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ON ORDER STATISTICS FOR RANDOM SAMPLE SIZE HAVING COMPOUND BINOMIAL AND POISSON DISTRIBUTIONS

1. Introduction. Suppose that $(X_1, X_2, \ldots, X_n)$ is a sample from population having distribution function $F(x)$ and probability density function (p.d.f.) $f(x)$. Let $X_1 \leqslant X_2 \leqslant \ldots \leqslant X_n$ be an ordered sample of size n. For fixed size n, extensive studies on order statistics have been made, while a literature on this subject in the case where n is a value of the random variable N is not so rich. Some studies on order statistics for random sample size can be found, e.g., in [1], [2] and [6].

In this note we study distributions and moments of order statistics in the case where N has a compound binomial distribution or a compound Poisson distribution. Particular cases of the population distributions (uniform on (0, 1]) and compounding ones (uniform on [0, 1], β- and γ-distributions) are also considered.

There are known some applications of compound distributions in a quality control and mathematical statistics (see, e.g., [3]-[5], [7], [8]). That is why we think our results would be useful in investigations of such a type. Moreover, our considerations lead to some combinatorial identities.

2. N has a compound binomial distribution. A random variable N is said to have the *compound binomial distribution* if the probability function of N is of the form

$$p(k; P) = \mathrm{P}[N = k] = \binom{n}{k} \int_0^1 p^k q^{n-k} f(p)\,dp, \tag{1}$$

$$k = 0, 1, 2, \ldots, n,\ 0 < p < 1,\ p+q = 1,$$

where $f(p)$ denotes the density function of the random variable P.

In what follows we write f_i for $f(x_i)$, F_i for $F(x_i)$ etc., and

$$A_i(n, p) = \sum_{k=i}^{n} \binom{n}{k} p^k q^{n-k},$$

$$S_i = S_i(n, p) = \int_0^1 A_i(n, p) f(p)\,dp = \mathrm{E}A_i(n, p) = \sum_{k=i}^{n} \frac{(n)_k}{k!} M_k,$$

where

$$M_k = \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \mathrm{E}P^{l+k} \quad \text{and} \quad (n)_k = n(n-1)\dots(n-k+1).$$

LEMMA 1. *If $X_1 \leqslant X_2 \leqslant \dots \leqslant X_N$ is an ordered sample of size N having the distribution* (1), *then*:

(a) *the conditional p.d.f. of X_i, $1 \leqslant i \leqslant N$, conditioned on the event $[N \geqslant i]$ is*

$$g(x_i) = \frac{(n)_i f_i}{(i-1)!S_i} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} F_i^{k+i-1} \mathrm{E}P^{k+i}; \tag{2}$$

(b) *the conditional p.d.f. of X_{N-i+1} conditioned on the event $[N \geqslant i]$ is*

$$g(x_{N-i+1}) = \frac{(n)_i f_{N-i+1}}{(i-1)!S_i} \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} (1-F_{N-i+1})^{k+i-1} \mathrm{E}P^{k+i}; \tag{3}$$

(c) *the joint conditional p.d.f. of X_i and X_j, $1 \leqslant i < j \leqslant N$, conditioned on the event $[N \geqslant j]$ is*

$$g(x_i, x_j) = \frac{(n)_j f_i f_j F_i^{i-1} [F_j - F_i]^{j-i-1}}{(i-1)!(j-i-1)!S_j} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} F_j^k \mathrm{E}P^{k+j}. \tag{4}$$

Proof. Let G be the conditional cumulative distribution function of X_i, conditioned on the event $[N \geqslant i]$, i.e. $G(x_i) = \mathrm{P}[X_i < x_i | N \geqslant i]$. For a given sample of size k let us put

$$H(x_i|k) = \mathrm{P}[X_i < x_i | N = k] \quad \text{and} \quad h(x_i|k) = H'(x_i|k).$$

