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ON BOUNDS FOR THE ASYMPTOTIC POWER AND ON PITMAN EFFICIENCIES OF THE CRAMÉR - VON MISES TEST

1. Introduction and power bounds. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from absolutely continuous distributions F and G, respectively. Several standard tests of the hypothesis H: $F(x) \equiv G(x)$ against the two-sided alternatives A: $G(x) = F(x-\theta)$, $\theta \neq 0$, are defined in terms of F. If, however, the true distributions of X and Y are $\Psi(x)$ and $\Psi(x-\theta)$, respectively, with Ψ not necessarily equal to F, then these standard tests are no longer optimal. Pitman ARE (asymptotic relative efficiency) will be used to compare the Cramér - von Mises test with standard optimal tests. In this section, we shall present the notation and derive bounds for the limiting power of the Cramér - von Mises test and an expression for the limiting power of the Neyman test.

Let $N = m+n$, $\lambda = m/N$, $1-\lambda = n/N$. We denote the empirical d.f.'s (distribution functions) of the X's and Y's by $F_m(x)$ and $G_n(x)$, respectively. The tests to be considered are based on the following statistics:

(i) The *Cramér - von Mises statistic*

$$M = \frac{mn}{m+n} \int\limits_{-\infty}^{\infty} \big(F_m(x) - G_n(x)\big)^2 dH_N(x),$$

where $H_N(x) = \lambda F_m(x) + (1-\lambda) G_n(x)$.

(ii) The *rank statistic*

$$T^* = \sum_{j=1}^{N} \mathrm{E}[g(V^{(j)})] Z_j,$$

where $V^{(j)}$ is the j-th order statistic in the joint sample of X's and Y's, Z_j equals zero or one according to whether $V^{(j)}$ is X or Y, and $g(x) = -f'(x)/f(x)$, f' being the derivative of f, the density of F. Two-tailed tests based on T^* are locally most powerful among unbiased rank tests.

(iii) The *Neyman* or c_a-*statistic*

$$T = \lambda \sum_{j=1}^{n} g(Y_j) - (1-\lambda) \sum_{i=1}^{m} g(X_i), \tag{1.1}$$

where the function g is defined in (ii). Two-tailed tests based on T are locally asymptotically most powerful among unbiased tests.

The test statistics defined above are for the two-sample case. For testing H: $F \equiv G$ against A: $G(x) = F(x-\theta)$, $\theta \neq 0$, the respective tests based on M, T, or T^* reject H if M, $|T|$, or $|T^* - \mathrm{E}T^*|$ is too large.

We assume that the d.f.'s F satisfy conditions 1-7 of [12], p. 1596.

The definition of ARE appears in [10]. Let $e \equiv e_{MT}(F; \Psi)$ denote the ARE computed under $\Psi(x)$ of the Cramér - von Mises test with respect to the Neyman test derived for $F(x)$, let $e^+ \equiv e_{TT^*}(F; \Psi)$ denote the ARE computed under $\Psi(x)$ of the Neyman test with respect to the rank test derived for $F(x)$, and let $e^* \equiv e_{MT^*}(F; \Psi)$ denote the ARE computed under $\Psi(x)$ of the Cramér - von Mises test with respect to the rank test derived for $F(x)$. In each case, $\Psi(x)$ is taken from a class of absolutely continuous d.f.'s with $\Psi'(x) = \psi(x)$ a.e.

Let $Y = \{Y(u), 0 \leqslant u \leqslant 1\}$ denote the Brownian bridge (see [3]). Φ will denote the d.f. of an $N(0,1)$ random variable, and φ will stand for its density. The upper p-level cut-off from Φ will be denoted by c_p:

$$1 - \Phi(c_p) = p. \tag{1.2}$$

Define the d.f. Q as

$$Q(x) = \mathrm{P}\left\{\int_0^1 Y^2(u)\,du \leqslant x\right\} \tag{1.3}$$

and let q_p denote the upper p-level cut-off:

$$1 - Q(q_p) = p. \tag{1.4}$$

Convergence in distribution will be denoted by

$$U_N \xrightarrow{\mathscr{D}} U.$$

Lastly, if x and y are square integrable functions whose domain is the unit interval, then

$$d(x, y) \equiv \left[\int_0^1 \big(x(u) - y(u)\big)^2 du\right]^{1/2}. \tag{1.5}$$

Note that d is a metric on the space of continuous functions on $[0, 1]$.

The following results will enable us to bound the limiting power of the Cramér - von Mises test. Rosenblatt [11], with the help of Fisz [5], proved that, under H: $F(x) \equiv G(x)$,

(1.6) $$M \xrightarrow{\mathscr{D}} \int_0^1 Y^2(u)\,du$$

as

$$\min\{m, n\} \to \infty \quad \text{and} \quad \frac{m}{N} \to \lambda \text{ with } 0 < \lambda < 1.$$

The following theorem, due to Anderson [1] and [2], p. 1152, is specialized to the case of translation alternatives and contains (1.6) as a special case.

