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APPLICATION OF MODIFIED MOMENTS IN THE NUMERICAL SOLUTION OF THE ABEL INTEGRAL EQUATION

Piessens and Verbaeten [8] have described a method for the numerical solution of the Abel integral equation, using Laplace transform techniques. The present paper shows how this powerful method can be modified to save a substantial portion of computational work. The modification is based on the application of modified moments (certain integrals involving Chebyshev polynomials of the second kind) which are computed by using a simple recurrence relation.

1. Introduction. We consider the Abel integral equation

$$\int_0^x (x^p - y^p)^{-a} f(y)\,dy = g(x) \qquad (0 \leqslant x \leqslant 1), \tag{1}$$

where $0 < a < 1$ and $p > 0$. This equation occurs in many physical and engineering problems. The solution of (1) is explicitly given by the inversion formula ([10], p. 41)

$$f(x) = \frac{p \sin a\pi}{\pi} \frac{d}{dx} \int_0^x (x^p - y^p)^{a-1} y^{p-1} g(y)\,dy. \tag{2}$$

However, it is very difficult to use this formula in some practical applications where g has a complicated form or no explicit mathematical expression for g is known. This is why many papers deal with the numerical solution of (1) (see [8], [9], and the references given therein).

In this paper we describe a simple and efficient method applicable in the case where g can be approximated accurately by g_n defined by

$$g_n(x) = x^\sigma \sum_{k=0}^{n}{}' a_k T_k^*(x^p) \qquad (0 \leqslant x \leqslant 1), \tag{3}$$

where $\sum'$ denotes the sum with the first term halved, T_k^* is the k-th shifted Chebyshev polynomial of the first kind, $T_k^*(t) = \cos k\theta$, $\cos\theta = 2t-1$, and σ $(\sigma > -p)$ is a parameter chosen so that the function $x^{-\sigma}g(x)$ is as smooth as possible on the interval $\langle 0, 1\rangle$.

The coefficients a_k can be obtained either by the classical recurrence relation or by interpolation methods ([3], [5], [7]) for the calculation of the Chebyshev coefficients of the function $G(t) := t^{-\sigma/p}g(t^{1/p})$ for $0 \leqslant t \leqslant 1$ (if g is given by an explicit mathematical formula), or by Clenshaw's curve fitting method [1] (if g is characterized by its values on a finite set of points).

Piessens and Verbaeten [8] have developed a method for the inversion of equation (1) in the case where $p = 1$ and the right-hand member is exactly of the form (3).

2. The Piessens-Verbaeten method. Given the approximation (3), Piessens and Verbaeten [8] have obtained, using Laplace transform techniques, the following approximation of the solution of (1):

$$f_n(x) = \frac{p\Gamma(\beta+1)}{\Gamma(1-a)\Gamma(a+\beta)} x^{p(a+\beta)-1} \sum_{k=0}^{n}{}' a_k q_k(x^p), \tag{4}$$

where $\beta = \sigma/p > -a$ and $q_k(t) = {}_3F_2(-k, k, \beta+1; 1/2, a+\beta; t)$ for $t \in \langle 0, 1\rangle$. (Strictly speaking, equation (4) is the generalized version of a formula given in [8] for $p = 1$.) It was shown that the sequence $\{q_k(t)\}$ satisfies the third-order recurrence relation

$$\begin{aligned}(k+a+\beta-1)(k-2)q_k(t) - [4(k+\beta)(k-2)t - 3k^2 - \\ -(a+\beta-9)k + 3(a+\beta-3)]q_{k-1}(t) - [4(k-\beta-3)(k-1)t - 3k^2 + \\ +(a+\beta+9)k - 6]q_{k-2}(t) + (k-a-\beta-2)(k-1)q_{k-3}(t) = 0.\end{aligned} \tag{5}$$

Examination of the asymptotic behaviour of all solutions of this equation led to the conclusion that $q_k(t)$ can be computed by forward recursion. The initial values for (5) are the following:

$$q_0(t) = 1, \quad q_1(t) = \frac{2(\beta+1)}{a+\beta}t - 1,$$

$$q_2(t) = \frac{8(\beta+1)(\beta+2)}{(a+\beta)(a+\beta+1)}t^2 - \frac{8(\beta+1)}{a+\beta}t + 1.$$

