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MOTION OF A SPHERE IN A STRATIFIED FLUID

0. Summary. The Laplace transform method is used to solve the initial value problem of the unsteady flow due to the motion of a sphere (of radius a) in a stratified fluid. First, we consider the case where the sphere moves in a horizontal direction perpendicular to the vertical axis (taken as the z-axis), the direction of the linear, stable stratification of the fluid. We find that the disturbances die out everywhere ultimately except in the column $|z| < a$, on the body and on the z-axis. The solution in the case where the sphere moves along the vertical axis shows that in the limiting steady state the singular surfaces found in the case of homogeneous rotating fluid are absent here. The perturbation in the fluid velocity tends to zero everywhere except on the horizontal tangential planes where the radial velocity has a finite limit.

1. Introduction. In a recent paper Bretherton [1] considered the two-dimensional unsteady flow due to the transverse motion of a circular cylinder in an unbounded rotating fluid (the axis of the cylinder being perpendicular to the axis of the rotation of the fluid) in which he analyzed the initial value problem interpreting the disturbances in terms of plane inertial waves. The unsteady, plane stratified flow analyzed by one of the authors [2] has brought out the similarity between these two types of flows along with certain essential differences. In this paper we consider the unsteady flow due to the motion of a sphere along and perpendicular to the vertical axis (z-axis), the direction of linear, stable stratification of the fluid.

When the sphere moves in the transverse direction (that is, in a horizontal direction), the disturbances die out everywhere in the fluid ultimately, except in the column $|z| \leqslant a$, on the body and on the z-axis. In the column $|z| < a$ the radial and transverse velocities tend to finite limits which depend on the vertical coordinate z, showing thereby that the fluid never moves two-dimensionally. The horizontal velocity in the column has a vertical gradient, a feature in contrast with the Taylor-Proudman constraint for a homogeneous rotating fluid (where the ultimate flow is steady and two-dimensional everywhere).

When the sphere moves with uniform velocity along the vertical direction, we find that the flow caused by the sphere in the column $|z| \leqslant a$ has a finite variable velocity. This velocity-gradient, set up in the liquid column $|z| \leqslant a$, is a direct consequence of the stratification and brings out the essential difference between stratified flow and the flow of a rotating fluid where the fluid inside the circumscribing cylinder moves as a rigid body with a sphere. Also, the singular surfaces found in the case of a rotating fluid are absent here.

2. Transverse motion of a sphere.

(a) *Governing equations and solution.* We suppose the fluid in the undisturbed state to be hydrostatic such that the initial density ϱ_0 is a linear function of the vertical coordinate z alone and if P_0 is the corresponding pressure, then P_0 can be determined from $dP_0/dz = \varrho_0 g$, where $\varrho_0 = \varrho_0' (1-\beta z)$, ϱ_0' and β being constants.

Taking the cylindrical polar coordinates (r, θ, z) such that z is measured positive in the direction opposing gravity, the linearized equations of motion for an incompressible, inviscid, non-diffusive, stratified fluid are

$$
(1) \qquad \left\{
\begin{aligned}
\varrho_0' \frac{\partial u}{\partial t} &= -\frac{\partial P}{\partial r}, \\
\varrho_0' \frac{\partial v}{\partial t} &= -\frac{\partial P}{r\partial \theta}, \\
\varrho_0' \frac{\partial w}{\partial t} &= -\frac{\partial P}{\partial z} - \varrho g, \\
\frac{\partial \varrho}{\partial t} + w\frac{\partial \varrho_0}{\partial z} &= 0, \\
\frac{1}{r}\frac{\partial}{\partial r}(ur) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} &= 0.
\end{aligned}
\right.
$$