We have

$$G(x_i) = \mathrm{P}[X_i < x_i | N \geqslant i] = \frac{1}{\mathrm{P}[N \geqslant i]} \sum_{k=i}^{n} \mathrm{P}[X_i < x_i | N = k]$$

$$= \frac{1}{\mathrm{P}[N \geqslant i]} \sum_{k=i}^{n} H(x_i|k) \mathrm{P}[N = k].$$

Hence, we get

$$g(x_i) = \frac{1}{\mathrm{P}[N \geqslant i]} \sum_{k=i}^{n} h(x_i|k) \mathrm{P}[N = k]. \tag{5}$$

Using (1) and the formula

$$h(x_i|k) = \frac{k!}{(i-1)!(k-i)!} F_i^{i-1} (1-F_i)^{k-i} f_i, \quad i = 1, 2, \dots, k,$$

we obtain

$$\sum_{k=i}^{n} h(x_i|k)\mathrm{P}[N = k]$$

$$= \sum_{k=i}^{n} \frac{k!}{(i-1)!(k-i)!} F_i^{i-1}(1-F_i)^{k-i} f_i \binom{n}{k} \int_0^1 p^k q^{n-k} f(p)\, dp$$

$$= n\binom{n-1}{i-1} F_i^{i-1} f_i \int_0^1 \sum_{r=0}^{n-i} \binom{n-i}{r} (1-F_i)^r p^{r+i} q^{n-i-r} f(p)\, dp$$

$$= \frac{(n)_i}{(i-1)!} F_i^{i-1} f_i \int_0^1 p^i (1-pF_i)^{n-i} f(p)\, dp$$

$$= \frac{(n)_i}{(i-1)!} F_i^{i-1} f_i \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} F_i^k \int_0^1 p^{i+k} f(p)\, dp$$

$$= \frac{(n)_i}{(i-1)!} f_i \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} F_i^{k+i-1} \mathrm{E}P^{k+i}.$$

Taking into account the relations

$$\mathrm{P}[N \geqslant i] = \sum_{k=i}^{n} \mathrm{P}[N = k] = \mathrm{E}A_i(n, P) = \sum_{k=i}^{n} \binom{n}{k} \mathrm{E}P^k(1-P)^{n-k}$$

$$= \sum_{k=i}^{n} \binom{n}{k} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \mathrm{E}P^{\;+k} = \sum_{k=i}^{n} \frac{(n)_k}{k!} \sum_{l=0}^{n-k} (-1)^l \binom{n-k}{l} \mathrm{E}P^{l+k},$$

we get (2).

Formulae (3) and (4) can be obtained in the analogous way, after using

$$h(x_{N-i+1}|k) = \frac{k!}{(i-1)!(k-i)!} (1-F_{N-i+1})^{i-1} F_{N-i+1}^{k-i} f_{N-i+1}$$

and

$$h(x_i, x_j|k) = \frac{k!}{(i-1)!(j-i-1)!(k-j)!} F_i^{i-1} [F_j - F_i]^{j-i-1} (1-F_j)^{k-j} f_i f_j$$

for $i < j$, respectively.

Remark 1. The smallest and the largest order statistics are the following:

$$g(x_1) = \frac{nf_1}{\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\mathrm{E}P^k}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}F_1^k\mathrm{E}P^{k+1},$$

$$g(x_N) = \frac{nf_N}{\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\mathrm{E}P^k}\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(1-F_N)^k\mathrm{E}P^{k+1}.$$

Remark 2. The considerations leading to the formulae above allow us to give

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{l=0}^{n-k}(-1)^l\binom{n-k}{l}\mathrm{E}P^{l+k} = \sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\mathrm{E}P^k.$$