3. The modified method. Replacing the function g on the right-hand side of (2) by g_n given by (3), we obtain the following approximation of the solution of (1):

$$f_n^*(x) = \frac{p\sin a\pi}{\pi}\sum_{k=0}^{n}{}' a_k\mu_k(x), \tag{6}$$

where

$$\mu_k(x) = \frac{d}{dx}\int_0^x (x^p-y^p)^{a-1}y^{\sigma+p-1}T_k^*(y^p)\,dy \qquad (k = 0, 1, \ldots).$$

Putting $t = x^p$ and $u = y^p/t$, we transform the last formula to the form

$$\mu_k(x) = x^{p-1}\frac{d}{dt}\Big[t^{a+\beta}\int_0^1 (1-u)^{a-1}u^\beta T_k^*(ut)\,du\Big]$$

$$= x^{p(a+\beta)-1}\int_0^1 (1-u)^{a-1}u^\beta[(a+\beta)T_k^*(ut)+utT_k^{*\prime}(ut)]\,du,$$

where $\beta = \sigma/p$.

Let us write

$$m_k(t) = \frac{\Gamma(a+\beta+1)}{\Gamma(a)\Gamma(\beta+1)}\int_0^1 (1-u)^{a-1}u^\beta U_k^*(ut)\,du \qquad (k=0,1,\ldots), \tag{7}$$

where U_k^* is the k-th shifted Chebyshev polynomial of the second kind,

$$U_k^*(t) = \sin[(k+1)\theta]/\sin\theta, \qquad \cos\theta = 2t-1.$$

The integrals occurring on the right-hand side of (7) for $k = 0, 1, \ldots$ are referred to as modified moments of the function $h(x) = (1-x)^{a-1}x^\beta$ with respect to the polynomials $\{p_k\}$, where $p_k(x) = U_k^*(tx)$, t being a parameter. Making use of the identities (see, e.g., [7], Chapter 2)

$$T_k^* = (U_k^*-U_{k-2}^*)/2, \qquad T_k^{*\prime} = 2kU_{k-1}^*,$$

$$zU_k^*(z) = [U_{k-1}^*(z)+2U_k^*(z)+U_{k+1}^*(z)]/4,$$

we get

$$\mu_k(x) = \frac{\Gamma(a)\Gamma(\beta+1)}{2\Gamma(a+\beta+1)}x^{p(a+\beta)-1}[(k-a-\beta)m_{k-2}(x^p)+2km_{k-1}(x^p)+$$

$$+(k+a+\beta)m_k(x^p)] \qquad (k = 0, 1, \ldots;\ m_{-2}(t) = m_0(t),\ m_{-1}(t) \equiv 0).$$

Substituting this in (6), we derive the formula

$$f_n^*(x) = \frac{p\Gamma(\beta+1)}{2\Gamma(1-a)\Gamma(a+\beta+1)}x^{p(a+\beta)-1}\sum_{k=0}^{n} b_k m_k(x^p), \tag{8}$$

where

$$b_k = (k+\alpha+\beta)a_k + 2(k+1)a_{k+1} + (k-\alpha-\beta+2)a_{k+2} \tag{9}$$

$$(k = 0, 1, \ldots, n;\ a_{n+1} = a_{n+2} = 0).$$

Using a technique based on that described in [4], we obtain the following recurrence relation, satisfied by the modified moments $m_k(t)$:

$$(k+\alpha+\beta)m_k(t) - [4(k+\beta)t - 3k - \alpha - \beta + 1]m_{k-1}(t) - \tag{10}$$

$$-[4(k-\beta-1)t - 3k + \alpha + \beta + 2]m_{k-2}(t) + (k-\alpha-\beta-1)m_{k-3}(t) = 0.$$

