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J. BARTOSZEWICZ (Wrocław)

INADMISSIBILITY OF TWO ESTIMATORS OF RELIABILITY IN THE EXPONENTIAL CASE

1. Introduction. Let N identical elements having the life-time distribution with probability density function

$$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x), \quad \lambda \in \Lambda = (0, \infty),$$

where $I_A(u)$ denotes the indicator of a set A, be placed simultaneously on life test with replacement of the failed elements and with duration of the observation until the fixed moment T. Let $D(t)$ denote the number of failures until the moment t and let $x_1, x_2, \ldots, x_{D(T)}$ be the moments of failures until the moment T. The problem of the unbiased estimation of the reliability function $R(t) = e^{-\lambda t}$, $t \geqslant 0$, on the basis of the truncated sample $(X_1, X_2, \ldots, X_{D(T)})$ has been studied by Bartoszewicz [1], Klimov [3], and Belaev and Smirnov [2]. In [1] the distribution of the minimal sufficient statistic $\big(D(T), S(T)\big)$, where

$$S(T) = \sum_{i=1}^{D(T)} X_i + \big(N - D(T)\big)T,$$

has been given. The joint probability density function of the statistic $(D, S) = \big(D(T), S(T)\big)$ is of the form

$$(1)\qquad f(d, s) = \begin{cases} e^{-\lambda NT} & \text{if } d = 0, \\ \binom{N}{d} \lambda^d e^{-\lambda s} V_d\big(s - (N-d)T; T\big) & \text{if } d = 1, 2, \ldots, N, \end{cases}$$

where

$$V_d(u; T) = \frac{1}{(d-1)!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} (u - iT)_+^{d-1}$$

and $a_+ = \max(0, a)$. It easy to see that the distribution of (D, S) is concentrated on the set $\{(d, s)\colon d = 0, 1, \ldots, N,\ s \in [(N-d)T, NT]\}$. Bartoszewicz [1] has also proved that the statistic (D, S) is incomplete except for the trivial case $N = 1$ and he has constructed, using the Rao-Blackwell theorem, the unbiased estimator of $R(t)$, $t \in [0, NT]$, being of the form

$$(2) \quad \hat{R}_1(D, S; t) = \mathrm{E}[\hat{R}_0 \mid D, S]$$

$$= \frac{\binom{N-D}{k}}{\binom{N}{k}} \left\{ 1 - \frac{D}{(N-k)(D-1)!} \times \right.$$

$$\left. \times \frac{\sum_{i=0}^{D-1} \binom{D-1}{i} \sum_{j=0}^{1} (-1)^{j+1} [S - j\tau - (N-D+i)T]_+^{D-1}}{V_D\big(S-(N-D)T; T\big)} \right\},$$

where $k = [t/T]$ (the integer part of the number t/T), $\tau = t - kT$, and

$$\hat{R}_0 = \frac{\binom{N-D}{k}}{\binom{N}{k}} \left[1 - \frac{D(\tau)}{N-k} \right].$$

The estimator $\hat{R}_1$ is better than the unbiased estimator $\hat{R}_0$ which is based on the empirical distribution function, using any strictly convex error loss.

Klimov [3] has given another estimator of $R(t)$, $t \in [0, NT]$, more elegant than $\hat{R}_1$. Klimov's estimator is of the form

$$(3) \quad \hat{R}_2(D, S; t) = \mathrm{E}\big[\delta(X_1, \ldots, X_N) \mid D, S\big] = \frac{V_D(S - Dt/N; T - t/N)}{V_D\big(S - (N-D)T; T\big)},$$

where

$$\delta(x_1, \ldots, x_N) = \begin{cases} 1 & \text{if } \min(x_1, \ldots, x_N) > t/N, \\ 0 & \text{otherwise.} \end{cases}$$

However, estimators $\hat{R}_1$ and $\hat{R}_2$ are not the uniformly minimum variance unbiased (UMVU) estimators of $R(t)$, $t \in [0, NT)$, if $N \geqslant 2$. Belaev and Smirnov [2] have given a characterization of the class of all unbiased estimators of zero based on the minimal sufficient statistic (D, S) and they have proved that if $N \geqslant 2$ and $t \in [0, NT)$, then the UMVU estimator of $R(t)$ does not exist. If $t = NT$, then the statistic $I_{\{0\}}(D)$ is the UMVU estimator of $R(t)$. In the trivial case $N = 1$ the UMVU estimator of $R(t)$ exists and is equal to the statistic $I_{[t,T]}(S)$, $t \leqslant T$. If $t > NT$ and $N \geqslant 1$, then the UMVU estimator of $R(t)$ does not exist.

