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THE INVERSION OF CYCLIC TRIDIAGONAL MATRICES

1. Introduction. In some problems that arise in the theory of cubic splines we deal with the inverses of cyclic tridiagonal matrices (see [1]). We would like to mention that an $(n \times n)$-matrix $A = (\alpha_{ij})$, where

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \ldots & 0 & 0 & a_1 \\ a_2 & b_2 & c_2 & \ldots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & a_{n-1} & b_{n-1} & c_{n-1} \\ c_n & 0 & 0 & \ldots & 0 & a_n & b_n \end{bmatrix}, \tag{1}$$

is said to be a *cyclic tridiagonal matrix* (see [2]).

The purpose of this paper is to derive some recurrence formulae for the elements of A^{-1}. In Sections 2 and 3 we give two different methods. The first one is based on the LU decomposition of A. In the case where $a_1 = c_n = 0$ the obtained results reduce to those given in [3]. In Section 3, using Evan's factorization of A (see [2]) we obtain another recursive method for the elements of A^{-1}. Three numerical algorithms and the arithmetical complexity for both methods under discussion are also included.

2. Inversion of A by using LU decomposition. To this end we assume $d_k \neq 0$, $k = 1, 2, \ldots, n$, where d_k denotes the k-th leading minor of A. In particular, positive definite matrices satisfy this assumption.

LEMMA 1. *There exists a unique factorization $A = LU$, where*

$$L = \begin{bmatrix} f_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ e_2 & f_2 & 0 & \ldots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & e_{n-1} & f_{n-1} & 0 \\ g_1 & g_2 & g_3 & \ldots & g_{n-2} & e_n & f_n \end{bmatrix}$$

and

$$U = \begin{bmatrix} 1 & -p_1 & 0 & \dots & 0 & 0 & q_1 \\ 0 & 1 & -p_2 & \dots & 0 & 0 & q_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 & -p_{n-2} & q_{n-2} \\ 0 & 0 & 0 & \dots & 0 & 1 & -p_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

The non-zero elements of matrices L and U satisfy the following recurrence formulae:

$$f_1 = b_1, \quad f_k = b_k + e_k p_{k-1} \ (k = 2, 3, \dots, n-1),$$

(2) $$f_n = b_n + e_n p_{n-1} - \sum_{l=1}^{n-2} g_l q_l,$$

(3) $$e_k = a_k \ (k = 2, 3, \dots, n-1), \quad e_n = a_n + p_{n-2} g_{n-2},$$

(4) $$p_k = -\frac{c_k}{f_k} \ (k = 1, 2, \dots, n-2), \quad p_{n-1} = -\frac{c_{n-1} - e_{n-1} q_{n-2}}{f_{n-1}},$$

(5) $$g_1 = c_n, \quad g_k = p_{k-1} g_{k-1} \ (k = 2, 3, \dots, n-2),$$

(6) $$q_1 = \frac{a_1}{f_1}, \quad q_k = -\frac{e_k q_{k-1}}{f_k} \ (k = 2, 3, \dots, n-2).$$

Proof. We can verify the correctness of (2)–(6) by multiplying L by U and then comparing the entries of LU with the suitable entries of A. This completes the proof.

It is well known that

(7) $$f_k = d_k / d_{k-1} \quad (k = 1, 2, \dots, n-1).$$

Hence $f_k \neq 0$ $(k = 1, 2, \dots, n-1)$. Therefore, in formulae (4) and (6) the division by zero does not occur. The leading minors of A also satisfy the following formula:

(8) $$d_k = b_k d_{k-1} - a_k c_{k-1} d_{k-2} \quad (k = 2, 3, \dots, n-1;\ d_0 = 1,\ d_1 = b_1).$$

For further aims we express the entries of L and U in terms of f_k $(k = 1, 2, \dots, n-1)$.