The sphere starts impulsively from rest at $t = 0$ and moves in the (r, θ)-plane along the x-direction with a uniform velocity U. Let u, v and w be the components of the fluid velocity relative to the coordinate axes fixed in the body. Since the disturbance is created impulsively, the initial perturbed motion is taken to be irrational with θ as its potential. Also the initial density perturbation is taken to be zero. Now, taking Laplace transforms of equations (1) and defining $\bar{N} = \bar{P} - \theta$, where $\bar{P} = P/\varrho_0'$, the governing equation for $\bar{N}$ can be written as

$$
(2) \qquad \frac{\partial^2 \bar{N}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{N}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{N}}{\partial \theta^2} + \frac{s^2}{s^2+\beta g}\frac{\partial^2 \bar{N}}{\partial z^2} = 0,
$$

where

$$
\bar{N} = \int_0^\infty e^{-st} N dt.
$$

The boundary conditions are $u \to -U\cos\theta$, $v \to U\sin\theta$, $w \to 0$ as $z \to \infty$ for fixed r and t, and, on the sphere $r^2+z^2 = a^2$, $ur+wz = 0$.

In terms of $\overline{N}$ these conditions become

$$\overline{N} \to 0 \quad \text{as } z \to \infty \tag{3}$$

for fixed r, and on the body

$$r\frac{\partial \overline{N}}{\partial r}+\frac{s^2}{s^2+\beta g}z\frac{\partial \overline{N}}{\partial z} = -Ur\cos\theta. \tag{4}$$

In view of the boundary condition (4), the θ-dependence of $\overline{N}$ can be chosen as $\overline{N} = Q(r,z)\cos\theta$. Then equations (2)-(4) reduce to

$$\frac{\partial^2 Q}{\partial r^2}+\frac{1}{r}\frac{\partial \theta}{\partial r}-\frac{Q}{r^2}+\frac{s^2}{s^2+\beta g}\frac{\partial^2 Q}{\partial z^2} = 0 \tag{5}$$

with the conditions $Q \to 0$ as $r^2+z^2 \to \infty$, and

$$r\frac{\partial Q}{\partial r}+\frac{s^2}{s^2+\beta g}z\frac{\partial Q}{\partial z} = -Ur. \tag{6}$$

The velocity components are given by

$$\bar{u} = -\left(\frac{U}{s}+\frac{1}{s}\frac{\partial Q}{\partial r}\right)\cos\theta,$$

$$\bar{v} = \left(\frac{U}{s}+\frac{Q}{rs}\right)\sin\theta,$$

$$\overline{w} = -\frac{s\cos\theta}{s^2+\beta g}\frac{\partial Q}{\partial z}.$$

Using the transformations

$$r = c\sqrt{1+\xi^2}\sqrt{1-\eta^2}, \qquad z = \sqrt{s^2/(s^2+\beta g)}\,c\xi\eta,$$

where $c = -ia\sqrt{\beta g}/s$, and the sphere is given by $\xi = i\sqrt{(s^2+\beta g)/\beta g}$, (5) and (6) become

$$\left[(1+\xi^2)\frac{\partial^2}{\partial \xi^2}+(1-\eta^2)\frac{\partial^2}{\partial \eta^2}+2\xi\frac{\partial}{\partial \xi}-2\eta\frac{\partial}{\partial \eta}-\frac{\xi^2+\eta^2}{(1+\xi^2)(1-\eta^2)}\right]Q = 0 \tag{7}$$

and

$$\left(\frac{\partial Q}{\partial \xi}\right)_{\xi=\xi_0} = \mu\sqrt{1-\eta^2}, \tag{8}$$

where $\mu = \dfrac{iUa}{s^2}\sqrt{\beta g(s^2+\beta g)}$ and $Q \to 0$ as $\xi \to \infty$.

The solution of (7) with conditions (8) is found as

$$Q = iaB(s)\left[\frac{2\xi}{\sqrt{1+\xi^2}} - i(1+\xi^2)\log\frac{\xi - i}{\xi + i}\right]\sqrt{1-\eta^2},$$

where

$$B(s) = \frac{iUs\sqrt{\beta g(s^2+\beta g)}}{D(s)}$$

and

$$D(s) = 2\sqrt{\beta g}(s^2-\beta g) + s^2\sqrt{s^2+\beta g}\log\frac{\sqrt{s^2+\beta g}-\sqrt{\beta g}}{\sqrt{s^2+\beta g}+\sqrt{\beta g}}.$$