COROLLARY 1. *If the random variable P has a uniform distribution on* $(0, 1)$, *then formulae* (2)-(4) *are of the form*

$$g(x_i) = \frac{(n+1)_{i+1}f_i}{(i-1)!(n-i+1)}\sum_{k=0}^{n-i}\frac{(-1)^k}{k+i+1}\binom{n-i}{k}F_i^{k+i-1},$$

$$g(x_{N-i+1}) = \frac{(n+1)_{i+1}f_{N-i+1}}{(i-1)!(n-i+1)}\sum_{k=0}^{n-i}\frac{(-1)^k}{k+i+1}\binom{n-i}{k}(1-F_{N-i+1})^{k+i-1},$$

$$g(x_i, x_j) = \frac{(n+1)_{j+1}f_if_jF_i^{i-1}[F_j-F_i]^{j-i-1}}{(i-1)!(j-i-1)!(n-j+1)}\sum_{k=0}^{n-j}\frac{(-1)^k}{k+j+1}\binom{n-j}{k}F_j^k.$$

Remark 3. The smallest and the largest order statistics are the following:

$$g(x_1) = (n+1)f_1\sum_{k=0}^{n-1}\frac{(-1)^k}{k+2}\binom{n-1}{k}F_1^k,$$

$$g(x_N) = (n+1)f_N\sum_{k=0}^{n-1}\frac{(-1)^k}{k+2}\binom{n-i}{k}(1-F_N)^k.$$

Remark 4. The considerations leading to the formulae above allow us also to give

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{l=0}^{n-k}\frac{(-1)^l}{k+l+1}\binom{n-k}{l} = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k+1}\binom{n}{k} = \frac{n}{n+1}.$$

COROLLARY 2. *If the random variable P has the β-distribution, i.e.*

$$(6)\quad f(p) = \frac{p^a q^{b-a}}{B(a+1, b-a+1)}, \quad 0 < p < 1,\ -1 < a < b+1, p+q = 1,$$

then

$$g(x_i) = \frac{(n)_i}{(i-1)!} f_i F_i^{i-1} D_i,$$

$$g(x_{N-i+1}) = \frac{(n)_i}{(i-1)!} f_{N-i+1}(1-F_{N-i+1})^{i-1} D_i^*,$$

$$g(x_i, x_j) = \frac{(n)_j}{(i-1)!(j-i-1)!} f_i f_j F_i^{i-1}(F_j - F_i)^{j-i-1} D_j,$$

where

$$D_i = \left[\sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} F_i^k B(k+a+i+1, b-a+1)\right] \times$$

$$\times \left[\sum_{k=i}^{n} \binom{n}{k} B(k+a+1, n-k+b-a+1)\right]^{-1}$$

and D_i^* is equal to D_i, $1-F_{N-i+1}$ standing for F_i.

LEMMA 2. *Let R be the range of an ordered sample $X_1 \leqslant X_2 \leqslant \ldots \leqslant X_N$, where N is a random variable distributed according to* (1). *Then*

$$(7)\quad g(r) = \frac{(n-1)\sum_{k=0}^{n-2} \binom{n-2}{k} \mathrm{E}(P^{k+2}) \int_{-\infty}^{\infty} [F(x+r) - F(x) - 1]^k f(x+r) f(x)\, dx}{\sum_{k=0}^{n-1} (-1)^{k+1} \frac{k}{k+1} \binom{n-1}{k} \mathrm{E}P^{k+1}}.$$

Proof. Since the p.d.f. of range R ([9], p. 248) is, for fixed k,

$$h(r|k) = k(k-1) \int_{-\infty}^{\infty} [F(x+r) - F(x)]^{k-2} f(x+r) f(x)\, dx,$$

and

$$S_2 = 1 - \mathrm{E}(1-P)^n - n\mathrm{E}P(1-P)^{n-1} = n \sum_{k=0}^{n-2} (-1)^{k+1} \frac{k}{k+1} \binom{n-1}{k} \mathrm{E}P^{k+1},$$

the p.d.f. of range R in the case where N is distributed according to (1) is given by (7).