LEMMA 2. *The elements of the matrices L and U may be written in the form*

(2′) $$f_1 = b_1, \quad f_k = b_k - \frac{a_k c_{k-1}}{f_{k-1}} \ (k = 2, 3, \dots, n-1),$$
$$f_n = b_n - \frac{a_n c_{n-1}}{f_{n-1}} + \frac{(-1)^{n+1}}{f_1 f_2 \dots f_{n-1}} \{\prod_{l=1}^{n} a_l + \prod_{l=1}^{n} c_l\} - c_n \sum_{k=1}^{n-1} \frac{a_k}{f_k} \prod_{l=1}^{k-1} \frac{a_l c_l}{f_l^2},$$

$$(3') \qquad e_k = a_k \quad (k = 2, 3, \ldots, n-1),$$
$$e_n = a_n + (-1)^n c_n \prod_{l=1}^{n-2} \frac{c_l}{f_l},$$

$$(4') \qquad p_k = -\frac{c_k}{f_k} \quad (k = 1, 2, \ldots, n-2),$$
$$p_{n-1} = -\frac{c_{n-1}}{f_{n-1}} + (-1)^{n-1} \prod_{l=1}^{n-1} \frac{a_l}{f_l},$$

$$(5') \qquad g_k = (-1)^{k+1} c_n \prod_{l=1}^{k-1} \frac{c_l}{f_l} \quad (k = 1, 2, \ldots, n-2),$$

$$(6') \qquad q_k = (-1)^{k+1} \prod_{l=1}^{k} \frac{a_l}{f_l} \quad (k = 1, 2, \ldots, n-2).$$

Proof. Formula (6′) follows immediately from (3) and (6). From (4) and (5) we have (5′). Applying (3) and (6′) to (4) we get (4′). From (4′) and (5′) we have (3′). Using (3′)–(6′) we obtain (2′). The proof is complete.

From (7) we have

$$f_1 f_2 \ldots f_{n-1} = \frac{d_1}{d_0} \frac{d_2}{d_1} \ldots \frac{d_{n-1}}{d_{n-2}} = d_{n-1}.$$

Since $\det(A) = d_n = f_1 f_2 \ldots f_n = d_{n-1} f_n$, we obtain $f_n = d_n/d_{n-1}$. Hence relation (7) is also true for $k = n$.

Now we want to express the determinant of A by the elements a_k, b_k, c_k and the leading minors d_k. Using (2′) and (7) we obtain

$$\frac{d_n}{d_{n-1}} = f_n = b_n - \frac{a_n c_{n-1}}{d_{n-1}} d_{n-2} - \frac{(-1)^n}{d_{n-1}} \{\prod_{l=1}^{n} a_l + \prod_{l=1}^{n} c_l\} - c_n \sum_{k=1}^{n-1} \frac{a_k}{f_k} \prod_{l=1}^{k-1} \frac{a_l c_l}{f_l^2} d_{n-1}^{-1},$$

$$d_n = b_n d_{n-1} - a_n c_{n-1} d_{n-2} - (-1)^n \{\prod_{l=1}^{n} a_l + \prod_{l=1}^{n} c_l\} - c_n d_{n-1} \sum_{k=1}^{n-1} \frac{a_k}{f_k} \prod_{l=1}^{k-1} \frac{a_l c_l}{f_l^2}.$$

Easy calculations show that

$$\sum_{k=1}^{n-1} \frac{a_k}{f_k} \prod_{l=1}^{k-1} \frac{a_l c_l}{f_l^2} = a_1 \sum_{k=1}^{n-1} \frac{1}{d_{k-1} d_k} \prod_{l=1}^{k} a_l \prod_{l=1}^{k-1} c_l.$$

Thus we obtain finally the following

LEMMA 3. *The determinant of A satisfies the formula*

$$(9) \qquad d_n = b_n d_{n-1} - a_n c_{n-1} d_{n-2} - (-1)^n \{\prod_{l=1}^{n} a_l + \prod_{l=1}^{n} c_l\}$$
$$- a_1 c_n d_{n-1} \sum_{k=1}^{n-1} \frac{1}{d_{k-1} d_k} \prod_{l=1}^{k} a_l \prod_{l=1}^{k-1} c_l.$$

