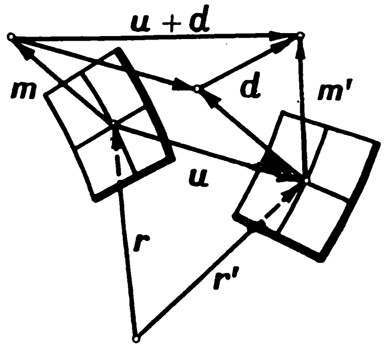

Fig. 2

into the surface $\boldsymbol{r}' = \boldsymbol{r}'(u^1, u^2)$, and let $\boldsymbol{u}$ be the displacement vector (Fig. 2). Then the *strained surface* is defined by the formula

$$\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u},$$

which is a function of the points of the strained middle inner surface of the shell.

3.1. Coefficients of the first and second differential forms of the strained surface. The coefficients of the first differential form of the strained surface g'_{ij}, after having introduced the concept of the momentless strain tensor γ_{ij} defined (see [4]) by

$$\gamma_{ij} = \tfrac{1}{2}(g'_{ij} - g_{ij}), \tag{3.1}$$

are given by the formula

$$g'_{ij} = g_{ij} + 2\gamma_{ij}. \tag{3.2}$$

Introducing the concept of the moment strain tensor ϱ_{ij}, the components of which are represented by the coefficients of the second differential form similarly to (3.1),

$$\varrho_{ij} = \tfrac{1}{2}(b'_{ij} - b_{ij}), \tag{3.3}$$

we are able to define the coefficients of the second differential form of the strained surface b'_{ij} by the formula

$$b'_{ij} = b_{ij} + 2\varrho_{ij}. \tag{3.4}$$

3.2. Connection of the components of displacement with the tensor of strain. We define the displacement vector $\boldsymbol{u}$ in the basis $\boldsymbol{r}_k$ and $\boldsymbol{m}$. Then

$$\boldsymbol{u} = w^k \boldsymbol{r}_k + w^3 \boldsymbol{m}, \tag{3.5}$$

where w^k and w^3 are counter-variant components of the displacement vector $\boldsymbol{u}$.

For minor displacements it can be assumed [4] that

$$\gamma_{ij} = \tfrac{1}{2}(\boldsymbol{r}_i \boldsymbol{u}_{,j} + \boldsymbol{r}_j \boldsymbol{u}_{,i}), \tag{3.6}$$

where the sign , (comma) denotes the common derivative. Introducing in (3.6) the proper derivatives evaluated from (3.5) we have the connection of the tensor of strain with the displacement vector,

$$\gamma_{ij} = \tfrac{1}{2}(w^k|_j g_{ik} + w^k|_i g_{jk}) - w^3 b_{ij},$$

where the vertical line signifies the co-variant derivative.

It is obvious that to determine the shape of the examined surface it is necessary to know the coefficients of the second differential form. If the surface under the examination is strained, it is not sufficient to know the displacement vector $\boldsymbol{u}$ only for its unmistakable definition, but it is necessary that the vector $\boldsymbol{d}$ (Fig. 2), which is strictly connected with the shape of the surface, be also given. Any point of the strained surface will be unmistakably defined with respect to the middle inner surface of the unstrained shell if the vector functions $\boldsymbol{u}$ and $\boldsymbol{d}$ are known. The introduction in this paper of the new concept of the vector $\boldsymbol{d}$, which is connected with turn, has an essential influence upon results obtained.

Let us consider now the connection of the moment strain tensor with displacements $\boldsymbol{u}$ and $\boldsymbol{d}$. Now, evaluate the tensor ϱ_{ij} for minor displacements.

Introduce in the formula $b'_{ij} = -\boldsymbol{r}'_i \boldsymbol{m}'_j$ the appropriate derivatives taken from the sums (see Fig. 2) $\boldsymbol{r}' = \boldsymbol{r} + \boldsymbol{u}$ and $\boldsymbol{m}' = \boldsymbol{m} + \boldsymbol{d}$. Then we obtain

$$b'_{ij} = b_{ij} - \boldsymbol{m}_j \boldsymbol{u}_i - \boldsymbol{r}_i \boldsymbol{d}_j - \boldsymbol{u}_i \boldsymbol{d}_j .$$