The velocity components at any general point in the fluid are therefore given by

$$(9)\quad \begin{cases} u = -U + \dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} \beta(s)\left[\frac{2\xi(\xi^2-\eta^2+2)}{(1+\xi^2)(\xi^2+\eta^2)} - i\log\frac{\xi-i}{\xi+i}\right]e^{st}\,ds, \\ v = \dfrac{U}{s} - \dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} B(s)\left[\frac{2\xi}{1+\xi^2} - i\log\frac{\xi-i}{\xi+i}\right]e^{st}\,ds, \\ w = \dfrac{1}{2\pi i}\displaystyle\int_{r-i\infty}^{r+i\infty} 4s^2B(s)\frac{\eta\sqrt{1-\eta^2}}{\sqrt{1+\xi^2}\,(\xi^2+\eta^2)}\,e^{st}\,ds, \end{cases}$$

where $r > 0$, and

$$\xi^2 = -\frac{1}{2a^2\beta g}\left[(r^2+z^2)\sqrt{s^2+l_1^2}\sqrt{s^2+l_2^2} + s^2(r^2+z^2) + \beta g(z^2+a^2)\right],$$

$$\eta^2 = -\frac{1}{2a^2\beta g}\left[(r^2+z^2)\sqrt{s^2+l_1^2}\sqrt{s^2+l_2^2} - s^2(r^2+z^2) - \beta g(z^2+a^2)\right],$$

$$(r^2+z^2)\sqrt{s^2+l_1^2}\sqrt{s^2+l_2^2}$$
$$= \left[s^4(r^2+z^2)^2 + 2s^2\beta g\left((r^2+z^2)(z^2+a^2) - 2a^2z^2\right) + (z^2-a^2)\beta^2g^2\right]^{1/4}.$$

(b) *General features of the flow.*

(i) Velocity at any general point for large time. The singularities in the integrands in (9) are $\pm il_1$, $\pm il_2$, $\pm i\sqrt{\beta g}$ and zero. The calculation of the contributions from $s = \pm il_1$ to u and w shows that they are of order $O(1/\sqrt{t})$, and to v of order $O(1/\sqrt{t^3})$, whereas the contributions from the branch points $\pm i\sqrt{\beta g}$ to all velocity components are of order $O(1/\sqrt{t^3})$.

Carrying out the calculations for $s = 0$ also, we find at any general point for large time

$$u \to \begin{cases} 0, & |z| > a, \\ \left(-U - U(a^2 - z^2)/r^2\right)\cos\theta, & |z| < a, \end{cases}$$

$$v \to \begin{cases} 0, & |z| > a, \\ \left(U - U(a^2 - z^2)/z^2\right)\sin\theta, & |z| < a, \end{cases}$$

$$w \to 0 \quad \text{for all } z.$$

Thus we note that the instantaneous velocities due to the motion of the sphere die out at all points in the region outside the column $|z| \leqslant a$, while the radial and transverse components are finite inside this column. We see also that in the region $|z| < a$ these horizontal velocities are not independent of the vertical coordinate and hence the flow is not two--dimensional. There exists a vertical gradient for the horizontal velocity, contrary to the case of transverse motion of a sphere in rotating liquid where the ultimate motion is steady and two-dimensional everywhere [3].

(ii) Velocity on the plane $|z| = a$. A similar calculation shows that

$$u\sec\theta = -U + O(1/\sqrt{t}), \qquad v\operatorname{cosec}\theta \doteq U + O(1/\sqrt{t^3}), \qquad w\sec\theta = O(1/\sqrt{t}),$$

which agrees, for $t \to \infty$, with the expression for the velocities inside the column when $|z| = a$.

(iii) Velocity on the circumscribing cylinder $r = a$. Now the singularities for $\bar{u}, \bar{v}$ and $\bar{w}$ are $s = \pm i\sqrt{\beta g}, \pm i\sqrt{\beta g}(z^2 - a^2)/(z^2 + a^2)$ and zero, respectively. The contributions from $\pm i\sqrt{\beta g}$ for large t to u, v and w are of the order $O(1/\sqrt{t^3})$, $O(1/\sqrt[4]{t})$ and $O(1/\sqrt[4]{t^3})$, respectively. Similarly, we can show that the other singularities lead to expressions of the order $O(1/t)$ as $t \to \infty$.