In Theorem 1, recursive formulae for obtaining the elements of $B = A^{-1}$ are given. Since $A = LU$, where $L = (l_{ij})$, $U = (u_{ij})$ $(i, j = 1, 2, \ldots, n)$, we have

$$B = U^{-1} L^{-1}.$$

Hence

$$UB = L^{-1} = (\gamma_{ij}) \quad (i, j = 1, 2, \ldots, n), \tag{10}$$

$$BL = U^{-1} = (\varrho_{ij}) \quad (i, j = 1, 2, \ldots, n). \tag{11}$$

From (10) and (11) we obtain recurrence formulae for the elements of $B = (\beta_{ij})$ $(i, j = 1, 2, \ldots, n)$

$$\sum_{k=i}^{n} u_{ik} \beta_{kj} = \gamma_{ij} \quad (i, j = 1, 2, \ldots, n), \tag{10'}$$

$$\sum_{k=j}^{n} \beta_{ik} l_{kj} = \varrho_{ij} \quad (i, j = 1, 2, \ldots, n). \tag{11'}$$

Now we present the announced theorem.

THEOREM 1. *Let $A = (\alpha_{ij})$ $(i, j = 1, 2, \ldots, n)$ be a cyclic tridiagonal matrix and let all leading minors of A be different from zero. Then the elements of the matrix B satisfy the recurrence formulae of the "first kind"*

$$\beta_{ij} = \begin{cases} \gamma_{nj} & (i = n; j = 1, 2, \ldots, n), \\ p_{n-1} \beta_{nj} + \gamma_{n-1,j} & (i = n-1; j = 1, 2, \ldots, n), \\ p_i \beta_{i+1,j} - q_i \beta_{nj} + \gamma_{ij} & (i = 1, 2, \ldots, n-2; j = 1, 2, \ldots, n), \end{cases} \tag{12}$$

where

$$\gamma_{ij} = \begin{cases} 1/f_n & (i = j = n), \\ -\dfrac{1}{f_j f_n}\left[g_j + \displaystyle\sum_{v=j+1}^{n-2} g_v (-1)^{v-j} \prod_{l=j+1}^{v} \frac{e_l}{f_l} + (-1)^{n-j-1} e_n \prod_{l=j+1}^{n-1} \frac{e_l}{f_l}\right] & \\ \qquad & (i = n;\ j = 1, 2, \ldots, n-1;\ g_{n-1} = 0), \\ \dfrac{(-1)^{i-j}}{f_j} \displaystyle\prod_{l=j+1}^{i} \frac{e_l}{f_l} & (i \geqslant j,\ i = 1, 2, \ldots, n-1), \\ 0 & (i < j,\ j = 2, 3, \ldots, n), \end{cases} \tag{13}$$

and the formulae of the "second kind"

$$\beta_{ij} = \begin{cases} \varrho_{in}/f_n & (j = n; i = 1, 2, \ldots, n), \\ \dfrac{\varrho_{i,n-1} - e_n \beta_{in}}{f_{n-1}} & (j = n-1; i = 1, 2, \ldots, n), \\ \dfrac{\varrho_{ij} - a_{j+1} \beta_{i,j+1} - g_j \beta_{in}}{f_j} & (j = 1, 2, \ldots, n-2; i = 1, 2, \ldots, n), \end{cases} \tag{14}$$

where

$$
(15) \qquad \varrho_{ij} = \begin{cases} \prod\limits_{k=i}^{n-1} p_k - \sum\limits_{k=i}^{n-2} q_k \prod\limits_{l=i}^{k-1} p_l & (j = n;\ i = 1, 2, \ldots, n), \\ \prod\limits_{k=i}^{j-1} p_k & (i \leqslant j,\ j = 1, 2, \ldots, n-1), \\ 0 & (i > j,\ i = 2, 3, \ldots, n). \end{cases}
$$