Neglecting the scalar product of $\boldsymbol{u}'_{,i}$ and $\boldsymbol{d}_{,j}$ as a minor quantity of higher order, using (3.3) and assuming the simplification, we obtain

$$\varrho_{ij} = -\tfrac{1}{2}(\boldsymbol{m}_{,j} \boldsymbol{u}_{,i} + \boldsymbol{r}_i \boldsymbol{d}_{,j}). \tag{3.7}$$

Let us resolve the vector $\boldsymbol{d}$ within the basis $\boldsymbol{r}_k$ and $\boldsymbol{m}$. We have

$$\boldsymbol{d} = -\delta^k \boldsymbol{r}_k - \delta^3 \boldsymbol{m}. \tag{3.8}$$

Then, let us evaluate the derivatives $\boldsymbol{u}_{,i}$ and $\boldsymbol{d}_{,j}$ from expressions (3.5) and (3.8) and introduce them in (3.7). We obtain

$$(3.9)\qquad \varrho_{ij} = \tfrac{1}{2}(w^k|_i b_{jk} + \delta^k|_j g_{ik}) - \tfrac{1}{2}(w^3 b_{ik} b_j^k + \delta^3 b_{ij}),$$

where the quantities δ^i are connected with the tensor of displacement w^i by the formulas

$$\delta^i = (w^k b_{kj} + w^3_{,j}) g^{ij} \quad \text{and} \quad \delta^3 = b_k^k w^3 - w^k|_k - 1 + \sqrt{\frac{g'}{g}}.$$

3.3. Geometrical representation of a layer parallel to the middle inner surface of the shell. Let a surface, every point of which is at a distance z from the middle inner surface of the shell structure, be called the *parallel layer*. We define this layer by a vector equation

$$\boldsymbol{R} = \boldsymbol{r} + z\boldsymbol{m}.$$

Let the coefficients of the first differential form of the parallel layer be stated by G_{ij}. They are defined (see [4]) by the formula $G_{ij} = \boldsymbol{R}_i \boldsymbol{R}_j$.

Employing formulas, given in manuals of shell structures theory, e.g. [2], we write

$$(3.10)\qquad G_{ij} = g_{ij} - 2zb_{ij} + z^2 b_{ik} b_j^k$$

or, equivalently,

$$G_{ij} = g_{ij}(1 - Kz^2) - 2b_{ij}(1 - Hz)z,$$

where $K = b/g$ is the Gauss curvature, and $H = \frac{1}{2} g^{kl} b_{kl}$ the mean curvature. The counter-variant component of tensor G^{ij} can be given by the formula

$$(3.11)\qquad G^{ij} = [g^{ij}(1 - Kz^2) - 2K\bar{b}^{ij}(1 - Hz)z]\frac{g}{G}.$$

The counter-variant tensor $\bar{b}^{ij}$ occurring in this formula is connected with b_{ij} by

$$\bar{b}^{11} = \frac{b_{22}}{b}, \quad \bar{b}^{12} = \bar{b}^{21} = -\frac{b_{12}}{b}, \quad \bar{b}^{22} = \frac{b_{11}}{b},$$

where $b = |b_{ij}|$ is the discriminant of the second differential form. For the discriminant G of the quadratic form of G_{ij} we have (see [2])

$$(3.12)\qquad G = g(1 - 2Hz + Kz^2)^2.$$

Neglecting in expressions (3.10), (3.11) and (3.12) the infinitely small quantities of second order, for minor displacements, we obtain

$$(3.13)\qquad G_{ij} = g_{ij} - 2zb_{ij}, \quad G^{ij} = (g^{ij} - 2Kz\bar{b}^{ij})\frac{g}{G}, \quad G = (l - 2Hz)^2 g.$$

3.4. Strain tensor of any layer parallel to the middle inner surface of the shell. Examining the strain of a parallel layer, we rely on the first principle of Kirchhoff which says that in the case of thin shells it can be assumed that fibres perpendicular to the middle inner surface of the shell retain the same position also after straining, not changing at the same time their lengths.

