ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
20, 1 (1988), pp. 117–123

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CLASS RECONSTRUCTION NUMBERS OF UNICYCLIC GRAPHS*

Abstract. The *reconstruction number* of a graph G is the minimum number of point-deleted subgraphs in the deck of G which are not contained in the deck of any other graph. The *reconstruction conjecture*, attributed originally to S. Ulam, is equivalent to the statement that the reconstruction number is well defined for every graph G with at least three points. If G belongs to a specified class $\mathscr{C}$ of graphs, the *conditional reconstruction number* $\mathrm{rn}(G|\mathscr{C})$ is the minimum number of point-deleted subgraphs of G that are not contained in the deck of any other graph in $\mathscr{C}$. We find all graphs which have conditional reconstruction number 1 when $\mathscr{C}$ is the class of unicyclic graphs and show that, for each unicyclic graph other than the cycle, two point-deleted subgraphs from the deck suffice to determine its rooted trees.

1. Introduction. A graph G with point set $V(G) = \{v_1, \ldots, v_p\}$ is *reconstructible* if $H \cong G$ whenever graph H with $V(H) = \{u_1, \ldots, u_p\}$ has $H-u_i \cong G-v_i$ for all i. The *deck* of a graph is the multiset of its point-deleted subgraphs $G-v_i$. The *reconstruction number* $\mathrm{rn}(G)$ of a reconstructible graph G is the minimum cardinality of a subdeck of G not contained in the deck of any other (non-isomorphic) graph. This invariant $\mathrm{rn}(G)$ was introduced in [5], where it was conjectured that almost all graphs have reconstruction number 3; this was recently proved by Bollobás [2].

Now, let $\mathscr{C}$ be a generic set of all graphs with a specified property. A graph $G \in \mathscr{C}$ is *$\mathscr{C}$-reconstructible* or *class-reconstructible* from k subgraphs if there is a k-subdeck of G not contained in the deck of any other graph $H \in \mathscr{C}$. Such a minimum value of k is the *$\mathscr{C}$-reconstruction number* of G or *class-reconstruction number* of G, denoted by $\mathrm{rn}(G|\mathscr{C})$. Harary and Lauri [3] determined the $\mathscr{C}$-reconstruction numbers when $\mathscr{C}$ is the class of all maximal planar graphs. Bange et al. [1] noted that when $\mathscr{C}$ is the class of total graphs, the $\mathscr{C}$-reconstruction number is always 1. They also showed that any singleton of the deck will serve. Harary and Lauri [4] studied the class of trees and conjectured that every tree has the class reconstruction number 1 or 2.

* Research supported in part by the Computing Research Laboratory of the New Mexico State University.

Recall that a *unicyclic graph* G is connected and has exactly one cycle. When all the lines of the cycle of G are removed, we get the *rooted trees* of G. Manvel [6] demonstrated that the graphs of the class $\mathscr{U}$ of all unicyclic graphs are reconstructible. The *unicyclic reconstruction number* of G is $\mathrm{rn}(G|\mathscr{U})$. We characterize unicyclic graphs for which $\mathrm{rn}(G|\mathscr{U}) = 1$ and show that if G is unicyclic but not a cycle, then two point-deleted subgraphs from its deck suffice to determine all its rooted trees.

2. Graphs with unicyclic reconstruction number one. Call a tree T a *pseudostar with vertex* v if no component of $T-v$ has more than two points.

THEOREM 1. *If* G *is unicyclic, then* $\mathrm{rn}(G|\mathscr{U}) = 1$ *if and only if* G *is isomorphic to one of the following*:

(i) *the union of a cycle and a pseudostar, with an added line joining a point of the cycle to the vertex of this pseudostar* (Fig. 1a);

(ii) *a copy of* K_3 *in which one point is identified with a vertex of a pseudostar* (Fig. 1b).