(iv) Velocity on the sphere. On the sphere we have

$$\bar{u} = \frac{4U(\beta g)^{3/2} z^2 s\cos\theta}{D(s^2a^2 + \beta g r^2)}, \qquad \bar{w} = \frac{-4U(\beta g)^{3/2} z r s\cos\theta}{D(s^2a^2 + \beta g r^2)},$$

$$\bar{v} = \frac{-4U(\beta g)^{3/2}\sin\theta}{sd}, \qquad \bar{N} = -\frac{U}{D}\left(D + 4(\beta g)^{3/2}\right) r\cos\theta,$$

where $D = D(s)$ as before.

Conversely, we get the velocity on the sphere for large time as

$$u \simeq \frac{4U(\beta g)^{3/2} z^2 \cos nt\cos\theta}{a^2\left(4\sqrt{\beta g}n^2 + D(in)\right)}$$

$$w \simeq \frac{-4U(\beta g)^{3/2} z r\cos nt\cos\theta}{a^2\left(4\sqrt{\beta g}n^2 + D(in)\right)},$$

$$v \simeq 2U,$$

where $n = \sqrt{\beta g r}/a$.

Hence the velocity on the sphere continues to oscillate indefinitely with finite amplitude. Only the transverse component of the velocity has a finite limit $2U$.

(v) Velocity on the axis $r = 0$. On the axis we have, for large time,

$$u \simeq \left\{ - U + O\left[\frac{2Ua}{zD(ik)} \sqrt{\beta g}(\beta g - k^2)\cos kt\right]\right\} \cos\theta,$$

$$v \simeq \left\{ U - O\left[\frac{2Ua}{zD(ik)} \sqrt{\beta g}(\beta g - k^2)\cos kt\right]\right\} \sin\theta,$$

$$w \simeq 0,$$

where $k = \beta g(z^2 - a^2)/z$.

Thus the flow is unsteady and the vertical velocity component tends to zero whereas the radial and transverse velocities continue to oscillate.

Thus we see that, in general, the disturbances die out everywhere in the field as $t \to \infty$ except in the column $|z| < a$, on the body and on the axis $r = 0$. In the column $|z| < a$ the radial and transverse velocities ultimately tend to finite limits which depend on the vertical coordinate z. This shows that the fluid never moves two-dimensionally and that the horizontal velocity has a vertical gradient, violating the Taylor-Proudman constraint of homogeneous, rotating fluids.

3. Vertical motion of the sphere.

(a) *Governing equations and solution.* The equations governing the flow are

$$\varrho_0' \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial r}, \quad \varrho_0' \frac{\partial w}{\partial t} = -\frac{\partial P}{\partial z} - \varrho g,$$

$$\frac{\partial P}{\partial t} + w\frac{\partial \varrho_0}{\partial z} = 0, \quad \frac{1}{r}\frac{\partial}{\partial r}(ur) + \frac{\partial w}{\partial z} = 0.$$

The sphere is assumed to start from rest impulsively and to move along the vertical direction with a uniform velocity U. With the same meaning for $\overline{N}$ as before, the governing equation for $\overline{N}$ is formed to be

$$\frac{\partial^2 \overline{N}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{N}}{\partial r} + \frac{s^2}{s^2 + \beta g}\frac{\partial^2 \overline{N}}{\partial z^2} = 0.$$

The boundary conditions are

$$u \to 0,\ w \to -U \quad \text{as } z \to \infty \text{ for fixed } r \text{ and } t,$$

$$r\bar{u} + z\bar{w} = 0 \quad \text{on } r^2 + z^2 = a^2,$$

which, expressed in terms of N, take the form

$$\frac{\partial \overline{N}}{\partial r}+\frac{s^2}{s^2+\beta g}z\frac{\partial \overline{N}}{\partial z} = -Uz \quad \text{on } r^2+z^2 = a^2,$$