THEOREM 1 (Anderson). *Let $\Psi(x)$ be a strictly increasing absolutely continuous d.f. with density $\psi(x)$. Assume that $\psi'(x)$ exists and is bounded for all x except finitely many and that for these values of x the left-hand and right-hand derivatives exist and are bounded. Fix a, β and λ with $0 < a < \beta < 1$ and $0 < \lambda < 1$. Let $\{\theta_k\}$ be a sequence of positive alternatives with $\theta_k \downarrow 0$ as $k \to \infty$. Let N_k be the minimal sample size necessary for the a-level Cramér - von Mises test, in which the ratio of X's to total sample size is approximately λ, to attain power β for the fixed alternative θ_k. Assume that $N_k^{1/2}\theta_k \to b > 0$ as $k \to \infty$. Then, as $k \to \infty$,*

(1.7) $$\frac{m}{N} \to \lambda \quad \textit{and} \quad M \xrightarrow{\mathscr{D}} \int_0^1 \{Y(u) + h(u)\}^2\,du,$$

where

(1.8) $$h(u) = [\lambda(1-\lambda)]^{1/2} b\,\psi\big(\Psi^{-1}(u)\big).$$

For a sequence of negative alternatives $\{\theta_k\}$, we assume that $N_k^{1/2}\theta_k \to b < 0$. Formulas (1.7) and (1.8) also hold for this case. Using (1.5) we may restate the preceding results: under the hypothesis H,

$$M \xrightarrow{\mathscr{D}} d^2(Y, 0)$$

while, under a sequence $\{\theta_k\}$ of alternatives,

$$M \xrightarrow{\mathscr{D}} d^2(Y, -h).$$

Combining (1.5) and (1.4), we obtain the limiting power of the Cramér - von Mises test (for a sequence of alternatives and Ψ satisfying Theorem 1) in the form

$$B_\Psi(b) \equiv \mathrm{P}\{d^2(Y, -h) \geqslant q_a\}.$$

PROPOSITION 1. *We have*

$$B_\Psi(b) \leqslant \mathrm{P}\left\{d(Y, 0) \geqslant q_\alpha^{1/2} - \left[\lambda(1-\lambda)b^2 \int_{-\infty}^{\infty} \psi^3(x)\,dx\right]^{1/2}\right\}, \tag{1.9}$$

$$B_\Psi(b) \geqslant \mathrm{P}\left\{d(Y, 0) \leqslant \left[\lambda(1-\lambda)b^2 \int_{-\infty}^{\infty} \psi^3(x)\,dx\right]^{1/2} - q_\alpha^{1/2}\right\}. \tag{1.10}$$

Proof. Inequality (1.9) follows from the triangle inequality applied to d and

$$d(h, 0) = \left[\lambda(1-\lambda)b^2 \int_{-\infty}^{\infty} \psi^3(x)\,dx\right]^{1/2}.$$

Inequality (1.10) follows from $|d(Y, 0) - d(h, 0)| \leqslant d(Y, -h)$ and the preceding representation of $d(h, 0)$.

The following representation is useful in deriving first a lower bound for the power of the Cramér - von Mises test and then an expression for the limiting power of the Neyman test. Let

$$L_\delta(\gamma) \equiv \Phi(-C_{\delta/2} - \gamma) + 1 - \Phi(C_{\delta/2} - \gamma). \tag{1.11}$$

Using (1.2) in conjunction with (1.11), it is easily seen that $L_\delta(\gamma)$ is minimized if $\gamma = 0$, $L_\delta(0) = \delta$, and there exists a unique positive γ^* satisfying $L_\delta(\gamma^*) = \beta$, $0 < \delta < \beta < 1$. For the Cramér - von Mises test, we obtain

PROPOSITION 2. *We have*

$$B_\Psi(b) \geqslant L_{\alpha'}(\gamma'), \tag{1.12}$$

where

$$c_{\alpha'/2} = (12 q_\alpha)^{1/2}, \qquad \alpha' = 2\left(1 - \Phi(c_{\alpha'/2})\right)$$

and

$$\gamma' = 12^{1/2} \int_0^1 h(u)\,du = [12\lambda(1-\lambda)]^{1/2} b \int_{-\infty}^{\infty} \psi^2(x)\,dx.$$

Proof. Since

$$d(Y, -h) \geqslant \left| \int_0^1 Y(u)\,du + \int_0^1 h(u)\,du \right|$$

and the integral

$$Z = \int_0^1 Y(u)\,du$$

has the $N(0, 1/12)$-distribution, (1.12) follows from a direct application of (1.11).