Applying the Belaev-Smirnov result we prove in this paper that the estimators $\hat{R}_1$ and $\hat{R}_2$ based on the minimal sufficient statistic and obtained in a very natural way are not even admissible on every compact subset of the parameter space $\Lambda = (0, \infty)$, using squared error loss. Before this we give a theorem on inadmissibility of unbiased estimators for incomplete exponential distribution families and we present the Belaev-Smirnov characterization of the class of unbiased estimators of zero based on the statistic (D, S).

2. Inadmissibility of unbiased estimators. In the sequel we use the following corollary to the well-known Lehmann-Scheffé theorem.

THEOREM 1. *Let $\gamma = \{P_\theta, \theta \in \Theta\}$, $\Theta \subset \mathbf{R}^n$, be the incomplete exponential family of distributions of the minimal sufficient statistic X, let $\mathcal{M}$ be the nonempty set of all unbiased estimators of zero based on X with finite variance for every $\theta \in \Theta$, and let $g(X)$ be an unbiased estimator of the real parametric function $\gamma(\theta)$, $\theta \in \Theta$, such that $\mathrm{E}_\theta[g(X)]^2 < \infty$ for every $\theta \in \Theta$. If there exists $\varphi_0 \in \mathcal{M}$ such that*

$$\mathrm{Cov}_\theta(\varphi_0, g) = \mathrm{E}_\theta[\varphi_0(X)g(X)] > 0 \ (or < 0) \quad for\ every\ \theta \in \Theta_0,$$

where Θ_0 is a compact subset of the set Θ, then the estimator $g(X)$ is inadmissible on Θ_0 using squared error loss.

The proof of this theorem may be found in [4], Chapter VII.

3. Unbiased estimators of zero. Belaev and Smirnov [2] and also Torgersen [5] have characterized the class of all unbiased estimators of zero with finite variance for the distribution family (1), $\lambda > 0$, if $N \geqslant 2$.

THEOREM 2. *A statistic $\varphi(D, S)$ is an unbiased estimator of zero with finite variance for every $\lambda > 0$ if and only if*

$$\varphi(d, s) = \frac{h(d, s)}{\binom{N}{d} V_d\big(s - (N-d)T; T\big)} \tag{4}$$

$$if\ d = 0, 1, \ldots, N\ and\ s \in [(N-d)T, NT],$$

where $h(d, \cdot)$ are square-integrable functions on intervals $[(N-d)T, NT]$, $d = 0, 1, \ldots, N$, and satisfy the following conditions:

$$h(0, NT) = 0, \tag{5}$$

$$\sum_{d=1}^{m} \frac{(-1)^{m-d}}{(m-d)!} \int_{(N-d)T}^{NT} x^{m-d} h(d, x)\,dx = 0, \quad m = 1, 2, \ldots, N-1, \tag{6}$$

(7)

$$h(N, s) = \begin{cases} 0 & \text{if } s \in [0, T], \\ -\sum_{d=1}^{N-1} \int_{(N-d)T}^{\max(s,(N-d)T)} h(d, x) \frac{(s-x)^{N-d-1}}{(N-d-1)!} dx & \text{if } s \in (T, NT]. \end{cases}$$

4. Inadmissibility of the estimator $\hat{R}_1$.

THEOREM 3. *If $N \geqslant 2$, then for every $t \in [0, NT)$ the estimator $\hat{R}_1$ defined by (2) is inadmissible on every compact subset of the parameter space $\Lambda = (0, \infty)$, using squared error loss.*

Proof. Theorem 1 implies that in order to prove this theorem it is sufficient to find for every $t \in [0, NT)$ an unbiased estimator of zero $\varphi_t(D, S)$ such that

$$\text{Cov}_\lambda(\varphi_t, \hat{R}_1) = \mathrm{E}_\lambda[\varphi_t(D, S)\hat{R}_1(D, S; t)] > 0 \quad \text{for every } \lambda > 0. \tag{8}$$