The initial values for (10) are

$$m_0(t) = 1, \quad m_1(t) = 4\frac{\beta+1}{\alpha+\beta+1}t - 2, \tag{11}$$

$$m_2(t) = 16\frac{(\beta+1)(\beta+2)}{(\alpha+\beta+1)(\alpha+\beta+2)}t^2 - 16\frac{\beta+1}{\alpha+\beta+1}t + 3.$$

Using the method of Denef and Piessens [2] we find the asymptotic behaviour of three independent solutions of (10), say, c_{1k}, c_{2k}, and c_{3k}. For $t = 0$, $0 < t < 1$, and $t = 1$ we have

(i) $c_{1k} = (-1)^k, \quad c_{2k} = (-1)^k k, \quad c_{3k} \sim (-1)^k k^{1-2\alpha-2\beta} \quad (k \to \infty),$

(ii) $c_{1k} \sim (-1)^k k^{-2\beta-1}, \quad c_{2k} \sim k^{-\alpha}\cos k\varphi, \quad c_{3k} \sim k^{-\alpha}\sin k\varphi$

$(k \to \infty,\ \varphi = \arccos(2t-1)),$

(iii) $c_{1k} = 1, \quad c_{2k} \sim k^{2\alpha-3}, \quad c_{3k} \sim (-1)^k k^{-2\beta-1} \quad (k \to \infty),$

respectively. This means that there is no strongly increasing solution among all solutions of (10). Thus $m_k(t)$ can be computed by forward recursion (see [6]).

Normally, the sequence $\{|a_k|\}$ tends quickly to zero. As $\{|m_k(t)|\}$ cannot increase strongly, the absolute error of $f_n^*(x)$ may be estimated by the expression

$$\frac{p\Gamma(\beta+1)(\alpha+\beta+n)}{2\Gamma(1-\alpha)\Gamma(\alpha+\beta+1)}|a_n m_n(x^p)|.$$

Obviously, f^* given by (8) is the exact solution of equation (1) for g being exactly of the form (3). The same is true for f_n defined by (4). Therefore, $f_n = f_n^*$, i.e. the Piessens-Verbaeten method and the present method are mathematically equivalent. By the way we have

$$q_k(t) = \frac{1}{2(\alpha+\beta)}[(k-\alpha-\beta)m_{k-2}(t) + 2km_{k-1}(t) + (k+\alpha+\beta)m_k(t)]$$

$$(k = 0, 1, \ldots).$$

However, in the present algorithm, if we assume that the coefficients b_k are precomputed, the numbers of multiplications and additions are both approximately reduced to 3/5 of the numbers in the Piessens-Verbaeten algorithm, and the number of divisions remains unchanged.

Case $\beta = 0$. Equations (8)-(11) can be used for any value of $\beta > -1$. However, in the case of $\beta = 0$ (i.e. in the case of $\sigma = 0$ in (3)) further simplifications can be made. For $\beta = 0$ formula (7) defines

$$m_k(t) = a\int_0^1 (1-u)^{a-1} U_k^*(tu)\,du \qquad (k = 0, 1, \ldots). \tag{12}$$

By a procedure analogous to that applied in the general case, we derive the equations

$$f_n^*(x) = \frac{p}{2\Gamma(1-a)\Gamma(1+a)} \sum_{k=0}^{n} b_k m_k(x^p), \tag{13}$$

$$b_k = (k+a)a_k + 2(k+1)a_{k+1} + (k-a+2)a_{k+2} \qquad (k = 0, 1, \ldots, n;\ a_{n+1} = a_{n+2} = 0), \tag{14}$$

$$(k+a)m_k(t) - 2(2t-1)km_{k-1}(t) + (k-a)m_{k-2}(t) = 2(-1)^k a \qquad (k = 2, 3, \ldots), \tag{15}$$

$$m_0(t) = 1, \qquad m_1(t) = \frac{4t}{a+1} - 2. \tag{16}$$

Notice that substituting $k-1$ for k in (15) and adding the obtained equation to (15), we get relation (10) for $\beta = 0$.

We have ascertained experimentally the numerical stability of the forward recursion method for computation of the moments (12) using the recurrence relation (15) with initial values (16).