We now turn our attention to the two-sample Neyman test based on the statistic T of (1.1). Let α, β, λ and $\{\theta_k\}$ be the same quantities as appear in Theorem 1. Let N'_k be the minimal sample size necessary for the α-level Neyman test based on T, in which the ratio of X's to total sample size is approximately λ, to attain power β for the fixed alternative θ_k. Assume that $N_k'^{1/2}\theta_k \to b^* > 0$ as $k \to \infty$. The statistic T has a normal limiting distribution when properly standardized. For k large, under the sequence of alternatives $\{\theta_k\}$, the mean of T is approximately equal to $N'_k\theta_k\lambda(1-\lambda)\mathrm{E}_\Psi g'(X)$, and the variance σ_k^2 of T is approximately $N'_k\lambda(1-\lambda)\mathrm{Var}_\Psi g(X)$. The results appear in [8] and [12]. Now, under H,

$$\frac{T}{\sigma_k} \xrightarrow{\mathscr{D}} N(0,1),$$

so that the α-level cut-off converges to $c_{\alpha/2}$ as $k \to \infty$. Using the large-sample mean and standard deviation, we get the limiting power of the Neyman test in the form

$$L_\alpha\left(\frac{b^*[\lambda(1-\lambda)]^{1/2}\mathrm{E}_\Psi g'(X)}{[\mathrm{Var}_\Psi g(X)]^{1/2}}\right). \tag{1.13}$$

Since the limiting power is β, by letting $d_{\alpha\beta}$ denote the positive root of $L_\alpha(\gamma) = \beta$, we obtain from (1.13)

$$b^{*2} = \frac{d_{\alpha\beta}^2}{\lambda(1-\lambda)}\,\frac{\mathrm{Var}_\Psi g(X)}{[\mathrm{E}_\Psi g'(X)]^2}. \tag{1.14}$$

Yu [12] has shown that

$$c_{\alpha/2} + c_{1-\beta+\alpha/2} \leqslant d_{\alpha\beta} \leqslant c_{\alpha/2} + c_{1-\beta}. \tag{1.15}$$

If $\alpha \leqslant 0.10$ and $\beta \geqslant 0.50$, then the right-hand side of (1.15) yields a very good approximation to $d_{\alpha\beta}$.

2. Behaviour of the ARE's e and e^*. Using the results of the previous section, we shall derive bounds for the ARE of the Cramér - von Mises test compared with the Neyman test $e \equiv e_{MT}(F; \Psi)$ and for the ARE of the former test compared with the rank test $e^* \equiv e_{MT^*}(F; \Psi)$. Then, we shall know how well or badly these standard optimal tests fare compared with the Cramér - von Mises test in terms of these ARE's. We shall assume that α, β and λ are fixed with $0 < \alpha < \beta < 1$ and $0 < \lambda < 1$. Let $\mathscr{H}$ denote the class of continuous d.f.'s Ψ which satisfy the assumptions of Theorem 1, while $\mathscr{H}(a, b)$ denotes the subclass of $\mathscr{H}$ consisting of those d.f.'s Ψ which have support on some finite interval (a, b).

We begin by obtaining an upper bound for e. Let b' denote the positive value of b which makes the right-hand side of (1.9) equal to β. Using the non-negativity of the random variable $d(Y, 0)$ and relation (1.3), we obtain

$$\beta = 1-Q\left(\left(q_\alpha^{1/2}-\left[\lambda(1-\lambda)b'^2\int_{-\infty}^{\infty}\psi^3(x)\,dx\right]^{1/2}\right)^2\right). \tag{2.1}$$

But, by (1.4), $\beta = 1-Q(q_\beta)$. Comparing this with (2.1), we obtain

$$b'^2 = \frac{(q_\alpha^{1/2}-q_\beta^{1/2})^2}{\lambda(1-\lambda)\int_{-\infty}^{\infty}\psi^3(x)\,dx}. \tag{2.2}$$

Since $B_\Psi(b) = \beta \geqslant B_\Psi(b')$, we have $b' \leqslant b$. Moreover, since

$$e = \frac{\lim\limits_{k\to\infty} N'_k\theta_k^2}{\lim\limits_{k\to\infty} N_k\theta_k^2} = \frac{b^{*2}}{b^2}, \tag{2.3}$$

we conclude that (using (2.2) and (1.14))

$$e \leqslant K_2 I_2(\Psi), \tag{2.4}$$

where

$$K_2 = \left(\frac{d_{\alpha\beta}}{q_\alpha^{1/2}-q_\beta^{1/2}}\right)^2$$

and

$$I_2(\Psi) \equiv \frac{\mathrm{Var}_\Psi g(X)}{[\mathrm{E}_\Psi g'(X)]^2}\int_{-\infty}^{\infty}\psi^3(x)\,dx. \tag{2.5}$$