First we consider the case where t is a fixed number from $[0, T]$. It is easy to notice from (2) that

$$\hat{R}_1(1, s; t) = \begin{cases} 1-1/N & \text{if } s \in \big((N-1)T, (N-1)T+t\big], \\ 1 & \text{if } s \in \big((N-1)T+t, NT\big], \end{cases} \tag{9}$$

and

(10)

$$\hat{R}_1(2, s; t) = \begin{cases} 1 - \frac{2}{N}\left[1 - \frac{2(T-t)}{NT-s}\right] & \text{if } s \in \big((N-1)T,\ (N-1)T+t\big], \\ 1 & \text{if } s \in \big((N-1)T+t, NT\big]. \end{cases}$$

We use these simple expressions to construct the estimator φ_t. It follows from (5) that

$$\varphi_t(0, NT) = 0. \tag{11}$$

Let us assume that

$$\varphi_t(3, s) \equiv \varphi_t(4, s) \equiv \ldots \equiv \varphi_t(N, s) \equiv 0. \tag{12}$$

Therefore, we have to define functions $\varphi_t(1, \cdot)$ and $\varphi_t(2, \cdot)$ which are not identically equal to zero, take the form (4), and satisfy (6) and (7). It is easy to prove that formulas (6) and (7) are of a simpler form in this case, namely

$$\int_{(N-1)T}^{NT} h_t(1, s)\, ds = 0 \tag{13}$$

and

$$(14)\qquad h_t(2,s) = \begin{cases} -\int\limits_{(N-1)T}^{\max(s,(N-1)T)} h_t(1,x)\,dx & \text{if } s\in[(N-1)T, NT],\\ 0 & \text{otherwise}.\end{cases}$$

From (9)-(14) and (1) we obtain

$$\mathrm{Cov}_\lambda(\varphi_t, \hat{R}_1) = \lambda^2 \int\limits_{(N-1)T}^{NT} W(s)e^{-\lambda s}ds,$$

where

$$W(s) = \begin{cases} \dfrac{1}{N}\left[1-\dfrac{2(T-t)}{NT-s}\right] \int\limits_{(N-1)T}^{\max(s,(N-1)T)} h_t(1,x)\,dx & \text{if } s\in\big[(N-1)T,(N-1)T+t\big),\\ -\dfrac{1}{N}\int\limits_{(N-1)T}^{(N-1)T+t} h_t(1,x)\,dx & \text{if } s\in[(N-1)T+t, NT].\end{cases}$$

Notice that if $s\in\big[(N-1)T, NT\big)$ and $t\in[0,T)$, then

$$1-\frac{2(T-t)}{NT-s}\begin{cases} >0 & \text{if } s\in\big[(N-1)T, \max\big(NT-T, (N-2)T+2t\big)\big),\\ <0 & \text{if } s\in\big(\max\big(NT-T, (N-2)T+2t\big), NT-T+t\big).\end{cases}$$

Therefore, the function $h_t(1,\cdot)$, being, e.g., of the form

$$(15)\qquad h_t(1,x) = \begin{cases} 1 & \text{if } x\in\big[(N-1)T, (N-1)T+(2t-T)_+/2\big),\\ -\dfrac{1+(2t-T)_+}{2t-(2t-T)_+} & \text{if } x\in\big[(N-1)T+(2t-T)_+/2, (N-1)T+t\big),\\ (T-t)/2 & \text{if } x\in[(N-1)T+t, NT],\end{cases}$$

makes $W(s)>0$ for all $s\in\big[(N-1)T, NT\big)$ except for $s=\max\big[(N-1)T, (N-2)T+2t\big)$ only. Hence the estimator φ_t defined by (4) and (11)-(15) satisfies (8). This completes the proof in the case $t\in[0,T]$.