Proof. We prove the formulae of the "first kind". From the equality $LL^{-1} = I$, where I is the identity matrix, we obtain the following system of linear equations satisfied by the j-th column of the matrix L^{-1}:

$$
\begin{aligned}
f_1 \gamma_{1j} &= 0, \\
e_2 \gamma_{1j} + f_2 \gamma_{2j} &= 0, \\
\ldots\ldots\ldots\ldots \\
e_j \gamma_{j-1,j} + f_j \gamma_{jj} &= 1, \\
\ldots\ldots\ldots\ldots \\
e_{n-1} \gamma_{n-2,j} + f_{n-1} \gamma_{n-1,j} &= 0, \\
g_1 \gamma_{1j} + g_2 \gamma_{2j} + \ldots + g_{n-2} \gamma_{n-2,j} + e_n \gamma_{n-1,j} + f_n \gamma_{nj} &= 0.
\end{aligned}
$$

Hence the recurrence formulae for the elements γ_{ij} are of the form

$$
\gamma_{1j} = \gamma_{2j} = \ldots = \gamma_{j-1,j} = 0, \qquad \gamma_{jj} = 1/f_j,
$$

$$
\gamma_{kj} = -\frac{e_k \gamma_{k-1,j}}{f_k} \qquad (k = j+1, j+2, \ldots, n-1),
$$

$$
\gamma_{nj} = -\frac{1}{f_n}\Big[\sum_{i=j}^{n-2} g_i \gamma_{ij} + e_n \gamma_{n-1,j}\Big].
$$

After simple calculations we obtain (13). From (10′) we have

$$
\beta_{ij} - p_i \beta_{i+1,j} + q_i \beta_{nj} = \gamma_{ij} \qquad (i = 1, 2, \ldots, n-2;\ j = 1, 2, \ldots, n),
$$

$$
\beta_{n-1,j} - p_{n-1} \beta_{nj} = \gamma_{n-1,j} \qquad (j = 1, 2, \ldots, n),
$$

$$
\beta_{nj} = \gamma_{nj} \qquad (j = 1, 2, \ldots, n).
$$

Thus we have obtained (12). The correctness of the formulae of the "second kind" can be proved analogously.

Using Theorem 1 we can formulate the following algorithm for obtaining the elements β_{ij}.

ALGORITHM 1. 1. We calculate the numbers f_i, p_i, e_i, g_i, q_i from the relations

$$f_1 = b_1, \quad f_i = b_i + e_i p_{i-1} \quad (i = 2, 3, \ldots, n-1),$$

$$f_n = b_n + e_n p_{n-1} - \sum_{l=1}^{n-2} g_l q_l;$$

$$e_i = a_i \quad (i = 2, 3, \ldots, n-1),$$

$$e_n = a_n + p_{n-2} g_{n-2};$$

$$p_i = -c_i/f_i \quad (i = 1, 2, \ldots, n-2),$$

$$p_{n-1} = -(c_{n-1} - e_{n-1} q_{n-2})/f_{n-1};$$

$$g_1 = c_n, \quad g_i = p_{i-1} g_{i-1} \ (i = 2, 3, \ldots, n-2);$$

$$q_1 = a_1/f_1, \quad q_i = -e_i q_{i-1}/f_i \ (i = 2, 3, \ldots, n-2).$$

2. We calculate the elements β_{ij} from the recurrence formulae

$$\beta_{nn} = 1/f_n, \quad \beta_{n,n-1} = -e_n/f_{n-1} f_n, \quad \beta_{n-1,n} = p_{n-1}/f_n,$$

$$\beta_{n-1,n-1} = p_{n-1} \beta_{n,n-1} + 1/f_{n-1},$$

$$\beta_{ij} = \begin{cases} \dfrac{-a_{j+1} \beta_{i,j+1} - g_j \beta_{in}}{f_n} & (i > j, j = 1, 2, \ldots, n-2), \\ p_i \beta_{i+1,j} - q_i \beta_{nj} + \delta_{ij}/f_i & (i \leqslant j, i = 1, 2, \ldots, n-2), \end{cases}$$

where δ_{ij} denotes Kronecker's delta.