Remark 5. The considerations given above lead us also to the formula

$$n\sum_{k=0}^{n-1}(-1)^{k+1}\frac{k}{k+1}\binom{n-1}{k}\mathrm{E}P^{k+1} = \sum_{k=2}^{n}\binom{n}{k}\sum_{l=0}^{n-k}(-1)^{l}\binom{n-k}{l}\mathrm{E}P^{k+l}.$$

COROLLARY 3. *In the case where P has the uniform distribution on* $(0,1)$ *the p.d.f. is of the form*

(8)
$$g(r) = n(n+1)\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{1}{k+3}\int_{-\infty}^{\infty}[F(x+r)-F(x)-1]^{k}f(x+r)f(x)\,dx.$$

Remark 6. The considerations given above lead us also to the formula

$$\sum_{k=0}^{n-1}(-1)^{k+1}\frac{k}{(k+1)(k+2)}\binom{n-1}{k} = -\frac{n-1}{n(n+1)}.$$

COROLLARY 4. *In the case where P has the β-distribution the p.d.f. is of the form*

(9)
$$g(r) = \Big[(n-1)\sum_{k=0}^{n-2}\frac{\Gamma(k+a+3)}{\Gamma(k+b+4)}\binom{n-2}{k}\times$$
$$\times\int_{-\infty}^{\infty}[F(x+r)-F(x)-1]^{k}f(x+r)f(x)\,dx\Big]\times$$
$$\times\Big[\sum_{k=0}^{n-1}(-1)^{k+1}\frac{k}{k+1}\binom{n-1}{k}\frac{\Gamma(k+a+2)}{\Gamma(k+b+3)}\Big]^{-1}.$$

3. N has a compound Poisson distribution. A random variable N is said to have the *compound Poisson distribution* if the probability function of N is of the form

(10)
$$p(k) = \mathrm{P}[N=k] = \int_{0}^{\infty}\frac{\lambda^{k}}{k!}e^{-\lambda}\,dG(\lambda),\qquad k = 0,1,2,\ldots,$$

where G denotes the distribution function of the parameter Λ.

Now we are going to present results analogous to those of Section 2 in the case where N has a compound Poisson distribution.

Let

$$A_i^{*}(\lambda) = \sum_{k=i}^{\infty}\frac{\lambda^{k}}{k!}e^{-\lambda}$$

and

$$S_i^* = S_i^*(\lambda) = \int_0^\infty A_i^*(\lambda)dG(\lambda) = \mathrm{E}A_i^*(\Lambda) = \sum_{k=i}^\infty \frac{M_k^*}{k!},$$

where

$$M_k^* = \sum_{l=0}^\infty \frac{(-1)^l}{l!} \mathrm{E}\Lambda^{k+l}.$$

LEMMA 3. *If* $X_1 \leqslant X_2 \leqslant \ldots \leqslant X_N$ *is an ordered sample of size* N *having the distribution* (10), *then*:

(a) *the conditional p.d.f. of* X_i, $1 \leqslant i \leqslant N$, *conditioned on the event* $[N \geqslant i]$ *is*

$$g(x_i) = \frac{f_i}{(i-1)!\,S_i^*} \sum_{k=0}^\infty (-1)^k \frac{F_i^{k+i-1}}{k!} \mathrm{E}\Lambda^{k+i}; \tag{11}$$

(b) *the joint conditional p.d.f. of* X_i *and* X_j, $1 \leqslant i < j \leqslant N$, *conditioned on the event* $[N \geqslant j]$ *is*

$$g(x_i, x_j) = \frac{f_i f_j F_i^{i-1}[F_j - F_i]^{j-i-1}}{(i-1)!(j-i-1)!\,S_j^*} \sum_{k=0}^\infty (-1)^k \frac{F_j^k}{k!} \mathrm{E}\Lambda^{k+j}. \tag{12}$$

Proof. Taking into account the formula

$$\mathrm{P}[N \geqslant i] = \sum_{k=i}^\infty \frac{1}{k!} \int_0^\infty \lambda^k e^{-\lambda} dG(\lambda) = \mathrm{E}A_i^*(\Lambda)$$

and making simple evaluations (some of them are analogous to those of Lemma 1), we get (11) and (12).

