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THE CHARACTERIZATION OF DISCRETE DISTRIBUTIONS BY CONDITIONAL DISTRIBUTIONS

1. Introduction. Several papers have appeared ([3], [5]-[7]) which deal with the distribution of independent random variables X and Y subject to the condition that $X+Y$ is fixed. In the present paper, inspired by a paper of Svensson [10], we consider similar problems omitting (in general) the assumption of independence.

2. General formulation. Let X and Y be random variables defined over the non-negative integers. Let the conditional distribution of X for $X+Y$ fixed be denoted by

$$p_m(n) = P(X = m \mid X+Y = n) \quad (m = 0, \ldots, n) \tag{1}$$

with probability generating function

$$\varphi_n(s) = \sum_{m=0}^{n} p_m(n) s^m \quad (n = 0, 1, 2, \ldots). \tag{2}$$

Suppose further that the probabilities and generating functions of the variables X, Y, $X+Y$ and (X, Y) are denoted by the following system:

$$a_n = P(X = n), \qquad \pi_X(s) = \sum_{n=0}^{\infty} a_n s^n,$$

$$b_n = P(Y = n), \qquad \pi_Y(s) = \sum_{n=0}^{\infty} b_n s^n,$$

$$c_n = P(X+Y = n), \qquad \pi_{X+Y}(s) = \sum_{n=0}^{\infty} c_n s^n;$$

$$d_{mn} = P(X = m, Y = n), \qquad \pi_{XY}(s, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} d_{mn} s^m t^n. \tag{3}$$

Equation (3) can be written in the form

$$\pi_{XY}(s, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_m(m+n) c_{m+n} s^m t^n$$

which is equivalent to

$$\pi_{XY}(s, t) = \sum_{k=0}^{\infty} \sum_{m=0}^{k} p_m(k) c_k s^m t^{k-m} \tag{4}$$

from which we obtain

$$\pi_{XY}(s, t) = \sum_{k=0}^{\infty} \varphi_k(s/t) t^k c_k. \tag{5}$$

The single variable generating functions for X and Y follow from (5):

$$\pi_X(s) = \pi_{XY}(s, 1) = \sum_{k=0}^{\infty} \varphi_k(s) c_k \tag{6}$$

and

$$\pi_Y(s) = \pi_{XY}(1, s) = \sum_{k=0}^{\infty} \varphi_k(1/s) s^k c_k. \tag{7}$$

3. Equidistribution. From (6) and (7) it follows that X and Y will be equidistributed (same form with same parameters) if

$$\varphi_k(s) = s^k \varphi_k(1/s) \tag{8}$$

is an identity in s for all k, that is if the conditional distribution is symmetrical:

$$p_m(n) = p_{n-m}(n) \quad (m = 0, \ldots, n).$$

The converse of this is not true, however, as the following simple example shows. Let $P(X = 0, Y = 1) = P(X = 1, Y = 2) = P(X = 2, Y = 0)$. Then X and Y are (rectangularly) equidistributed with unit mean, although $p_0(1) = 1 \neq p_1(1)$, $p_2(2) = 1 \neq p_0(2)$, $p_1(3) = 1 \neq p_2(3)$.

4. Independence. The formula

$$p_m(n) c_n = a_m b_{n-m} \quad (m = 0, \ldots, n) \tag{9}$$

is equivalent to the assumption of independence between X and Y. If we substitute (9) with $m = n = k$ into (6), we obtain

$$\pi_X(s) = \sum_{k=0}^{\infty} \varphi_k(s) \frac{b_0 a_k}{p_k(k)}.$$

Setting $s = 1$, we find the value of b_0 and are able to express the probabilities and generating functions for X explicitly in terms of those for the conditional distribution (1)-(2):

$$\pi_X(s) = \left\{ \sum_{k=0}^{\infty} \frac{\varphi_k(s) a_k}{p_k(k)} \right\} \Bigg/ \left\{ \sum_{k=0}^{\infty} \frac{a_k}{p_k(k)} \right\}. \tag{10}$$

Similarly, we find that

$$\pi_Y(s) = \left\{ \sum_{k=0}^{\infty} \varphi_k(1/s) s^k \frac{b_k}{p_0(k)} \right\} \Big/ \left\{ \sum_{k=0}^{\infty} \frac{b_k}{p_0(k)} \right\}$$

and, setting $s = t$ in (5), that

$$\pi_{X+Y}(s) = \left\{ \sum_{k=0}^{\infty} s^k \frac{b_k}{p_0(k)} \right\} \Big/ \left\{ \sum_{k=0}^{\infty} \frac{b_k}{p_0(k)} \right\}.$$