$$N \to 0 \quad \text{as } z \to \infty.$$

Transforming to the (ξ, η)-coordinates, where

$$r = c\sqrt{1+\xi^2}\sqrt{1-\eta^2} \quad \text{and} \quad z\sqrt{(s^2+\beta g)/s^2} = c\xi\eta,$$

the governing equation and boundary conditions reduce to

$$\left[(1+\xi^2)\frac{\partial^2}{\partial \xi^2}+(1-\eta^2)\frac{\partial^2}{\partial \eta^2}+2\xi\frac{\partial}{\partial \xi}-2\eta\frac{\partial}{\partial \eta}\right]\overline{N} = 0, \tag{10}$$

with

$$\left(\frac{\partial \overline{N}}{\partial \xi}\right)_{\zeta=\xi_0} = \frac{i\sqrt{\beta g}\,Ua\sqrt{s^2+\beta g}}{s^2}\eta, \quad \overline{N} \to 0 \text{ as } \xi \to \infty.$$

We find the solution of (10), and then $\bar{u}$ and $\overline{w}$. Finally, the inverse transforms give the pressure and velocity at any general point of the fluid.

(b) *General features of the flow.*

(i) At any general point in the flow field. The singularities of the integrands in the Laplace inversion integrals are at $s = \pm i\sqrt{\beta g}$, $\pm il_1$, $\pm il_2$ and zero. The contributions from the branch points $\pm il_1$ and $\pm il_2$ to u and w are $O(1/\sqrt{t})$, whereas those from $\pm i\sqrt{\beta g}$ are $O(1/\sqrt{t^3})$. The singularity $s = 0$ gives for large $t \to \infty$

$$u \simeq \begin{cases} Uz/2r+O(1/t), & |z| < a, \\ 0, & |z| > a, \end{cases}$$

$$w \simeq -U+O(1/t) \quad \text{for all } z.$$

Hence, in general, the ultimate flow is steady and the perturbations tend to zero at any general point except in the column of liquid between the parallel planes $|z| = a$, where the radial velocity acquires a finite limit depending on the position coordinates showing that the velcoity has a gradient in that column.

(ii) On the circumscribing cylinder $r = a$. Here the contributions from $s = \pm i\sqrt{\beta g}$ are of the order $O(t^{-5/4})$ for u and of $O(t^{-3/4})$ for w.

(iii) On the tangential planes $|z| = a$. Now the radial velocity approaches a finite limit $Ua/2r$ and the vertical velocity tends to $-U+O(1/t)$. A similar procedure shows that the velocity oscillates on the sphere; on the axis the unidirectional flow along the vertical direction continues to oscillate.

Thus we conclude that if a sphere moves in an unbounded, stratified fluid with a uniform velocity along the vertical direction, starting from rest impulsively, then the perturbation tends to zero everywhere in the flow except in the column $|z| < a$, where the liquid moves with a finite variable velocity and the perturbed velocity continues to oscillate indefinitely on the body and on the axis. The ultimate velocity gradient set up in the liquid column $|z| < a$, being a direct consequence of stratification, brings out an essential difference between stratified flow and the flow of a homogeneous rotating fluid, where the fluid inside the circumscribing cylinder moves as a rigid body along with the body. Another important difference between the two types of flows is the absence of singular surfaces (where the velocity becomes infinite) in the case of stratified flow.

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RUCH KULI W ROZWARSTWIONEJ CIECZY

STRESZCZENIE

W pracy rozpatruje się przepływ niestacjonarny, wywołany przez ruch kul w cieczy rozwarstwionej, jako zagadnienie początkowe. Najpierw rozważany jest przypadek, gdy kula porusza się w kierunku poziomym, prostopadłym do pionowej osi ustalonego liniowego rozwarstwienia cieczy. Pokazano, że zaburzenia wywołane ruchem kuli zanikają z upływem czasu wszędzie poza słupem $z = a$, powierzchnią ciała i osią z. Z postaci rozwiązania wynika, że gdy kula porusza się wzdłuż osi pionowej, przy przejściu do granicznego stanu stacjonarnego nie pojawiają się powierzchnie osobliwe, występujące w przypadku jednorodnej, obracającej się cieczy.