Remark 7. For the smallest order statistic we have

$$g(x_1) = \frac{f_1 \mathrm{E}\Lambda e^{-F_1\Lambda}}{1 - \mathrm{E}e^{-\Lambda}} = \frac{f_1 \sum_{k=0}^\infty \frac{(-1)^k}{k!} F_1^k \mathrm{E}\Lambda^{k+1}}{\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k!} \mathrm{E}\Lambda^{k+1}}.$$

COROLLARY 5. *If the random variable* Λ *is distributed according to*

$$f(\lambda) = \begin{cases} \dfrac{a^v}{\Gamma(v)} \lambda^{v-1} e^{-a\lambda} & \text{for } \lambda > 0, \\ 0 & \text{for } \lambda \leqslant 0, \end{cases} \tag{13}$$

where $a > 0$, $v > 0$, *then*

$$g(x_i) = \frac{f_i F_i^{i-1}}{(i-1)!} E_i$$

and

$$g(x_i, x_j) = \frac{f_i f_j F_i^{i-1}[F_j - F_i]^{j-i-1}}{(i-1)!(j-i-1)!} E_j$$

with

$$E_i = \left[\sum_{k=i}^{\infty} (-1)^{k-i}(k)_i \binom{v+k-1}{k} \frac{F_i^{k-i}}{a^k}\right] \times$$

$$\times \left[\sum_{k=i}^{\infty} \frac{1}{k!} \sum_{l=k}^{\infty} (-1)^{l-k}(l)_k \binom{v+l-1}{l} \frac{1}{a^l}\right]^{-1}.$$

One can prove the following

LEMMA 4. *Let R be the range of an ordered sample $X_1 \leqslant X_2 \leqslant \ldots \leqslant X_N$, where N is a random variable distributed according to* (10). *Then*

$$(14) \qquad g(r) = \frac{\sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{E}(\Lambda^{k+2}) \int_{-\infty}^{\infty} [F(x+r) - F(x)]^k f(x+r) f(x)\, dx}{1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} (\mathrm{E}\Lambda^{k+1} + \mathrm{E}\Lambda^k)}.$$

Remark 8. One can also prove

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)!} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \mathrm{E}\Lambda^{l+2+k} = 1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} (\mathrm{E}\Lambda^{k+1} + \mathrm{E}\Lambda^k).$$

COROLLARY 6. *In the case where Λ is distributed according to* (13) *we have*

$$g(r) = \frac{\sum_{k=2}^{\infty} \frac{(k-1)k}{a^k} \binom{v+k-1}{k} \int_{-\infty}^{\infty} [F(x+r) - F(x)]^{k-2} f(x+r) f(x)\, dx}{1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{a^k} [a+v+k] \binom{v+k-1}{k}}.$$

Remark 9. For $a > 0$ and $v > 0$ we also have

$$1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{a^{k+1}} [a+v+k] \binom{v+k-1}{k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(k+2)!} \sum_{l=0}^{\infty} \frac{(-1)^l}{a^{l+k+2}} (l+k+2)_{k+2} \binom{v+l+k+1}{l+k+2}.$$

4. A sample from a population uniformly distributed in $(0,1)$. We now consider a sample $(X_1, X_2, \ldots, X_N)$ of size N from a population having the uniform distribution in $(0,1)$, i.e. $F(x) = x$ for $x \in (0,1)$. In this case we can prove the following

THEOREM 1. *If* $X_1 \leqslant X_2 \leqslant \ldots \leqslant X_N$ *is an ordered sample from* $F(x) = x$, $x \in (0,1)$, *and* N *has the compound binomial distribution* (1), *then*

$$g(r) = \frac{(n-1)\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(1-r)^{k+1}\mathrm{E}P^{k+2}}{\sum_{k=0}^{n-1}(-1)^{k+1}\frac{k}{k+1}\binom{n-1}{k}\mathrm{E}P^{k+1}}. \tag{15}$$