5. Conditional distribution binomial. If we assume that

$$p_m(n) = \binom{n}{m} r^m (1-r)^{n-m} \qquad (m = 0, \dots, n), \tag{11}$$

then

$$\varphi_n(s) = (1-r+rs)^n. \tag{12}$$

Substituting (12) into (5) we obtain

$$\pi_{XY}(s, t) = \pi_{X+Y}\{(1-r)t + rs\} \tag{13}$$

and thus

$$\pi_{XY}(s, t) = \pi_X\{(t - rt + rs - 1 + r)/r\};$$

furthermore,

$$\pi_{X+Y}(s) = \pi_X\{(s-1+r)/r\} \tag{14}$$

and

$$\pi_Y(s) = \pi_X\{(s - rs + 2r - 1)/r\}. \tag{15}$$

Equations (13) and (14) are given by Svensson [10], who also gives a constructive proof for the implication in the opposite direction:

SVENSSON'S THEOREM. *If X is a random variable over the non-negative integers, then there exists a two-dimensional random variable (X, Y) satisfying* (12) *if and only if* (14) *is a probability generating function.*

If the conditions of Svensson's Theorem are satisfied, the relationships between the probabilities a_n, b_n and c_n can be found by expanding the series; (14), for example, yields

$$c_n = (1-\Omega)^n \sum_{m=0}^{\infty} \binom{m+n}{n} a_{m+n} \Omega^m \qquad (n = 0, \dots), \tag{16}$$

where $\Omega = (r-1)/r$.

The corresponding calculation for (15) is simplified by noting that it is of the form of (14) with r replaced by $r/(1-r)$. Hence the b_n also satisfy (16), with $\Omega = (2r-1)/r$.

EQUIDISTRIBUTION. *X and Y are equidistributed when $r = 1/2$; X and $X+Y$ are equidistributed only in the degenerate case $r = 1$; Y and $X+Y$ are equidistributed only in the degenerate case $r = 0$.*

INDEPENDENCE. *X and Y are independent if and only if X and Y are both Poisson with parameters λ and μ satisfying*

$$\lambda/\mu = r/(1-r). \tag{17}$$

This result is given by Moran [6] and Chatterji [3] and is called by Svensson *The Moran-Chatterji Theorem* [10]. Chatterji gives a number of interesting variants of the theorem for cases in which $r = r_n$ in (11); it turns out that r is indeed independent of n in any case, but the method of proof differs considerably, as Svensson points out.

In our formulation, the result follows directly from equation (10); for the substitution (12) leads to the functional equation

$$\pi_X(1/r)\pi_X(s) = \pi_X\{(1-r+rs)/r\}$$

with solution (cf. Aczél [1], Theorem 2, p. 67)

$$\pi_X(s) = \exp(-\lambda+\lambda s)$$

and the corresponding calculation for $\pi_Y(s)$ yields (17).

Special cases. If X is negative binomial with parameters a and k:

$$\pi_X(s) = \{a/(1+a-s)\}^k;$$

then Y is also negative bionomal with parameters $ar/(1-r)$ and k; $X+Y$ is negative binomial with parameters ar and k, and the joint probability generating function of X and Y is

$$\pi_{XY}(s,t) = \{ar/(1+ar-t+rt-rs)\}^k.$$

Svensson's condition is satisfied for all a and k.

6. Conditional distribution rectangular. If we assume that

$$p_m(n) = (1+n)^{-1} \quad (m = 0, \ldots, n),$$

then

$$\varphi_n(s) = (1-s^{n+1})/(1-s)(1+n) \tag{18}$$

and (5) becomes

$$\pi_{XY}(s,t) = (t-s)^{-1}\int_s^t \pi_{X+Y}(u)\,du.$$

Setting $s = 1$ and then $t = 1$, we obtain $\pi_X(s) = \pi_Y(s)$ which follows from (8) in any event, and $s = t$ yields