To obtain a lower bound for e, let $\hat{b}$ be the positive solution for b which makes the right-hand side of (1.10) equal to β. Again, by the non-negativity of $d(Y, 0)$ and (1.3), we obtain

$$\beta = Q\left(\left(\left[\lambda(1-\lambda)\hat{b}^2\int_{-\infty}^{\infty}\psi^3(x)\,dx\right]^{1/2}-q_\alpha^{1/2}\right)^2\right). \tag{2.6}$$

By (1.4), $\beta = Q(q_{1-\beta})$. Using this and (2.6), we get

$$\hat{b}^2 = \frac{(q_\alpha^{1/2}+q_{1-\beta}^{1/2})^2}{\lambda(1-\lambda)\int_{-\infty}^{\infty}\psi^3(x)\,dx}.$$

From (2.3), (1.14) and the fact that $B_\Psi(\hat{b}) \geqslant \beta = B_\Psi(b)$ implies $\hat{b} \geqslant b$ we conclude

$$K_1 I_2(\Psi) \leqslant e, \tag{2.7}$$

where

$$K_1 = \left(\frac{d_{\alpha\beta}}{q_{\alpha}^{1/2}+q_{1-\beta}^{1/2}}\right)^2.$$

The result of Proposition 2 may be used to obtain a different lower bound for e. Let b^+ denote the positive solution for b which makes the right-hand side of (1.12) equal to β. Denote this same solution for γ by $d_{\alpha'\beta}$. This yields

$$b^{+2} = \frac{d_{\alpha'\beta}^2}{12\lambda(1-\lambda)[\int_{-\infty}^{\infty}\psi^2(x)\,dx]^2}. \tag{2.8}$$

Since $B_{\Psi}(b^+) \geqslant \beta = B_{\Psi}(b)$, we have $b^+ \geqslant b$. Hence, (1.14), (2.3) and (2.8) yield

$$K_0\frac{\mathrm{Var}_{\Psi}\, g(X)}{[\mathrm{E}_{\Psi}\, g'(X)]^2}\cdot 12\Big[\int_{-\infty}^{\infty}\psi^2(x)\,dx\Big]^2 \leqslant e, \tag{2.9}$$

where

$$K_0 = \left(\frac{d_{\alpha\beta}}{d_{\alpha'\beta}}\right)^2.$$

It is easy to prove that $K_0 \leqslant 1$ by showing $\alpha' \leqslant \alpha$, where α' may be computed for a given α from Proposition 2.

For the ARE of the Neyman test compared with the rank test, Mikulski [8] proved the following result for one-sided alternatives and Yu [12] extended it to two-sided alternatives:

$$e^+ = \frac{[\mathrm{E}_{\Psi} g'(X)]^2}{\mathrm{Var}_{\Psi} g(X)}\,\frac{\mathrm{Var}_F\, g(X)}{[\int_{-\infty}^{\infty} J'(\Psi(x))\,\psi^2(x)\,dx]^2},$$

where

$$J(u) = g(F^{-1}(u)) \quad \text{and} \quad J'(u) = \frac{d}{du}J(u).$$

Since $e^* = ee^+$, the bounds on e^* are

$$e^* \leqslant K_2 I(f) I_3(\Psi), \tag{2.10}$$

$$K_1 I(f) I_3(\Psi) \leqslant e^*, \tag{2.11}$$

$$K_0\left(\frac{\int_{-\infty}^{\infty}\psi^2(x)\,dx}{\int_{-\infty}^{\infty} J'(\Psi(x))\,\psi^2(x)\,dx}\right)^2\cdot 12 I(f) \leqslant e^*,$$

where $I(f) = \mathrm{Var}_F\, g(X)$ is Fisher's information for F which is independent of Ψ,

$$I_3(\Psi) \equiv \frac{\int_{-\infty}^{\infty} \psi^3(x)\,dx}{\left[\int_{-\infty}^{\infty} J'(\Psi(x))\psi^2(x)\,dx\right]^2},$$

and the constants K_0, K_1 and K_2 have previously been defined.

We now turn our attention to using these bounds for the ARE's to describe how well or how badly the Cramér - von Mises test may be compared with the standard optimal tests. At first, we consider the case $F \equiv \Phi$, the $N(0,1)$ d.f. Here, $g(x) = x$ so that $g'(x) = 1$. For this special case, $I_1(\Psi)$ denotes the functional $I_2(\Psi)$, i.e.

$$I_1(\Psi) \equiv \mathrm{Var}_\Psi X \int_{-\infty}^{\infty} \psi^3(x)\,dx. \tag{2.12}$$

Since the expression $I_1(\Psi)$ is scale and translation invariant, we consider a standardized distribution Ψ. Thus, we would like to optimize $\int_{-\infty}^{\infty} \psi^3(x)\,dx$ subject to the constraints

$$1 = \int_{-\infty}^{\infty} \psi(x)\,dx = \int_{-\infty}^{\infty} x^2\psi(x)\,dx \quad \text{and} \quad 0 = \int_{-\infty}^{\infty} x\psi(x)\,dx.$$

By the method of Lagrange multipliers, it suffices to optimize

$$\int_{-\infty}^{\infty} \left(\psi^3(x) - a\psi(x) - bx\psi(x) - cx^2\psi(x)\right)dx, \tag{2.13}$$

where a, b and c are undertermined multipliers.