Moreover, for $m \geqslant 1$ *we have*

$$\mathrm{E}X_i^m = \frac{(n)_i}{(i-1)!S_i}\sum_{k=0}^{n-i}(-1)^k\frac{1}{m+k+i}\binom{n-i}{k}\mathrm{E}P^{k+i} \tag{16}$$

and

$$\mathrm{E}X_i^m = \mathrm{E}X_i^{m-1} - \frac{(n)_i}{(i-1)!S_i}\sum_{k=0}^{n-i}\frac{(-1)^k}{(m+k+i)(m+k+i-1)}\binom{n-i}{k}\mathrm{E}P^{k+i}, \tag{17}$$

and for $i<j$

$$\mathrm{E}X_iX_j = \frac{i(n)_j}{j!S_j}\sum_{k=0}^{n-j}(-1)^k\frac{1}{k+j+2}\binom{n-j}{k}\mathrm{E}P^{k+j}. \tag{18}$$

Proof. (15) follows directly from (7), whereas (16)-(18) can be obtained by simple evaluations after using Lemma 1.

COROLLARY 7. *If in the considered case* P *is uniformly distributed in* $(0,1)$, *then*

$$g(r) = n(n+1)\sum_{k=0}^{n-2}(-1)^k\frac{1}{k+3}\binom{n-2}{k}(1-r)^{k+1}.$$

Moreover, for $m \geqslant 1$ *we have*

$$\mathrm{E}X_i^m = \frac{(n+1)_{i+1}}{(i-1)!(n-i+1)}\sum_{k=0}^{n-2}(-1)^k\frac{1}{(m+k+i)(k+i+1)}\binom{n-i}{k}$$

and

$$\mathrm{E}X_i^m = \mathrm{E}X_i^{m-1} - \frac{(n+1)_{i+1}}{(i-1)!(n-i+1)}\sum_{k=0}^{n-2}(-1)^k\frac{1}{(m+k+i-1)(m+k+i)(k+i+1)}\binom{n-i}{k},$$

and for $i<j$

$$\mathrm{E}X_iX_j = \frac{i(n+1)_{j+1}}{j!(n-j+1)}\sum_{k=0}^{n-j}(-1)^k\frac{1}{(k+j+2)(k+j+1)}\binom{n-j}{k}.$$

COROLLARY 8. *If in the considered case P has the β-distribution* (6), *then*

$$g(r) = \frac{(n-1)\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{\Gamma(k+a+3)}{\Gamma(k+b+4)}(1-r)^{k+1}}{\sum_{k=0}^{n-1}(-1)^{k+1}\frac{k}{k+1}\binom{n-k}{k}\frac{\Gamma(k+a+2)}{\Gamma(k+b+3)}}.$$

Moreover, for $m\geqslant 1$ *we have*

$$\mathrm{E}X_i^m = \frac{(n)_i}{(i-1)!\sum_{k=0}^{n-i}\binom{n}{k+i}B(k+i+a+1,\, n-k-i+b-a+1)}\times$$

$$\times\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\frac{1}{m+k+i}B(k+i+a+1,\, b-a+1)$$

and

$$\mathrm{E}X_i^m = \mathrm{E}X_i^{m-1} - \frac{(n)_i}{(i-1)!}\left[\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\frac{B(k+i+a+1,\, b-a+1)}{(m+k+i)(m+k+i-1)}\right]\times$$

$$\times\left[\sum_{k=0}^{n-i}\binom{n}{k+i}B(k+i+a+1,\, n-k-i+b-a+1)\right]^{-1},$$

and for $i<j$

$$\mathrm{E}X_iX_j = \frac{i(n)_j}{j!}\left[\sum_{k=0}^{n-j}(-1)^k\frac{1}{k+j+2}\binom{n-j}{k}B(k+j+a+1,\, b-a+1)\right]\times$$

$$\times\left[\sum_{k=0}^{n-j}\binom{n}{k+j}B(k+j+a+1,\, n-k-j+b-a+1)\right]^{-1}.$$