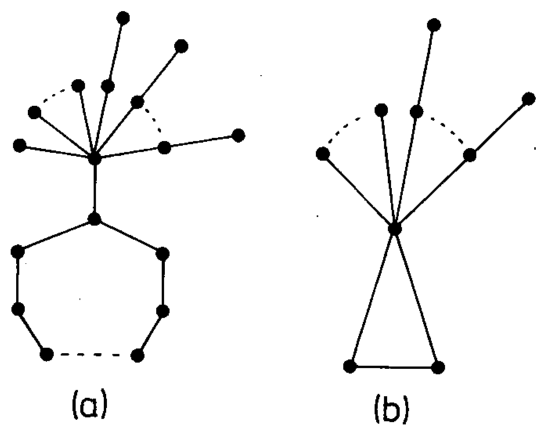

Fig. 1. The graphs with unicyclic reconstruction number one

Proof. It is easily verified that if G is a graph constructed as in (i) or (ii) and v is the vertex of the pseudostar, then $G-v$ determines G uniquely among all unicyclic graphs.

Now, let G be any unicyclic graph other than those described in (i) and (ii), and suppose that G contains a point v such that $G-v$ allows unique identification of G among all unicyclic graphs. Then v must be either in the cycle of G or adjacent to a point of the cycle, for otherwise $G-v$ does not determine the number of points which are adjacent to points of the cycle.

In case the cycle of G has length four or more, v cannot be on the cycle, for otherwise the length of the cycle is not uniquely determined by $G-v$. Therefore v must be a non-cycle point which is adjacent to a point of the cycle. Also, v is the only such point because the number of points adjacent to the cycle cannot otherwise be identified by $G-v$. Finally, no component of

$G-v$ other than the one containing the cycle can have more than two points; if it did, then the maximum distance to v among the points in this component is again left undetermined by $G-v$. Therefore, if the cycle length is at least four, the graph G must have the structure described in (i).

A similar argument holds when the cycle in G has length three, except that v may now be a point of the cycle. In this case, no component of $G-v$ can contain more than two points in order for the length of the cycle to be determined, and so G must have the structure given in (ii).

It is easy to establish the unicyclic reconstruction number of a cycle. From Theorem 1 we already know that $\mathrm{rn}(C_3|\mathscr{U})=1$.

THEOREM 2. *For each* $k \geqslant 4$, $\mathrm{rn}(C_k|\mathscr{U})=3$.

Proof. Each point-deleted subgraph of C_k is isomorphic to the path P_{k-1}. The unicyclic graph of order k obtained from C_{k-1} by adding a point adjacent to exactly one point of the cycle also has two point-deleted subgraphs that are isomorphic to P_{k-1}, so $\mathrm{rn}(C_k|\mathscr{U}) \geqslant 3$. It is also easy to check that three point-deleted subgraphs will determine C_k uniquely, so that $\mathrm{rn}(C_k|\mathscr{U})=3$.

Fig. 2 displays the two unicyclic graphs of order 4: C_4 and $C_3 \cdot K_2$, thus illustrating the general case.

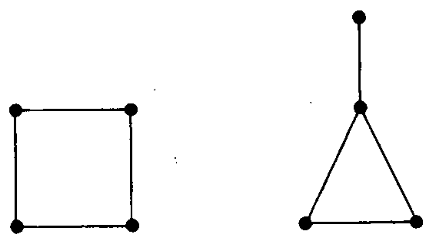

Fig. 2. The two unicyclic graphs of order 4

3. Bounds for other unicyclic graphs. Since we are concerned only with unicyclic graphs, we know that the graph G has the form shown in Fig. 3, where the T_i are its (possibly trivial) rooted trees.

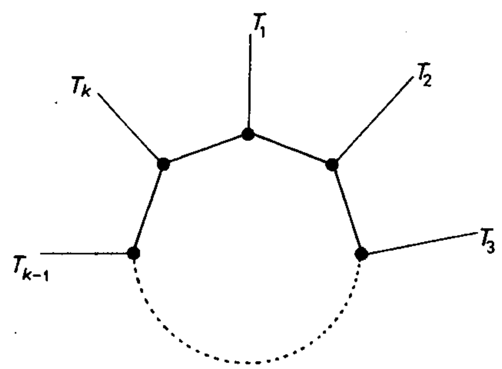

Fig. 3. The unicyclic graph G

The *depth* of any point v of the unicyclic graph G is the minimum distance from v to the cycle (cycle points have depth 0). The *maximum depth* of G is then the maximum among all its point depths.

THEOREM 3. *If the maximum depth of G is one, then* $\mathrm{rn}(G|\mathscr{U}) \leqslant 3$.