LEMMA 1. *We have*

$$\inf_{\Psi\in\mathscr{K}} I_1(\Psi) = \frac{3}{4\pi^2}.$$

Proof. In order to minimize the functional (2.13), it is clear that one should choose $\psi(x)$ as large as possible for all x such that $0 < \psi^2(x) \leqslant a' + b'x + c'x^2$. Similarly to a proof in [6], we show that by a judicious choice of Lagrange multipliers there is a unique d.f. Ψ which is standardized and satisfies $\psi^2(x) \leqslant a' + b'x + c'x^2$.

Let

$$\psi(x) = k\sqrt{a^2 - x^2}, \qquad -a \leqslant x \leqslant a.$$

The three side conditions yield $a = 2$ and $k = 1/2\pi$. Let the Lagrange multipliers satisfy

$$a' = \frac{1}{\pi^2}, \quad b' = 0, \quad c' = \frac{1}{4\pi^2}.$$

Then

$$\psi_0(x) = \frac{1}{2\pi}\sqrt{4-x^2}, \qquad -2 \leqslant x \leqslant 2,$$

is the unique standardized density which satisfies $\psi^2(x) \leqslant a' + b'x + c'x^2$ for all x for which $\psi(x) > 0$. Hence, the density ψ_0 must minimize (2.13) and thus yields a minimum for $I_1(\Psi)$. This minimum value is

$$\int_{-2}^{2} \psi_0^3(x)\,dx = \frac{3}{4\pi^2}.$$

LEMMA 2. *We have*

$$\sup_{\Psi \in \mathscr{H}} I_1(\Psi) = +\infty.$$

Proof. Let $D > 1$. Let Ψ_D be the d.f. corresponding to the following density:

$$\psi_D(x) = \begin{cases} \varphi(x), & |x| \geqslant 2, \\ 0.0540, & |x| \leqslant 1 \text{ or } x_D \leqslant |x| \leqslant 2, \\ \dfrac{0.44304}{x_D - 1} + 0.054, & \dfrac{x_D+5}{6} < |x| \leqslant \dfrac{5x_D+1}{6}, \\ \dfrac{2.65824}{(x_D-1)^2}|x| + 0.054 - \dfrac{2.65824}{(x_D-1)^2}, & 1 \leqslant |x| \leqslant \dfrac{x_D+5}{6}, \\ \dfrac{-2.65824}{(x_D-1)^2}|x| + 0.054 + \dfrac{2.65824}{(x_D-1)^2}x_D, & \dfrac{5x_D+1}{6} < |x| < x_D. \end{cases}$$

The d.f. $\Psi_D \in \mathscr{H}$, and if $1 < x_D < 1 + 0.182/D^{1/2}$, then $I_1(\Psi_D) > D$. Thus

$$\sup_{\Psi \in \mathscr{H}} I_1(\Psi) = +\infty.$$

THEOREM 2. *We have*

(a)
$$\sup_{\Psi \in \mathscr{H}} e_{MT}(\Phi; \Psi) = +\infty,$$

(b)
$$\inf_{\Psi \in \mathscr{H}} e_{MT}(\Phi; \Psi) \geqslant K_1 \frac{3}{4\pi^2},$$

(c)
$$\inf_{\Psi \in \mathscr{H}} e_{MT}(\Phi; \Psi) \geqslant 0.864 K_0.$$

Proof. Rewriting (2.7) for $F \equiv \Phi$, we obtain

$$K_1 I_1(\Psi) \leqslant e_{MT}(\Phi; \Psi). \tag{2.14}$$

(a) follows from (2.14) and Lemma 2, and (b) follows from (2.14) and Lemma 1.

(c) Rewriting (2.9) for $F \equiv \Phi$, we obtain

$$K_0 e_{Wt}(\Phi; \Psi) \leqslant e_{MT}(\Phi; \Psi),$$

where $e_{Wt}(\Phi; \Psi)$ is the ARE of the Mann-Whitney-Wilcoxon test compared with the two-sample t-test. Hodges and Lehmann [6] find the d.f. which minimizes $e_{Wt}(\Phi; \Psi)$. Since this d.f. is in $\mathscr{H}$, the desired result follows.