$$\pi_{X+Y}(s) = \pi_X(s)+(s-1)\pi'_X(s). \tag{19}$$

Thus, the joint probability generating function can be written as

$$\pi_{XY}(s,t) = (t-s)^{-1}\{(t-1)\pi_X(t)-(s-1)\pi_X(s)\}. \tag{20}$$

The condition corresponding to Svensson's for the binomial conditional is very simple: for discrete distribution of X, there exists a two-dimensional random variable satisfying (18) if and only if $a_n > a_{n+1}$ for all n, i.e. the distribution of X must be unimodal at the origin. This follows from the explicit relations connecting the various probabilities, which are not difficult to obtain:

$$\begin{aligned} c_n &= (n+1)(a_n-a_{n+1}) \quad (n=0,1,\ldots), \\ d_{mn} &= c_{m+n}/(m+n+1) = a_{m+n}-a_{m+n+1}. \end{aligned} \tag{21}$$

Independence. Substituting (18) into (10) yields

$$\pi_X(s) = \frac{1-s\pi_X(s)}{\{1+E(X)\}(1-s)};$$

with the substitution $\varrho = E(X)/[1+E(X)]$ the standard form

$$\pi_X(s) = (1-\varrho)/(1-\varrho s)$$

follows. Therefore, the distribution of both X and Y is geometric and with the same parameter, when they are independent with conditional distribution rectangular. This result is due to Ferguson [5] and is given as a problem by Feller ([4], third edition only, p. 237, problem 7).

Special cases. If X is negative binomial with parameters a and k, the condition $a_n > a_{n+1}$ will be satisfied whenever $a+1>k$, and in particular for the geometric $k=1$. Then $X+Y$ is not negative binomial (unless $k=1$); in fact, equation (19) gives

$$\pi_{X+Y}(s) = \{a/(1+a-s)\}^k\left[\frac{k(s-1)+(1-s+sa)}{1-s+sa}\right]$$

corresponding to $c_n = a_n\{(1+a+na-k)/(1+a)\}$.

Similarly, setting $a = \varrho/(1-\varrho)$, equation (20) yields

$$\pi_{XY}(s,t) = \{(1-\varrho)/(1-\varrho s)(1-\varrho t)\}^k\left\{\frac{(t-1)(1-\varrho s)^k-(s-1)(1-\varrho t)^k}{t-s}\right\}$$

which yields independence of X and Y when $k=1$.

Non-trivial distributions can also be obtained by assuming that X is a Poisson distributed variable with parameter λ. Of course Y has the same distribution, but $X+Y$ is not Poisson distributed,

$$\pi_{X+Y}(s) = (s\lambda-\lambda+1)\exp(-\lambda+\lambda s),$$

and the joint probability generating function is

$$\pi_{XY}(s,t) = (t-s)^{-1}\{(t-1)\exp(-\lambda+\lambda t)-(s-1)\exp(-\lambda+\lambda s)\}.$$

7. Applications. *Accident analysis.* Rasch [9] has shown that a before-and-after problem in accident analysis can be greatly simplified by consideration of the conditional distribution. If accidents are assumed to occur at instants in time which form a Poisson process with parameter λ before some "improvement" is made in the road system, and with parameter $r\lambda$ afterwards, then, by virtue of equation (17) testing for the effectiveness of the improvement can be reduced to the problem of testing a binomial parameter. In this way the large number of road and time segments can be reduced by grouping together all those for which the sum of the before and after values is constant.

Inventory theory. Bissinger [2] and Prichard and Eagle ([8], Section 9.3, p. 197-205) consider a probability transformation of the form

$$q_n = (1-p_0)^{-1}\sum_{j=n+1}^{\infty} p_j/j. \tag{22}$$

There are several features of this transformation which are rather awkward. In the first place, the initial probability p_0 does not occur in the triangular array of the elements of $\sum p_j/j$ and must be compensated for in the factor $(1-p_0)$. Thus, the relationship (22) between p_n and q_n corresponds to a truncated form for q_n rather than to the natural form

$$q_n = \sum_{j=n}^{\infty} p_j/(j+1).$$

This is equivalent to equation (21).

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STRESZCZENIE

W literaturze znane są prace, w których podano rozkłady niezależnych zmiennych losowych X i Y pod warunkiem, że $X+Y$ jest ustalone ([3], [5]-[7]). W niniejszej pracy autor, opierając się na wynikach otrzymanych przez Svenssona [10], rozważa podobne problemy bez założenia niezależności zmiennych losowych X i Y.