THEOREM 2. *If $X_1\leqslant X_2\leqslant\ldots\leqslant X_N$ is an ordered sample from $F(x)=x$, $x\in(0,1)$, and N has the compound Poisson distribution* (10), *then*

$$g(r) = \frac{1-r}{1+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(\mathrm{E}\Lambda^{k+1}+\mathrm{E}\Lambda^k)}\sum_{k=0}^{\infty}\frac{r^k}{k!}\mathrm{E}\Lambda^{k+2}. \tag{19}$$

Moreover, for $m \geqslant 1$ *we have*

$$\mathrm{E}X_i^m = \frac{1}{(i-1)!\,S_i^*} \sum_{k=0}^{\infty} \frac{(-1)^k}{(m+k+i)k!}\,\mathrm{E}\Lambda^{k+i} \tag{20}$$

and

$$\mathrm{E}X_i^m = \mathrm{E}X_i^{m-1} - \frac{1}{(i-1)!\,S_i^*} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,\frac{\mathrm{E}\Lambda^{k+i}}{(m+k+i-1)(m+k+i)}, \tag{21}$$

and for $i < j$

$$\mathrm{E}X_i X_j = \frac{i}{j!\,S_j^*} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+j+2)}\,\mathrm{E}\Lambda^{k+j}. \tag{22}$$

Proof. (19) follows directly from (14), and (20)-(22) can be obtained by simple evaluations after using Lemma 3.

COROLLARY 9. *If in the considered case* Λ *is distributed according to* (13), *then*

$$g(r) = \frac{(1-r)\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{a^{k+2}} \binom{v+k+1}{k+2} r^k}{1+\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{a^{k+1}} [a+v+k] \binom{v+k-1}{k}}.$$

Moreover, for $m \geqslant 1$ *we have*

$$\mathrm{E}X_i^m = \frac{\sum_{k=i}^{\infty} \frac{(-1)^{k-i}(k)_i}{m+k} \binom{v+k-1}{k} \frac{1}{a^k}}{(i-1)!\sum_{k=i}^{\infty} \frac{1}{k!} \sum_{l=k}^{\infty} (-1)^{l-k}(l)_k \binom{v+l-1}{l} \frac{1}{a^l}}$$

and

$$\mathrm{E}X_i^m = \mathrm{E}X_i^{m-1} - \frac{\sum_{k=i}^{\infty} \frac{(-1)^{k-i}(k)_i}{(m+k-1)(m+k)} \binom{v+k-1}{k} \frac{1}{a^k}}{(i-1)!\sum_{k=i}^{\infty} \frac{1}{k!} \sum_{l=k}^{\infty} (-1)^{l-k}(l)_k \binom{v+l-1}{l} \frac{1}{a^l}},$$

and for $i < j$

$$\mathrm{E}X_i X_j = \frac{i\sum_{k=j}^{\infty} \frac{(-1)^{k-}\,(k)_j}{k+2} \binom{v+k-1}{k} \frac{1}{a^k}}{j!\sum_{k=j}^{\infty} \frac{1}{k!} \sum_{l=k}^{\infty} (-1)^{l-k}(l)_k \binom{v+l-1}{l} \frac{1}{a}}.$$

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O STATYSTYKACH PORZĄDKOWYCH DLA PRÓBY O LOSOWEJ LICZEBNOŚCI, MAJĄCEJ ZŁOŻONY ROZKŁAD DWUMIANOWY I ZŁOŻONY ROZKŁAD POISSONA

STRESZCZENIE

W pracy podaje się wzory na gęstości warunkowe statystyk porządkowych, gdy próba jest pobrana z populacji o rozkładzie $F(x)$, a liczebność próby jest zmienną losową o złożonym rozkładzie dwumianowym lub Poissona. Dla próby pobranej z populacji o rozkładzie jednostajnym na przedziale $(0, 1)$ podano wzory na momenty statystyk porządkowych.