LEMMA 3. *We have*

$$\sup_{\Psi \in \mathscr{H}} I_2(\Psi) = +\infty.$$

Proof. From the proof of Lemma 2 it is clear that there exists a d.f. Ψ_D in $\mathscr{H}(-2, 2)$ with $I_1(\Psi_D) > D$ for any given $D > 1$. Thus

$$\sup_{\Psi \in \mathscr{H}(-2,2)} I_1(\Psi) = +\infty.$$

From the definition and properties of g for the assumed d.f. F, there exists an interval (A, B) such that

$$\inf_{A<x<B} g'(x) = \varepsilon > 0.$$

Since g' is bounded on any finite interval, we have

$$[\mathrm{E}_\Psi g'(X)]^2 \leqslant \sup_{A<x<B} [g'(X)]^2 = L^2 < \infty \quad \text{for each } \Psi \in \mathscr{H}(A, B).$$

For each $x \in (A, B)$, $g(x) = g(A) + (x-A)g'(\bar{x})$, the inequality $A < \bar{x} < x$ implies

$$\varepsilon^2 \operatorname{Var}_\Psi X \leqslant \operatorname{Var}_\Psi g(X) \quad \text{for each } \Psi \in \mathscr{H}(A, B).$$

From the preceding, (2.5) and (2.12) we obtain

$$\frac{\varepsilon^2}{L^2} I_1(\Psi) \leqslant I_2(\Psi). \tag{2.15}$$

The desired result now follows from the scale and translation invariance of $I_1(\Psi)$ and from the fact that

$$\sup_{\Psi \in \mathscr{H}(-2,2)} I_1(\Psi) = +\infty.$$

THEOREM 3. (a) $\sup\limits_{\Psi\in\mathscr{H}} e = +\infty$.

(b) $\inf\limits_{\Psi\in\mathscr{H}} e = 0$ *if g is bounded.*

(c) $\inf\limits_{\Psi\in\mathscr{H}} e > 0$ *if g is unbounded and g' is bounded away from zero.*

Proof. (a) follows from (2.7) and Lemma 3.

(b) Let $R(a, b)$ denote the d.f. of the uniform distribution on (a, b). It is known that $R(a, b) \in \mathscr{H}(a, b)$. We find that

$$I_2(R(a, b)) = \frac{\mathrm{Var}_{R(a,b)} g(X)}{[g(b)-g(a)]^2}.$$

Since $\mathrm{Var}_{R(a,b)} g(X) \to 0$ as $b \to \infty$, we have $\inf\limits_{\Psi\in\mathscr{H}} I_2(\Psi) = 0$ for g bounded. The desired result follows from (2.4).

(c) Since $g(x) = g(0) + xg'(\bar{x})$, we obtain $0 < \bar{x} < x$ or $x < \bar{x} < 0$, and since $g'(x) \geqslant \varepsilon > 0$ for all x, we get

$$\varepsilon^2 \mathrm{Var}_\Psi X \leqslant \mathrm{Var}_\Psi g(X) \quad \text{for each } \Psi \in \mathscr{H}.$$

Moreover, since $g'(x) \leqslant L < \infty$ for all x, $[\mathrm{E}_\Psi g'(X)]^2 \leqslant L^2$ for each $\Psi \in \mathscr{H}$. Thus (2.15) holds for each $\Psi \in \mathscr{H}$. From this result, (2.7) and Lemma 1, the desired result follows.

Note that for part (c), we could have used Theorem 2 (c) and (2.9). We now turn our attention to e^*.

LEMMA 4. *We have*

$$\sup_{\Psi\in\mathscr{H}} I_3(\Psi) = +\infty.$$

Proof. By our assumptions on F there exist u_1 and u_2, $u_1 < 0 < u_2$, with $f(u_1) = f(u_2)$. Let

$$\psi_0(x) = \begin{cases} f(x), & x \in (-\infty, u_1] \cup [u_2, \infty), \\ K|x|^{-1/3} + f(u_1), & x \in (u_1, 0) \cup (0, u_2), \end{cases}$$

with

$$K = \frac{2}{3}\,\frac{F(u_2) - F(u_1) - (u_2 - u_1) f(u_1)}{u_1^{2/3} + u_2^{2/3}}.$$

Then

$$0 < \int_{-\infty}^{\infty} J'(\Psi_0(x))\,\psi_0^2(x)\,dx \leqslant C_0 < \infty$$

with

$$C_0 = \int_{-\infty}^{\infty} g'(x) f(x)\,dx +$$

$$+ \sup_{u_1 \leqslant x \leqslant u_2} \frac{g'(x)}{f(x)} [K^2 \cdot 3(u_2^{1/3} - u_1^{1/3}) + 3Kf(u_1)(u_2^{2/3} + u_1^{2/3}) + f^2(u_1)(u_2 - u_1)],$$