Now let t be a fixed number from $[T, NT)$, $k=[t/T]$, and $\tau=t-kT$. It is easy to see from (2) that

$$\hat{R}_1(d,s;t)=0 \qquad \text{if } d=N-k \text{ and } s\in(kT,t] \text{ and}$$
$$\qquad\qquad \text{if } d=N-k+1, N-k+2, \ldots, N \text{ and } s \text{ is arbitrary}$$

and

$$\hat{R}_1(N-k,s;t)>0 \quad \text{if } s\in(t, NT].$$

Therefore, for any unbiased estimator of zero φ_t we have

$$\mathrm{Cov}_\lambda(\varphi_t, \hat{R}_1) = \sum_{d=1}^{N-k-1} \int_{(N-d)T}^{NT} \varphi_t(d, s)\hat{R}_1(d, s; t)f(d, s)ds + \\ + \int_t^{NT} \varphi_t(N-k, s)\hat{R}_1(N-k, s; t)f(N-k, s)ds.$$

We find an estimator φ_t such that (8) is also satisfied in this case.
It is obvious that $\varphi_t(0, NT) = 0$. We assume also that

$$\varphi_t(1, s) \equiv \varphi_t(2, s) \equiv \ldots \equiv \varphi_t(N-k-1, s) \equiv 0. \tag{16}$$

The problem is to find a function $\varphi_t(N-k, \cdot)$ positive for $s \in [t, NT]$ and satisfying (4), (6), and (7). The functions $\varphi_t(i, \cdot)$, $i = N-k+1, \ldots, N$, can be arbitrary, but they have to satisfy (4), (6), and (7). For convenience we put

$$\varphi_t(N-k+1, s) \equiv \varphi_t(N-k+2, s) \equiv \ldots \equiv \varphi_t(N-1, s) \equiv 0, \tag{17}$$

and hence only $\varphi_t(N, \cdot)$, defined by $\varphi_t(i, \cdot)$, $i < N$, and by (4), (6), and (7), cannot be identically zero. Under the above assumptions we have

$$\mathrm{Cov}_\lambda(\varphi_t, \hat{R}_1) = \int_t^{NT} h_t(N-k, s)\hat{R}_1(N-k, s; t)\lambda^{N-k}e^{-\lambda s}ds, \tag{18}$$

where $h_t(N-k, s)$ satisfies k equalities

$$\int_{kT}^{NT} x^{j-1}h_t(N-k, x)dx = 0, \quad j = 1, 2, \ldots, k, \tag{19}$$

and also

$$h_t(N, s) = -\int_{kT}^{\max(s,kT)} h_t(N-k, x)\frac{(s-x)^{k-1}}{(k-1)!}dx, \quad s \in (T, NT]. \tag{20}$$

It is obvious that $\mathrm{Cov}_\lambda(\varphi_t, \hat{R}_1) > 0$ for every $\lambda > 0$ if $h_t(N-k, s) > 0$ for $s \in [t, NT]$, and hence we may put

$$h_t(N-k, s) = c > 0 \quad \text{if } s \in [t, NT]. \tag{21}$$

Such a function $h_t(N-k, \cdot)$, square-integrable, orthogonal to the function set $\{1, x, \ldots, x^{k-1}\}$ on $[kT, NT]$, and constant on $[t, NT]$, exists and may be easily constructed. From (19) and (21) we obtain the system of equalities

$$\int_{kT}^{t} x^{j-1}h_t(N-k, x)dx = c_j,$$

where $c_j = -(c/j)[(NT)^j - t^j]$, $j = 1, 2, \ldots, k$. Let

$$A_j = \left[kT + \frac{j-1}{k}\tau,\ kT + \frac{j}{k}\tau\right), \quad j = 1, 2, \ldots, k,$$

and

$$a_{ij} = \int_{A_j} x^{i-1}\,dx, \quad i, j = 1, 2, \ldots, k.$$

Then $[kT, t) = \bigcup_{j=1}^{k} A_j$. It is easy to see that we may put

$$(22) \qquad h_t(N-k, s) = \begin{cases} \eta_j & \text{if } s \in A_j,\ j = 1, 2, \ldots, k, \\ c & \text{if } s \in [t, NT], \end{cases}$$

where $\eta_1, \eta_2, \ldots, \eta_k$ satisfy the system of linear equations

$$(23) \qquad \sum_{j=1}^{k} a_{ij}\eta_j = c_i, \quad i = 1, 2, \ldots, k.$$

The system (23) has a unique solution because the function set $\{1, x, \ldots, x^{k-1}\}$ is linearly independent. Hence for the estimator φ_t, defined by (4), (16), (17), (22), and (21), inequality (8) holds for every $\lambda > 0$. This completes the proof of Theorem 3.