The above algorithm requires $\frac{5}{2}n^2 + \frac{13}{2}n - 15$ multiplications and $n^2 + 3n + 4$ additions.

We present now a simpler version of Algorithm 1 assuming that A is a symmetric cyclic tridiagonal matrix.

ALGORITHM 2. 1. We calculate the quantities f_i, p_i, e_i, g_i, q_i from the formulae

$$f_1 = b_1, \quad f_i = b_i + e_i p_{i-1} \ (i = 2, 3, \ldots, n-1),$$

$$f_n = b_n + e_n p_{n-1} - \sum_{l=1}^{n-2} q_l g_l;$$

$$e_i = c_{i-1} \quad (i = 2, 3, \ldots, n-1),$$

$$e_n = c_{n-1} + p_{n-2} g_{n-2};$$

$$p_i = -c_i/f_i \quad (i = 1, 2, \ldots, n-2),$$

$$p_{n-1} = -e_n/f_{n-1};$$

$$g_1 = c_n, \quad g_i = p_{i-1} g_{i-1} \ (i = 2, 3, \ldots, n-2);$$

$$q_1 = c_n/f_1, \quad q_i = g_i/f_i \ (i = 2, 3, \ldots, n-2).$$

2. We calculate the elements β_{ij} from the relations

$$\beta_{nn} = 1/f_n, \quad \beta_{n-1,n} = p_{n-1}/f_1,$$
$$\beta_{n,n-1} = \beta_{n-1,n}, \quad \beta_{n-1,n-1} = 1/f_{n-1} + p_{n-1}\beta_{n-1,n},$$
$$\beta_{ij} = \begin{cases} p_i\beta_{i+1,j} - q_i\beta_{nj} + \delta_{ij}/f_i & (i \leqslant j,\ i = 1, 2, \ldots, n-2), \\ \beta_{ji} & (i > j,\ j = 1, 2, \ldots, n-2). \end{cases}$$

The above algorithm follows from Theorem 1 and from the condition $A = A^T$, where A^T denotes the transpose of A. We prove only the correctness of the relations for p_{n-1} and q_i. Applying (4) and (5′) to (3) we have

$$e_n = a_n + p_{n-2}g_{n-2} = c_{n-1} - \frac{c_{n-2}}{f_{n-2}}(-1)^{n-1}c_n\prod_{l=1}^{n-3}\frac{c_l}{f_l}$$
$$= c_{n-1} - (-1)^{n-1}c_n\prod_{l=1}^{n-2}\frac{c_l}{f_l}.$$

Hence

$$c_{n-1} = e_n + (-1)^{n-1}c_n\prod_{l=1}^{n-2}c_l/f_l.$$

Using the above result together with (6′) and (3) to (4) we obtain

$$p_{n-1} = -\frac{e_n + (-1)^{n-1}c_n\prod_{l=1}^{n-2}\frac{c_l}{f_l} - c_{n-2}(-1)^{n-1}\frac{c_n}{f_1}\prod_{l=2}^{n-2}\frac{c_{l-1}}{f_l}}{f_{n-1}} = -\frac{e_n}{f_{n-1}}.$$

By virtue of (5′) the relation (6′) may be transformed to the form

$$q_i = (-1)^{i+1}\frac{c_n}{f_1}\prod_{l=2}^{i}\frac{c_{l-1}}{f_l} = (-1)^{i+1}\frac{c_n}{f_1}\prod_{l=1}^{i-1}\frac{c_l}{f_l} = \frac{g_i}{f_i} \quad (i = 2, 3, \ldots, n-2).$$

Algorithm 2 requires $n^2+7n-12$ multiplications and $n^2/2+\frac{7}{2}n-6$ additions.