$I_3(\Psi_0) = +\infty$, but $\Psi_0 \notin \mathscr{H}$. Let $D > 0$. We construct a density ψ_ε such that $\Psi_\varepsilon \in \mathscr{H}$ and $I_3(\Psi_\varepsilon) > D$. Choose a positive integer k such that

$$10^{-k} \leqslant \min\{|u_1|, u_2\},$$

k depending on the assumed distribution F. Let

$$0 < \varepsilon < \min\{|u_1| - 10^{-k},\ u_2 - 10^{-k}\}.$$

Let

$$\psi_\varepsilon(x) = \begin{cases} \psi_0(x), & x \in (-\infty, u_1] \cup [u_1 + 10^{-k}, -\varepsilon] \cup [\varepsilon, u_2 - 10^{-k}] \cup [u_2, \infty), \\ \psi_0(-\varepsilon), & x \in (-\varepsilon, \varepsilon), \end{cases}$$

and let ψ_ε be a linear function in each of the four regions

$$(u_1, u_1 + \tfrac{1}{2}\cdot 10^{-k}), \quad (u_1 + \tfrac{1}{2}\cdot 10^{-k}, u_1 + 10^{-k}),$$
$$(u_2 - 10^{-k}, u_2 - \tfrac{1}{2}\cdot 10^{-k}), \quad (u_2 - \tfrac{1}{2}\cdot 10^{-k}, u_2)$$

with the linear functions chosen so that ψ_ε is a continuous density function.

Now

$$\sup_A \psi_\varepsilon(x) \leqslant 2\cdot 10^k + f(u_1)$$

with

$$A = \{x \mid x \in [u_1, u_1 + 10^{-k}] \cup [u_2 - 10^{-k}, u_2]\}.$$

Hence

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty} J'\big(\Psi_\varepsilon(x)\big)\psi_\varepsilon^2(x)\,dx \\ &\leqslant C_0 + \int_{u_1}^{u_1+10^{-k}} J'\big(\Psi_\varepsilon(x)\big)\psi_\varepsilon^2(x)\,dx + \int_{u_2-10^{-k}}^{u_2} J'\big(\Psi_\varepsilon(x)\big)\psi_\varepsilon^2(x)\,dx \\ &\leqslant C_0 + \sup_{n_1 \leqslant x \leqslant u_2} \frac{g'(x)}{f(x)} \big[2\cdot 10^{-k}\big(2\cdot 10^k + f(u_1)\big)^2\big] = C_0 + C_1 < \infty. \end{aligned}$$

Thus

$$I_3(\Psi_\varepsilon) \geqslant \frac{\int_{-\infty}^{\infty} \psi_\varepsilon^3(x)\,dx}{(C_0 + C_1)^2} \geqslant \frac{-\ln\varepsilon}{(C_0 + C_1)^2} \geqslant D$$

if and only if

$$\varepsilon < \exp[-(C_0 + C_1)^2 D].$$

Hence, if

$$0 < \varepsilon < \min\{|u_1| - 10^{-k},\ u_2 - 10^{-k},\ \exp[-(C_0 + C_1)^2 D]\},$$

then $\Psi_\varepsilon \in \mathscr{H}$ and $I_3(\Psi_\varepsilon) > D$. Since D is arbitrary, we have

$$\sup_{\Psi \in \mathscr{H}} I_3(\Psi) = +\infty.$$

The first part of the following lemma is easily proved by applying Jensen's inequality to the strictly convex function $h(u) = u^2$.

LEMMA 5. (a) *Let Ψ be an absolutely continuous d.f. for which*

$$\int_{-\infty}^{\infty} \psi^2(x)\,dx < \infty.$$

Then

$$\frac{\int_{-\infty}^{\infty} \psi^3(x)\,dx}{\left[\int_{-\infty}^{\infty} \psi^2(x)\,dx\right]^2} \geqslant 1.$$

(b) *The equality holds in* (a) *if and only if Ψ is the uniform d.f. where (a, b) is a finite interval.*

Proof. (b) Sufficiency. Let $\Psi \equiv R_{(a,b)}$. Then direct calculations yield

$$\left[\int_{-\infty}^{\infty} \psi^2(x)\,dx\right]^2 = \frac{1}{(b-a)^2} = \int_{-\infty}^{\infty} \psi^3(x)\,dx.$$

Necessity. Since $h(u) = u^2$ is strictly convex, the inequality in (a) is strict unless $\psi(X)$ is constant a.e. Thus

$$\int_{-\infty}^{\infty} \psi^3(x)\,dx = \left[\int_{-\infty}^{\infty} \psi^2(x)\,dx\right]^2 \Rightarrow \psi(x) = K \text{ a.e.} \Rightarrow \Psi \equiv R_{(a,b)}$$

for some interval (a, b).

THEOREM 4. (a) $\sup\limits_{\Psi \in \mathscr{H}} e^* = +\infty$.