If we compare this method and the method of Section 2, we see that the reduction factors for the numbers of multiplications and additions are both approximately equal to 1/3.

Remark. Since $U_k^*(x) = (-1)^k(k+1)\,{}_2F_1(-k, k+2;\ 3/2;\ x)$, we have

$$\begin{aligned} m_k(t) &= (-1)^k(k+1)B(a, \beta+1)\int_0^1 (1-u)^{a-1} u^{\beta}\,{}_2F_1(-k, k+2;\ 3/2; tu)\,du \\ &= (-1)^k(k+1)\,{}_3F_2(-k, k+2, \beta+1;\ 3/2, a+\beta+1;\ t) \end{aligned}$$

(see [5], Vol. 1, Section 3.6). Equations (10) and (15) can be also obtained by using a theorem from [5], Vol. 2, Section 12.2.

4. Numerical examples.

Example 1. Let us consider the equation

$$\int_0^x (x^p - y^p)^{-1/2} f(y)\,dy = \exp(x^p) - 1 \qquad (0 \leqslant x \leqslant 1).$$

The exact solution is

$$f(x) = p\,\pi^{-1/2} x^{p-1} \exp(x^p)\operatorname{erf}(x^{p/2}).$$

Using Table 3.3 from [5] we find coefficients $a_0, a_1, \ldots, a_{13}$ such that

$$e^t - 1 \cong t \sum_{k=0}^{13}{}' a_k T_k^*(t) \qquad (0 \leqslant t \leqslant 1).$$

Notice that $a_{13} = 5.6_{10}{-19}$. Formulas (4) and (8) for $n = 13$ and $\beta = 1$ ($\sigma = p$) give results presented in the Table, where

$$d = \max_{1 \leqslant k \leqslant 10} |f_n(k/10) - f(k/10)| \quad \text{and} \quad d^* = \max_{1 \leqslant k \leqslant 10} |f_n^*(k/10) - f(k/10)|.$$

TABLE

p	d	d^*
.1	$1.1_{10}-17$	$3.4_{10}-18$
.3	$1.3_{10}-17$	$3.9_{10}-18$
.5	$9.5_{10}-18$	$3.0_{10}-18$
1.0	$1.7_{10}-17$	$5.3_{10}-18$
2.0	$3.4_{10}-17$	$1.1_{10}-17$

Let t and t^* denote the times of calculation of $f_n(x)$ and $f_n^*(x)$, respectively. The average of the ratio t^*/t was 0.65. The calculations were carried out on the ODRA 1305 computer of the Institute of Computer Science, University of Wrocław, using double precision arithmetic.

Example 2. An integral equation of great importance in plasma-ysics can be transformed to the form (see [8])

$$\int_0^x (x-y)^{-1/2} f(y)\,dy = g(x) \qquad (0 \leqslant x \leqslant 1),$$

where

$$g(x) = \frac{10}{11}\sqrt{\pi}\, x^{-1/2} \exp\left[1.21\left(1 - \frac{1}{x}\right)\right].$$

The exact solution is

$$f(x) = x^{-3/2} \exp[1.21(1 - 1/x)].$$

Let J_n be the polynomial of degree at most n which interpolates the function $G(x) = x^{-\sigma}g(x)$ $(\sigma > -1)$ at the points $x_k = [1-\cos(k\pi/n)]/2$ $(k = 0, 1, \ldots, n)$. It is well known (see, e.g., [3] or [7], Chapter 7) that

$$J_n(x) = {\sum_{k=0}^{n}}' a_k T_k^*(x) \qquad (0 \leqslant x \leqslant 1),$$

where

$$a_k = \frac{2-\delta_{kn}}{n} {\sum_{j=0}^{n}}'' G(x_j) T_j^*(x_k) \qquad (k = 0, 1, \ldots, n).$$

The symbol $\sum''$ denotes the sum with both the first and the last terms halved.

For $\sigma = 0$ and $n = 25$ the absolute errors at $x = 0(.1)1$ of the Piessens-Verbaeten and modified methods (formulas (13)-(16)) are the same and do not exceed $2.1_{10}-5$.

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