Let the strain of the parallel layer be described by the vector $\boldsymbol{U}$ which is defined (see [3]) by the equation

$$\boldsymbol{U} = \boldsymbol{u} + z\boldsymbol{d}. \tag{3.14}$$

The vector equation of the strained layer is of the form

$$\boldsymbol{R}' = \boldsymbol{R} + \boldsymbol{U}. \tag{3.15}$$

We evaluate the coefficients G'_{ij} of first differential form of the strained parallel layer by the formula $G'_{ij} = \boldsymbol{R}'_i \boldsymbol{R}'_j$. Making use of (3.14) and (3.15), neglecting the product $\boldsymbol{U}_{,i}\boldsymbol{U}_{,j}$ as a minor quantity of higher order and employing quantities (3.6) and (3.7), we obtain

$$G'_{ij} = G_{ij} + 2\gamma_{ij} - 2(\varrho_{ij} + \varrho_{ji})z + (\boldsymbol{m}_{,i}\boldsymbol{d}_{,j} + \boldsymbol{m}_{,j}\boldsymbol{d}_{,i})z^2.$$

Introducing the concept of the bending strain tensor ϑ_{ij}, the components of which are defined by the formula

$$\vartheta_{ij} = \tfrac{1}{2}(\boldsymbol{m}_{,i}\boldsymbol{d}_{,j} + \boldsymbol{m}_{,j}\boldsymbol{d}_{,i}),$$

employing at the same time the symmetry of the tensor ϱ_{ij}, we have

$$G'_{ij} = G_{ij} + 2\gamma_{ij} - 4\varrho_{ij}z + 2\vartheta_{ij}z^2.$$

The strain tensor of any parallel layer, defined by the difference

$$\gamma^*_{ij} = \tfrac{1}{2}(G'_{ij} - G_{ij}),$$

is given by the expression

$$\gamma^*_{ij} = \gamma_{ij} - 2\varrho_{ij}z + \vartheta_{ij}z^2. \tag{3.16}$$

4. PHYSICAL RELATIONS

Physical connections establish the interrelations between strains and stresses upon the basis of the principle of Hooke. In technical publications dealing with shell structures these relations are given as a rule for curvate systems of the curvilinear coordinates. The unquestionable advantage of a curvate system lies in its being an orthogonal system of coordinates not only on the middle inner surface of the shell, but in any parallel layer also. Whereas transferring the characteristics of a curvate system to any orthogonal system related to the middle inner surface of the shell not always gives accurate results.

Although the curvate system is advantageous for physical connections it need not to be the most proper system for defining other quantities and dependences occurring in a shell structure. The proper system of coordinates for rectilinearly drawn shells is a system basing on a group of rectilinear generators since such a system affords the simplest parametrisation of this group of shells.

4.1. Stresses and strains. We get the connections between stresses and strains for thin shells from the generalized principle of Hooke. In any curvilinear system of coordinates these connections are expressed by the formula

$$\tau^{ij} = [\lambda^* G^{ij} G^{mn} + \mu (G^{im} G^{jn} + G^{in} G^{jm})] \gamma^*_{mn} \tag{4.1}$$

or

$$\tau^{ij} = \lambda^* G^{ij} G^{mn} \gamma^*_{mn} + 2\mu \gamma^{*ij},$$

where τ^{ij} is the counter-variant tensor of the stress and γ^{*ij} is the counter-variant tensor defined in the basis G^{ij}. The parameter λ^* is connected with the constants of elasticity, the factors of Lame λ and μ, by the formula

$$\lambda^* = \frac{2\mu\lambda}{\lambda + 2\mu}. \tag{4.2}$$

Relation (4.2) results from the simplifying supposition adopted in the linear theory of shells [2] assuming that the stresses perpendicular to the middle inner surface of the shell are of minor quantity and can be neglected (the assumption of Kirchhoff).

Including in (4.1) expressions (3.13) and (3.16) and neglecting products comprising z^2 and higher powers of z as minor quantities of second order, we have

$$\begin{aligned} \tau^{ij} = \Big\{ & [\lambda^* g^{ij} g^{mn} + \mu (g^{im} g^{jn} + g^{in} g^{jm})] \gamma_{mn} - \\ & - 2z \Big[\big(\lambda^* g^{ij} g^{mn} + \mu (g^{im} g^{jn} + g^{in} g^{jm})\big) \varrho_{mn} + \\ & + K \big(\lambda^* (g^{ij} b^{mn} + g^{mn} b^{ij}) + \mu (g^{im} b^{jn} + g^{jn} b^{im} + g^{in} b^{jm} + g^{jm} b^{in})\big) \gamma_{mn} \Big] \Big\} \left(\frac{g}{G}\right)^2. \end{aligned} \tag{4.3}$$