3. Inversion of a symmetric cyclic tridiagonal matrix by using Evan's factorization. Let A be a symmetric positive definite matrix of the form

$$A = \begin{bmatrix} b_1 & c_1 & 0 & \ldots & 0 & c_n \\ c_1 & b_2 & c_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_n & 0 & 0 & \ldots & c_{n-1} & b_n \end{bmatrix}. \tag{16}$$

Then, as shown by Evans and Benson (see [2]), there exists a unique factorization of the form

$$A = DT^T TD, \tag{17}$$

where $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, and T is a unique upper triangular matrix such that

$$T = \begin{bmatrix} 1 & e_1 & 0 & \ldots & 0 & f_1 \\ 0 & 1 & e_2 & \ldots & 0 & f_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 & e_{n-1}+f_{n-1} \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix},$$

and T^T denotes the transpose of T. The elements of the matrices D and T are given by

$$d_1 = \sqrt{b_1}, \quad d_j = [b_j-(c_{j-1}/d_{j-1})^2]^{1/2} \ (j = 2, 3, \ldots, n-1),$$

$$e_j = \frac{c_j}{d_j d_{j+1}} \quad (j = 1, 2, \ldots, n-2),$$

$$d_n = \left\{b_n - \frac{c_n^2}{d_1^2}(1+e_1^2+e_1^2 e_2^2+ \ldots +e_1^2 e_2^2 \ldots e_{n-3}^2) - \left[\frac{c_{n-1}}{d_{n-1}}+(-1)^n \frac{e_1 e_2 \ldots e_{n-2} c_n}{d_1}\right]^2\right\}^{1/2},$$

$$e_{n-1} = \frac{c_{n-1}}{d_{n-1} d_n}, \quad f_1 = \frac{c_n}{d_1 d_n}, \quad f_j = -e_{j-1} f_{j-1} \ (j = 2, 3, \ldots, n-1).$$

Let $B = (\beta_{ij})$ $(i, j = 1, 2, \ldots, n)$ denote the inverse of A. From (17) we obtain the relations

$$B = D^{-1} T^{-1} (T^T)^{-1} D^{-1}, \quad DBD = T^{-1} (T^T)^{-1}.$$

Hence

$$TDBD = (T^{-1})^T = (\theta'_{ij}) \quad (i, j = 1, 2, \ldots, n), \tag{18}$$

$$DBDT^T = T^{-1} = (\theta_{ij}) \quad (i, j = 1, 2, \ldots, n). \tag{19}$$

We can now state and prove the main result of this section.

THEOREM 2. *Let A be a symmetric positive definite matrix of the form (16). Then the entries β_{ij} satisfy the recurrence formulae of the "first kind"*

(20)
$$\beta_{ij} = \begin{cases} \theta'_{nj}/d_j d_n & (i = n; j = 1, 2, \ldots, n), \\ \dfrac{\theta'_{ij} - e_i d_{i+1} d_j \beta_{j+1,j} - f_i d_j d_n \beta_{nj}}{d_i d_j} & (i = 1, 2, \ldots, n-1; j = 1, 2, \ldots, n) \end{cases}$$

and the formulae of the "second kind"

(21)

$$\beta_{ij} = \begin{cases} \theta_{in}/d_i d_n & (j = n;\ i = 1, 2, \ldots, n), \\ \dfrac{\theta_{ij} - e_j d_{j+1} d_i \beta_{i,j+1} - f_j d_i d_n \beta_{in}}{d_i d_j} & (j = 1, 2, \ldots, n-1;\ i = 1, 2, \ldots, n), \end{cases}$$

where

(22)

$$\theta_{ij} = \begin{cases} (-1)^i \prod_{k=i}^{n-1} e_k + \prod_{k=i}^{n-1} f_k (-1)^{k-i-1} \prod_{l=i}^{k-1} e_l & (j = n;\ i = 1, 2, \ldots, n-1), \\ (-1)^{j-i} \prod_{k=i}^{j-1} e_k & (i \leqslant j;\ j = 1, 2, \ldots, n-1), \\ 0 & (i > j;\ i = 2, 3, \ldots, n) \end{cases}$$

and $\theta'_{ij} = \theta_{ji}$.