5. Inadmissibility of the estimator $\hat{R}_2$.

THEOREM 4. *If $N \geqslant 2$, then for every $t \in [0, NT)$ the estimator $\hat{R}_2$ defined by* (3) *is inadmissible on every compact subset of the parameter space Λ, using squared error loss.*

Proof. Similarly as in the proof of Theorem 3, we show that for every $t \in [0, NT)$ there exists an unbiased estimator of zero $\varphi_t(D, S)$ such that

$$(24) \qquad \mathrm{Cov}_\lambda(\varphi_t, \hat{R}_2) > 0 \quad \text{for every } \lambda > 0.$$

Let t be a fixed number from $[0, NT)$. Notice that

$$(25) \qquad \hat{R}_2(d, s; t) = 0$$

$$\text{if } d = 0, 1, \ldots, N \text{ and } s \notin \left((N-d)T + d\,\frac{t}{N},\ NT\right].$$

Consider an unbiased estimator of zero φ_t such that

$$(26) \qquad \varphi_t(0, NT) \equiv \varphi_t(1, s) \equiv \ldots \equiv \varphi_t(N-2, s) \equiv 0$$

and the functions $\varphi_t(N-1, \cdot)$ and $\varphi_t(N, \cdot)$ are not identically equal to zero. Then from (25) we obtain

$$\text{(27)}\qquad \operatorname{Cov}_\lambda(\varphi_t, \hat{R}_2) = \int_{T+\frac{N-1}{N}t}^{NT} h_t(N-1, s)\hat{R}_2(N-1, s; t)\lambda^{N-1}e^{-\lambda s}ds + \int_{\max(t,T)}^{NT} h_t(N, s)\hat{R}_2(N, s; t)\lambda^{N}e^{-\lambda s}ds,$$

where the functions $h_t(N-1, \cdot)$ and $h_t(N, \cdot)$ satisfy (6) and (7). In order to make the covariance (27) positive for every $\lambda > 0$ it is sufficient to find a function $h_t(N-1, \cdot)$, square-integrable on $[T, NT]$, positive on $[T+(N-1)t/N, NT]$, satisfying the condition

$$\int_T^{NT} h_t(N-1, s)ds = 0,$$

and such that

$$h_t(N, s) = -\int_T^{\max(s,(N-d)T)} h_t(N-1, x)dx > 0 \quad \text{if } s \in [\max(t, T), NT].$$

It is easy to verify that such a function takes, e.g., the form

$$\text{(28)}\qquad h_t(N-1, s) = \begin{cases} 0 & \text{if } s \in \big[0, \max(t, T)\big), \\ -\dfrac{1}{2\left[T+\dfrac{N-1}{N}t-\max(t, T)\right]} & \text{if } s \in \left[\max(t, T), T+\dfrac{N-1}{N}t\right), \\ \dfrac{1}{2(N-1)(T-t/N)} & \text{if } s \in \left[T+\dfrac{N-1}{N}t, NT\right], \end{cases}$$

and the function $h_t(N, \cdot)$ is of the form

$$\text{(29)}\qquad h_t(N, s) = \begin{cases} 0 & \text{if } s \in \big[0, \max(t, T)\big), \\ \dfrac{s-\max(t, T)}{2\left[T+\dfrac{N-1}{N}t-\max(t, T)\right]} & \text{if } s \in \left[\max(t, T), T+\dfrac{N-1}{N}t\right), \\ \dfrac{T+\dfrac{N-1}{N}t-s}{2(N-1)(T-t/N)}+\dfrac{1}{2} & \text{if } s \in \left[T+\dfrac{N-1}{N}t, NT\right]. \end{cases}$$

Hence the estimator φ_t defined by (4), (26), (28), and (29) satisfies inequality (24) for every $\lambda > 0$. This completes the proof of Theorem 4.

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MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
50-384 WROCŁAW

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