Introducing the concepts of tensors of momentless stress $\bar{\tau}^{ij}$ and of moment stress $\hat{\tau}^{ij}$, defined by the expressions

$$\bar{\tau}^{ij} = \lambda^* g^{ij} g^{mn} \gamma_{mn} + 2\mu \gamma^{ij}$$

and

$$\hat{\tau}^{ij} = \lambda^* g^{ij} g^{mn} \varrho_{mn} + 2\mu \varrho^{ij} + K [\lambda^* (g^{ij} b^{mn} + g^{mn} b^{ij}) \gamma_{mn} + 2\mu (b^{jn} \gamma^i_n + b^{in} \gamma^j_n)],$$

we are able to express formula (4.3) in the form

$$\tau^{ij} = (\bar{\tau}^{ij} - 2z\hat{\tau}^{ij})\left(\frac{g}{G}\right)^2. \tag{4.4}$$

4.2. Inner forces. We evaluate the sectional forces (stresses and moments) on the basis of formulas given in [4]. Then the counter-variant tensors of tensions N^{ij} and the counter-variant tensors of moments M^{ij} are expressed by

$$\begin{aligned} N^{ij} &= \int_{-h}^{h} \sqrt{\frac{G}{g}}\,(\delta^j_k - zb^j_k)\bar{\tau}^{ik}\,dz, \\ M^{ij} &= \int_{-h}^{h} \sqrt{\frac{G}{g}}\,(\delta^j_k - zb^j_k)\tau^{ik} z\,dz. \end{aligned} \tag{4.5}$$

Putting (4.4) in expression (4.5), employing at the same time the discriminant G from (3.13) and neglecting minor quantities of higher order, we have

$$\begin{aligned} N^{ij} &= \int_{-h}^{h} \frac{1}{(1-2Hz)^3}\,[\bar{\tau}^{ij} - zF^{ij}]\,dz, \\ M^{ij} &= \int_{-h}^{h} \frac{z}{(1-2Hz)^3}\,[\bar{\tau}^{ij} - zF^{ij}]\,dz, \end{aligned} \tag{4.6}$$

where $F^{ij} = 2\hat{\tau}^{ij} + b^j_k\bar{\tau}^{ik}$. The integrated expressions (4.6) are equal

$$N^{ij} = J_0\bar{\tau}^{ij} - J_1F^{ij}, \qquad M^{ij} = J_1\bar{\tau}^{ij} - J_2F^{ij}, \tag{4.7}$$

where the integrals J_n, for a simplified version under the assumption that $2hH \ll 1$, are given by

$$J_0 = 2h, \quad J_1 = 4h^3H, \quad J_2 = \frac{2}{3}h^3.$$

Eliminating the function F^{ij} from the system of equations (4.7) we have

$$N^{ij} = 2h[1 - 3(2hH)^2]\bar{\tau}^{ij} + 6HM^{ij},$$

and neglecting the minor quantity of second order in brackets we can write

$$N^{ij} = 2h\bar{\tau}^{ij} + 6HM^{ij}. \tag{4.8}$$

Since $2h$ represents the thickness of the shell, the product $2h\bar{\tau}^{ij}$ is the counter-variant tensor of the momentless force. Let this tensor be $\bar{N}^{ij}$; then $\bar{N}^{ij} = 2h\bar{\tau}^{ij}$ which put in (4.8) makes

$$N^{ij} = \bar{N}^{ij} + 6HM^{ij}.$$

The sectional forces (tensions) are then the sums composed of influences of the momentless work $\bar{N}^{ij}$ and of the moment work $\hat{N}^{ij} = 6HM^{ij}$.

5. EQUATIONS OF EQUILIBRIUM

Let us express the general scheme of equilibrium equations for shells in the vector notation (on the basis of [4]) by

$$
\begin{gathered}
N^{ij}|_i - Q^i b_i^j + P^j = 0, \qquad N^{ij} b_{ij} + Q^j|_j + P^3 = 0, \\
M^{ij}|_i - Q^j = 0,
\end{gathered} \tag{5.1}
$$

where the quantities Q^i are the counter-variant tensors of shearing forces, and P^j and P^3 are counter-variant load tensors. If we state the outward oad by the vector $\boldsymbol{P}$, then

$$
\boldsymbol{P} = P^i \boldsymbol{r}_i + P^3 \boldsymbol{m}.
$$

Then, the system of equations (5.1) set down for rectilinearly drawn shells for which the geometrical quantities are stated by formulas (2.4) and (2.5) is the following:

$$
\left\{
\begin{aligned}
&\frac{\partial \sqrt{g} N^{11}}{\partial u^1} + \frac{\partial \sqrt{g} N^{12}}{\partial u^2} + 2\Gamma_{12}^1 \sqrt{g} N^{12} + \Gamma_{22}^1 \sqrt{g} N^{22} + \\
&\qquad + \frac{1}{\sqrt{g}} [g_{12} b_{12} Q^1 - (g_{22} b_{12} - g_{12} b_{22}) Q^2] + \sqrt{g} P^1 = 0, \\
&\frac{\partial \sqrt{g} N^{12}}{\partial u^1} + \frac{\partial \sqrt{g} N^{22}}{\partial u^2} + 2\Gamma_{12}^2 \sqrt{g} N^{12} + \Gamma_{22}^2 \sqrt{g} N^{22} + \\
&\qquad + \frac{1}{\sqrt{g}} [b_{12} Q^1 + (b_{22} - g_{12} b_{12}) Q^2] + \sqrt{g} P^2 = 0, \\
&2 b_{12} N^{12} + b_{22} N^{22} + P^3 = 0, \\
&\frac{\partial \sqrt{g} M^{11}}{\partial u^1} + \frac{\partial \sqrt{g} M^{21}}{\partial u^2} + \Gamma_{12}^1 \sqrt{g} (M^{12} + M^{21}) + \Gamma_{22}^1 \sqrt{g} M^{22} - \sqrt{g} Q^1 = 0, \\
&\frac{\partial \sqrt{g} M^{12}}{\partial u^1} + \frac{\partial \sqrt{g} M^{22}}{\partial u^2} + \Gamma_{12}^2 \sqrt{g} (M^{1\prime} + M^{21}) + \Gamma_{22}^2 \sqrt{g} M^{22} - \sqrt{g} Q^2 = 0.
\end{aligned}
\right. \tag{5.2}
$$

In expressions (5.2) we have assumed the symmetry of the counter-variant tensor of stresses $N^{ij} = N^{ji}$.

Transition from tensor coordinates to physical coordinates, i.e. related to the unitary basis of the middle inner surface of the shell, is accomplished as follows (1):

$$N_{ij}^{\urcorner} = \sqrt{\frac{g_{jj}}{g^{ii}}}\, N^{ij}, \qquad Q_i^{\urcorner} = \sqrt{\frac{1}{g^{ii}}}\, Q^i,$$

$$M_{i1}^{\urcorner} = -\sqrt{\frac{gg^{11}}{g^{ii}}}\, M^{i2}, \qquad M_{i2}^{\urcorner} = \sqrt{\frac{gg^{22}}{g^{ii}}}\, M^{i1},$$

$$P_i^{\urcorner} = \sqrt{g_{ii}}P^i, \qquad P_3^{\urcorner} = P^3.$$

6. EQUATIONS OF CONTINUITY

In order that the first and second differential forms of the strained surface bear the same character in certain surrounding of any point of that area, it is necessary and, at the same time, it is sufficient that the discriminant of the first form g' be positively determined and that the coefficients g'_{ij} and b'_{ij} satisfy the equations of Gauss and Codazzi. The discriminant g' evaluated from (3.2) after neglecting quantities of higher order is given by the formula $g' = g(1+2g^{mn}\gamma_{mn})$.

The equation of Gauss of a strained surface has the form (see [1])

$$R_{1212} = b'_{11}b'_{22} - b'_{12}b'_{21},$$

where R_{1212} is the Riemann tensor of a surface. The tensor R_{1212} can be stated by means of coefficients of the first differential form. Employing the results of [4] for the infinitesimal state of a strain we write

$$R_{1212} = \overset{\circ}{R}_{1212} + 2\gamma_{12}|_{12} - \gamma_{11}|_{22} - \gamma_{22}|_{11}, \tag{6.1}$$

where $\overset{\circ}{R}_{1212}$ is the Riemann tensor of a surface before the straining. It is known from differential geometry that $\overset{\circ}{R}_{1212} = b$. Substituting then in (6.1) the discriminants of the second differential forms for the Riemann tensors, we have

$$b' = b + 2\gamma_{12}|_{12} - \gamma_{12}|_{22} - \gamma_{22}|_{11}. \tag{6.2}$$

On the other hand, the discriminant b' can be evaluated from expression (3.4). Neglecting products containing infinitely small quantities of higher order and raising discriminants ij in basis b_{ij}, similarly as it was done in (3.11), we have

$$b' = b(1+2\bar{b}^{mn}\varrho_{mn}). \tag{6.3}$$

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(1) j is not to be summed up after i, and the sign $\urcorner$ means physical coordinate.

