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CHEBYSHEV OPTIMAL STARTING APPROXIMATION BY FAMILIES WITH THE WEAK BETWEENNESS PROPERTY

1. Introduction. Let X be a compact subset of the closed interval $[a, b]$ and let $C(X)$ denote the space of all continuous real-valued functions defined on X, normed by

$$\|f\| = \max\{|f(x)|: x \in X\}.$$

Rice ([10], [11]), Meinardus and Schwedt ([7], [8]) and Dunham ([2], [3]) have developed three distinct theories of approximations to functions from $C(X)$ by non-linear families of functions: varisolvent, asymptotically convex and having the betweenness property, respectively.

In this paper we present a theory of non-linear approximation which includes the last two theories and, under the additional assumption that either $a \notin X$ or $b \notin X$, also the first theory.

Let the family G of real functions $g(A, x)$ defined for $x \in X$ and $A \in P$, where P is an arbitrary subset of E^n, be given. We assume that $g(A, x)$ is a continuous function of $n+1$ variables in the Cartesian product of P and X. Now we define varisolvent functions introduced by Rice in [10].

Definition 1. A family G of functions $g(A, x)$, $A \in P$, is said to have *property* Z *of degree m* at $B \in P$ if for every $A \in P$ the function $g(B, x) - g(A, x)$ has at most $m-1$ zeroes on X or vanishes identically.

Definition 2. A family G of functions $g(A, x)$, $A \in P$, is called *locally solvent of degree m* at $B \in P$ if for every set of m distinct points x_i $(i = 1, \ldots, m)$ in X and for every prescribed $\varepsilon > 0$ there exists a $\delta = \delta(B, \varepsilon, x_1, \ldots, x_m) > 0$ such that the inequality $|y_i - g(B, x_i)| < \delta$ for $i = 1, \ldots, m$ implies the existence of a $C \in P$ satisfying $g(C, x_i) = y_i$ for $i = 1, \ldots, m$ and $\|g(B, \cdot) - g(C, \cdot)\| < \varepsilon$.

Definition 3. A family G is called *varisolvent* if for each $A \in P$ there exists an integer $m = m(A)$ such that G is locally solvent of degree m at A and has property Z of degree m and if there exists no integer $k > m$ with the above properties. The number $m(A)$ is called *degree of varisolvency* of the family G at A.

Definition 4 (Meinardus and Schwedt [8]). A family G of functions $g(A,x)$, $A \in P$, is *asymptotically convex* if for each A, B in P and each $t \in [0,1]$ there exist a parameter value $A(t) \in P$ and a continuous real-valued function $h(x,t)$, defined on $X \times [0,1]$ and satisfying $h(x,0) > 0$ for all $x \in X$, such that

$$\big\|\big(1-th(x,t)\big)g(A,x)+th(x,t)g(B,x)-g\big(A(t),x\big)\big\| = o(t) \quad \text{as } t \to 0.$$

Definition 5 (Dunham [2] and [3]). A subset G of $C(X)$ has the *betweenness property* if for any two elements g_0 and g_1 there exists a λ-set $\{H_\lambda\}$ of elements of G such that $H_0 = g_0$, $H_1 = g_1$ and, for all $x \in X$ and $0 \leqslant \lambda \leqslant 1$, $H_\lambda(x)$ is either a strictly monotonic function of λ or a constant.

2. Families with the weak betweenness property.

Definition 6. A subset G of $C(X)$ has the *weak betweenness property* if for any two distinct elements g and h in G and for every closed subset D of X with $h(x) \neq g(x)$ for all $x \in D$ there exists a sequence $\{g_i\}$ of elements of G such that

(i) $\lim_{i\to\infty} \|g-g_i\| = 0$,

(ii) the numbers $g_i(x)$, where $x \in D$ and $i = 1, 2, \ldots$, lie strictly between $g(x)$ and $h(x)$ (i.e. either $g(x) < g_i(x) < h(x)$ or $h(x) < g_i(x) < g(x)$)

By Definitions 5 and 6 and Lemma 1 in [2] it is obvious that th. subset G of $C(X)$ has the weak betweenness property if it has the betweenness property.