Proof. From the equality $TT^{-1} = I$ we obtain the following system of linear equations satisfied by the i-th row of T:

$$\begin{aligned} \theta_{ii} &= 1, \\ \theta_{i,i+1} + e_i \theta_{i+1,i+1} &= 0, \\ \ldots\ldots\ldots\ldots & \\ \theta_{i,n-1} + e_i \theta_{i+1,n-1} &= 0, \\ \theta_{in} + e_i \theta_{i+1,n} + f_i \theta_{nn} &= 0. \end{aligned}$$

Hence

$$\theta_{ij} = \begin{cases} -e_i \theta_{i+1,n} - f_i & (j = n;\ i = 1, 2, \ldots, n-1), \\ -e_i \theta_{i+1,j} & (i < j;\ j = 2, 3, \ldots, n-1), \\ 1 & (i = j;\ i = 1, 2, \ldots, n), \\ 0 & (i > j;\ i = 2, 3, \ldots, n). \end{cases}$$

After easy calculations we get (22). By virtue of (18) we have

$$d_i \beta_{ij} d_j + e_i d_{i+1} \beta_{i+1,j} d_j + f_i d_n \beta_{nj} d_j = \theta'_{ij} \quad (i = 1, 2, \ldots, n-1;\ j = 1, 2, \ldots, n),$$

$$d_n \beta_{nj} d_j = \theta'_{nj} \quad (j = 1, 2, \ldots, n).$$

Hence we obtain the formulae of the "first kind". Formulae (21) and (22) can be established in the same manner. The proof is complete.

Algorithm 3 follows directly from Theorem 2.

ALGORITHM 3. 1. We calculate the numbers d_j, e_j, f_j from the relations

$$d_1 = \sqrt{b_1}, \quad d_j = [b_j - (c_{j-1}/d_{j-1})^2]^{1/2} \ (j = 2, 3, \ldots, n-1),$$

$$e_j = \frac{c_j}{d_j d_{j+1}} \quad (j = 1, 2, \ldots, n-2),$$

$$d_n = \left\{ b_n - \frac{c_n^2}{d_1^2}(1 + e_1^2 + e_1^2 e_2^2 + \ldots + e_1^2 e_2^2 \ldots e_{n-3}^2) - \left[\frac{c_{n-1}}{d_{n-1}} + (-1)^n \frac{e_1 e_2 \ldots e_{n-2} c_n}{d_1} \right]^2 \right\}^{1/2},$$

$$e_{n-1} = \frac{c_{n-1}}{d_{n-1} d_n}, \quad f_1 = \frac{c_n}{d_1 d_n}, \quad f_j = -e_{j-1} f_{j-1} \ (j = 2, 3, \ldots, n-1).$$

2. We calculate the elements β_{ij} from the formulae

$$\beta_{nn} = 1/d_n^2,$$

$$\beta_{ij} = \begin{cases} \dfrac{\delta_{ij} - e_i d_{i+1} d_j \beta_{i+1,j} - f_i d_j d_n \beta_{nj}}{d_i d_j} & (i \leqslant j, i = 1, 2, \ldots, n-1), \\ \beta_{ji} & (i > j, i = 2, 3, \ldots, n). \end{cases}$$

The above algorithm requires $\frac{5}{2}n^2 + \frac{27}{2}n - 17$ multiplications, $n^2/2 + \frac{7}{2}n - 4$ additions and n square roots. The complexity of Algorithm 3 is greater than the complexity of Algorithm 2.

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