Proof. If G has exactly one non-cycle point, then $\mathrm{rn}(G|\mathscr{U}) = 1$ by Theorem 1. Thus let G have at least two non-cycle points u and v. Then $G-u$ and $G-v$ together verify that the maximum depth of G is one, for any point at depth two in G would be evident in at least one of these two subgraphs either at distance two from the cycle or in a different component than the cycle.

Case 1. If G contains a point w on its cycle such that the two points of the cycle adjacent to w either both have degree 2 or both have degree greater than 2, then we claim that $G-w$ (in conjunction with $G-u$ and $G-v$) uniquely identifies G. For, the cycle length is known from $G-u$, and there is only one way (up to isomorphism) to introduce a point to $G-w$ to get a unicyclic graph with maximum depth 1 and the proper cycle length (Fig. 4a,b).

Case 2. Now, when G contains no such point w, on tracing its cycle, we must alternate between pairs of points with degree 2 and pairs of points with degree at least 3 (Fig. 4c).

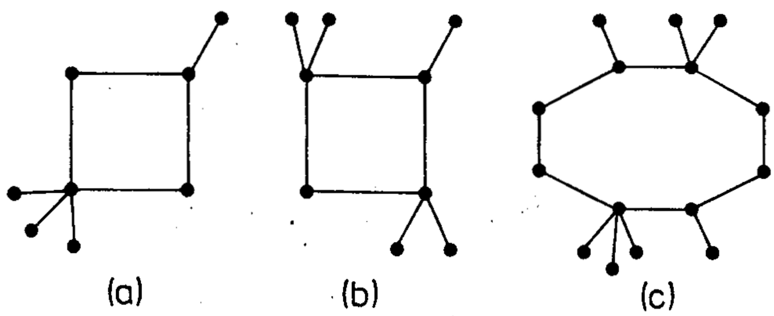

Fig. 4. Unicyclic graphs with depth 1

The result is easily verified in this case when the cycle length is 4, so consider the cycle length at least 8. We may not know from $G-u$ and $G-v$ that G has the form presented in Fig. 4c, but we do know that the cycle cannot contain more than three consecutive points with degree greater than 2. Let x be a cycle point with $\deg x > 2$. Then $G-x$ (again together with $G-u$ and $G-v$) identifies G uniquely, for there is only one possible way to add a point to $G-x$ and get a maximum-depth-one unicyclic graph that does not contain four consecutive cycle points that have degree greater than 2.

4. Graphs with maximum depth greater than one. When the maximum depth of the graph G is greater than one, the reconstruction of the rooted trees T_i poses more of a problem. In order to show that the multiset of rooted trees T_i (but not necessarily their sequential order around the cycle) can be reconstructed from only two point-deleted subgraphs, we need Theorem A from [4]. Let T be a tree and v an endpoint of T adjacent to point x. If there is another point $y \neq x$ in T such that the tree obtained from T by removing one line and adding another, $T-vx+vy$, is isomorphic to T, then v is called a *replaceable point* of T.

THEOREM A ([4]). *If T is a tree which is not a path, then T has an endpoint which is not replaceable.*

For ease of exposition we will state the next result, which we require for Theorem 4, in terms of a rooted tree rather than a unicyclic graph. Recall that the *depth* of a point *in a rooted tree* is the distance of that point to the root. Also, when x is a point of a rooted tree T, we will call the components of $T-x$ induced by the descendants of x (in T) the *sub-branches of x in T*.

LEMMA. *Let T be a rooted tree with root r. If T contains a point of depth 3, then there exist two points s and x, both having depth at least 2, such that T can be uniquely determined from $T-s$ and $T-x$.*

Proof. Since the root r is labelled, it suffices to show that the sub-branches $T_1, T_2, \ldots, T_k$ of r in T can be uniquely determined from the two point-deleted subgraphs. Among all points of depth 2 which have at least one descendant, let s be one with the minimum number of descendants. Among all sub-branches of s in T, let B be a sub-branch of minimum order.

Case 1. When B is not a path, let x be an endpoint of B that is not replaceable, as guaranteed by Theorem A (it is possible that x is adjacent to s).