(b) $\inf\limits_{\Psi \in \mathscr{H}} e^* = 0$ *if g is unbounded.*

(c) $\inf\limits_{\Psi \in \mathscr{H}} e^* > 0$ *if g is bounded and* $\sup\limits_{0<u<1} J'(u) = L < \infty$.

Proof. (a) The desired result follows from (2.11) and Lemma 4.

(b) $\Psi \equiv R_{(a,b)} \in \mathscr{H}(a, b)$ with

$$\int_{-\infty}^{\infty} \psi^3(x)\,dx = \frac{1}{(b-a)^2}$$

while

$$\int_{-\infty}^{\infty} J'\big(\Psi(x)\big)\psi^2(x)\,dx = \int_a J'\left(\frac{x-a}{b-a}\right)\frac{1}{(b-a)^2}\,dx$$

$$= \frac{1}{b-a}\,[g(\infty)-g(-\infty)] = +\infty,$$

since g is unbounded and strictly increasing. Thus $I_3(R_{(a,b)}) = 0$, and the desired result follows from (2.10).

(c) Since J' is bounded, we have

$$\int_{-\infty}^{\infty} J'\big(\Psi(x)\big)\psi^2(x)\,dx \leqslant L \int_{-\infty}^{\infty} \psi^2(x)\,dx.$$

Thus

$$\frac{\int_{-\infty}^{\infty} \psi^3(x)\,dx}{L^2\left[\int_{-\infty}^{\infty} \psi^2(x)\,dx\right]^2} \leqslant I_3(\Psi).$$

Applying Lemma 5 to the preceding, we obtain

$$\frac{1}{L^2} \leqslant \inf_{\Psi\in\mathscr{H}} I_3(\Psi).$$

From this last inequality and (2.11) the desired result is obtained.

3. Discussion. Since the Neyman test T and the locally most powerful among unbiased rank (LMPR) test T^* are derived and optimal for a specific distribution F, it is not too surprising that there are distributions Ψ which satisfy the assumptions of Theorem 1 and make the efficiencies e or e^* as large as needed. However, these efficiencies were not optimized by referring to some scaled or translated (or both) version of F. This procedure might have been proved to be useful in optimizing one or both of the efficiencies; however, it would not be treating the test statistics fairly. Specifically, one would use a "studentized" version of Neyman's test in practice, obtained by dividing T of (1.3) by a consistent estimator of its standard deviation. Statistics in this class would be scale invariant, at least for $g(x) = ax^{2k+1}$, $k = 0, 1, 2, \ldots$ Such studentizing makes the notation more cumbersome and makes the proofs more difficult without changing the results. With this in mind, the paper follows the example of Chernoff and Savage [4], Mikulski [8], Yu [12], and Kalish and Mikulski [7] in not presenting the studentized version of T and in not optimizing efficiencies by recourse to scaling and changing location. Even so, it is possible to produce distributions Ψ which force the ARE's e or e^* to infinity. Moreover, the distributions Ψ used in forcing e^* to be large are not that unlike F. This points to the fact that there is a valid question about optimality of rank test in the presence of slight changes of the assumed distribution F.

On the other hand, the Cramér - von Mises test may behave infinitely worse than either Neyman's test (if g is bounded) or the LMPR test (if g is unbounded). If g and J' are both bounded, then the Cramér - von Mises test cannot behave infinitely worse than the rank test. This is also the case for Cramér - von Mises test compared with Neyman's test when g is unbounded but has a bounded derivative. Such additional assumptions on g are similar to those appearing in [12]. These results on minimizing the efficiencies indicate that the Cramér - von Mises test for location is not a panacea test in the sense that it should be used everywhere for everything. However, the results on optimizing e and e^* indicate that the Cramér - von Mises test should be considered a strong competitor when distributions of the type that maximize e and e^* are suspected. How well or how badly, in terms of Pitman ARE, does the Cramér - von Mises test behave compared with reasonable competitors for an assumed F has been established.

We have two open problems. One concerns the sample sizes necessary in order for limiting behaviour to assert itself. Another deals with empirical construction of the distribution function of the limiting random variable in (1.7). These questions and others will be the topics of future papers on the Cramér - von Mises test.

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O OGRANICZENIACH ASYMPTOTYCZNEJ MOCY I O ASYMPTOTYCZNEJ EFEKTYWNOŚCI TESTU CRAMÉRA - VON MISESA

STRESZCZENIE

W pracy zbadano własności testu Craméra - von Misesa w problemie dwu próbek. Wyprowadzono ograniczenia na asymptotyczną moc testu oraz porównano asymptotyczną efektywność tego testu względem testu rangowego i testu Neymana. Interesujący jest przykład, który pokazuje, że test Craméra - von Misesa jest efektywniejszy od każdego z dwu pozostałych testów. Przypadek rozkładu normalnego jest rozpatrywany oddzielnie.