Comparing (6.3) with (6.2) results in

$$2b\bar{b}^{mn}\varrho_{mn} = 2\gamma_{12}|_{12} - \gamma_{11}|_{22} - \gamma_{22}|_{11}. \tag{6.4}$$

We write the equation of Codazzi for a strained surface in the tensor formulation [1] as

$$b'_{ij}\dagger_k - b_{ik}\dagger_j = 0, \tag{6.5}$$

where the symbol $\dagger$ denotes a co-variant derivative evaluated on the strained surface. There are only the following two independent equations of Codazzi (6.5):

$$b'_{11}\dagger_2 - b'_{12}\dagger_1 = 0 \quad \text{and} \quad b'_{22}\dagger_1 - b_{21}\dagger_2 = 0. \tag{6.6}$$

Putting expression (3.4) in (6.6), we have explicite

$$\begin{aligned} 2(\varrho_{11}\dagger_2 - \varrho_{12}\dagger_1) + b_{11}\dagger_2 - b_{12}\dagger_1 &= 0, \\ 2(\varrho_{22}\dagger_1 - \varrho_{21}\dagger_2) + b_{22}\dagger_1 - b_{21}\dagger_2 &= 0. \end{aligned} \tag{6.7}$$

The knowledge of Christoffel symbols Γ'^k_{ij} of a strained surface makes it possible to evaluate the derivatives of expressions (6.7). For an infinitesimal state of a strain these symbols equal

$$\Gamma'^k_{ij} = \Gamma^k_{ij} - \gamma^{km}\Gamma_{ijm} + g^{km}\gamma_{ijm},$$

where Γ_{ijm} are Christoffel symbols of the first kind, and

$$\gamma_{ijm} = \gamma_{jm,i} + \gamma_{mi,j} - \gamma_{ij,m}.$$

Knowing the symbols Γ'^k_{ij}, we evaluate the derivatives of expression (6.7) and, after neglecting minor quantities of higher order, we obtain the following scheme of Codazzi equations:

$$\begin{aligned} &2(\varrho_{11}|_2 - \varrho_{12}|_1) - 2H(\gamma_{11}|_2 - \gamma_{12}|_1) + b^k_2\gamma_{k1}|_1 - \\ &\qquad\qquad - b^k_1\gamma_{k2}|_1 + (b^k_2\Gamma^m_{11} - b^k_1\Gamma^m_{12})\gamma_{km} = 0, \\ &2(\varrho_{22}|_1 - \varrho_{21}|_2) - 2H(\gamma_{22}|_1 - \gamma_{21}|_2) + b^k_1\gamma_{k2}|_2 - b^k_2\gamma_{kl}|_2 + \\ &\qquad\qquad + (b^k_1\Gamma^m_{22} - b^k_2\Gamma^m_{12})\gamma_{km} = 0. \end{aligned} \tag{6.8}$$

The evolving shell surfaces of group I parametrized in the orthogonal system of coordinates simplify equations (6.8) as follows:

$$\begin{aligned} \varrho_{11}|_2 - \varrho_{12}|_1 &= H(\gamma_{11}|_2 - 2\gamma_{12}|_1) = 0, \\ \varrho_{22}|_1 - \varrho_{21}|_2 &= H(\gamma_{22}|_1 + \Gamma^2_{12}\gamma_{22}) = 0. \end{aligned}$$

Equations (6.8) together with formula (6.4) are called *equations of continuity for shell structures*. These equations must be absolutely satisfied for an infinitesimal state of strain if the strained middle inner surface of the shell has to be a regular surface.

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OGÓLNY UKŁAD RÓWNAŃ POWŁOK PROSTOKREŚLNYCH

STRESZCZENIE

Praca omawia ogólny układ równań powłok prostokreślnych. Przyjęty mode matematyczny, opisujący stan naprężenia w powłoce, oparto na linio wej teorii powłok przy czym założono, że ośrodek materialny, z którego utworzone są powłoki, jest ośrodkiem Hooke'a. Model ten prowadzi do układu liniowych równań o pochodnych cząstkowych, zwanych *równaniami równowagi*, oraz do liniowych, różniczkowych związków między funkcjami opisującymi stan odkształcenia powłoki a współrzędnymi wektora przemieszczenia powierzchni środkowej powłoki. Równania te, uzupełnione algebraicznymi związkami między napięciami i momentami a funkcjami opisującymi stan odkształcenia powłoki, wynikającymi z przyjętego modelu ośrodka, prowadzą do układu równań opisujących statyczną pracę powłoki.