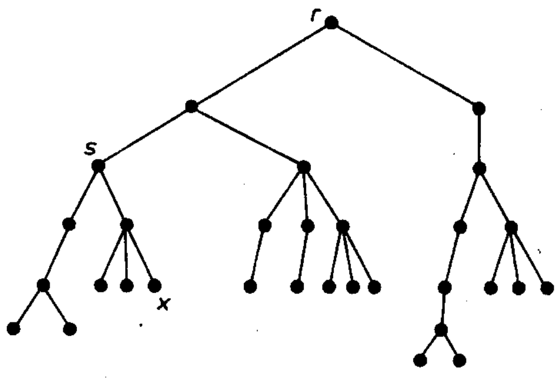

Fig. 5. Choosing points to delete from G

We now make the following observations.

(1) In both $T-s$ and $T-x$, we can distinguish (up to isomorphism) which of the sub-branches of r have had a point deleted from it. In $T-s$ (resp., $T-x$) it will be the sub-branch of r whose multiplicity of appearance

among the sub-branches is one greater than the corresponding multiplicity in $T-x$ (resp., $T-s$).

(2) From $T-s$ we know the number of descendants of each point of depth 2, but even more, we know the structure of all the sub-branches of the points of depth 2.

(3) Consider $T-x$. As noted in (1), we can distinguish (up to isomorphism) which sub-branch of r has had x deleted from it. By the chosen minimum properties of s and x, we can identify s in $T-x$ (again, up to isomorphism of the appropriate sub-branches), and also from which sub-branch of s that x has been deleted. By Theorem A and the choice of x, this sub-branch can be restored to B in only one way, even with the adjacency to s now included.

Thus the tree T can be reconstructed uniquely.

Case 2. When B is a path, the same proof will hold by choosing x to be the point of B adjacent to s. In this case Theorem A is not applicable, but the tree T is again the only obtainable reconstruction.

Let G be a unicyclic graph and consider the subgraphs obtained by deleting non-cycle points. Suppose that in these subgraphs we replace the lines of the cycle C_k by the lines of the star $K_{1,k}$, identifying the endpoints of the star with the cycle points and labelling the center of the star r (see Fig. 6).

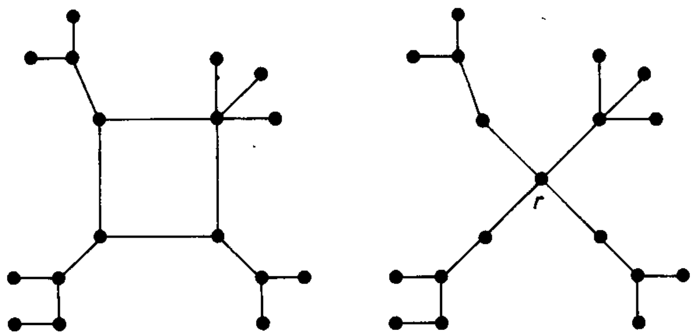

Fig. 6. Replacing a cycle by a rooted star

It is clear that the only information lost in this transformation is the order in which the cycle points, and hence the rooted trees $T_1, T_2, \ldots, T_k$ appear around the cycle. Thus, by the previous Lemma, we obtain the following result on unicyclic graphs.

THEOREM 4. *If G is a unicyclic graph which contains a point of depth 2, then the rooted trees $T_1, T_2, \ldots, T_k$ can be reconstructed uniquely from two point-deleted subgraphs of G.*

There are many subfamilies of unicyclic graphs for which the $\mathcal{U}$-

reconstruction number can be determined exactly or at least tightly bounded. One example is the following

THEOREM 5. *Let G be unicyclic with rooted trees $T_1, T_2, \ldots, T_k$. If G contains a non-cycle point $v \in T_i$ and T_i^* is what is left of T_i in the unicyclic component of $G-v$, and $T_i^* \not\cong T_j$ for any j, then* $\mathrm{rn}(G|\mathscr{U}) \leqslant 3$.

Proof. If the depth of G is one, the result follows from Theorem 3. Otherwise, by Theorem 4, $T_1, T_2, \ldots, T_k$ can be determined from two point-deleted subgraphs of G. Now in $G-v$ it is possible to identify which branch has had the point v deleted from it, so the original graph G can then be reconstructed.

Certainly, stronger results can be obtained as much of the information given in the point-deleted subgraphs has barely been used (especially, the order in which the rooted trees appear along the cycle). We conjecture that if G is not one of the graphs described in Theorem 1 or 2, then $\mathrm{rn}(G|\mathscr{U}) = 2$.

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Received on 1986.06.15