THEOREM 1. *Let us suppose that either $a \notin X$ or $b \notin X$. Then every varisolvent family G on the interval $[a,b]$ has the weak betweenness property.*

Proof. Denote by U a non-empty set defined by

$$U = [a, \inf\{x\colon x \in X\}) \cup (\sup\{x\colon x \in X\}, b].$$

Let $g(A,x)$ and $g(B,x)$ be two arbitrary distinct elements of G and let m be the degree of varisolvency of G at $g(A,x)$, $x \in [a,b]$. Hence there exist k $(k \leqslant m-1)$ zeroes x_j $(j = 1, \ldots, k)$ of $g(A,x)-g(B,x)$ in $[a,b]$. Let D be any closed subset of X such that $g(A,x) \neq g(B,x)$ for all $x \in D$. Moreover, let z and x_j $(j = k+1, \ldots, m-1)$ be arbitrarily prescribed distinct points in D and U, respectively. By Definition 2, for every $\varepsilon_i = 1/i$ $(i = 1, 2, \ldots)$ we may choose a $\delta_i \leqslant \varepsilon_i$ and a suitable y_i lying strictly between $g(A,z)$ and $g(B,z)$ and convergent to $g(A,z)$, such that there exist parameters A_i in P satisfying

$$g(A_i, x_j) = g(A, x_j), \quad j = 1, \ldots, m-1,$$

$$g(A_i, z) = y_i \quad \text{and} \quad \|g(A,\cdot)-g(A_i,\cdot)\| < \frac{1}{i}.$$

Hence

$$\lim_{i\to\infty} \|g(A,\cdot)-g(A_i,\cdot)\| = 0$$

and condition (i) in Definition 6 is satisfied.

It is known (see [11], p. 4) that all zeroes x_j of the function $g(A,x)-g(A_i,x)$ must be simple, i.e. this function changes its sign at x_j. Hence and from the fact that $g(A_i,z)$ lies strictly between $g(A,z)$ and $g(B,z)$ and the functions $g(A,x)-g(B,x)$ and $g(A,x)-g(A_i,x)$ have the same number of zeroes in $[a,b]\setminus U$ we infer that $g(A_i,x)$ lies strictly between $g(A,x)$ and $g(B,x)$ for all $x\in D$. Hence in Definition 6 we may accept g_i to be equal to $g(A_i,\cdot)$, which completes the proof.

THEOREM 2. *Every asymptotically convex family G has the weak betweenness property.*

Proof. Let $g(A,x)$ and $g(B,x)$ be two arbitrary distinct elements in G. It follows from Definition 4 that for every $\varepsilon>0$ there exists a $t_0>0$ such that

$$-\varepsilon t< g(A(t),x)-g(A,x)-th(x,t)[g(B,x)-g(A,x)]<\varepsilon t$$

or all $t\in(0,t_0]$ and $x\in X$. Hence we obtain immediately

$$\lim_{i\to\infty} \|g(A(t_i),\cdot)-g(A,\cdot)\| = 0$$

or any sequence $t_i\to 0$. Moreover, we may suppose, decreasing t_0 if necessary, that $h(x,t)>0$ for all $t\in[0,t_0]$ and $x\in X$. Let D be an arbitrary fixed closed subset of X such that $g(A,x)\neq g(B,x)$ for all $x\in D$. Without loss of generality we may suppose that ε is chosen so that

$$0<\varepsilon<\min_{t\in[0,t_0],x\in D} h(x,t)\,|g(A,x)-g(B,x)|.$$

Then for all $x\in D$ and $t\in(0,t_0]$ we have either

$$g(B,x)-g(A,x)>0 \quad\text{and}\quad g(A(t),x)-g(A,x)>0$$

or

$$g(B,x)-g(A,x)<0 \quad\text{and}\quad g(A(t),x)-g(A,x)<0.$$

Hence, decreasing t_0 if necessary, we infer that $g(A(t),x)$ lies strictly between $g(A,x)$ and $g(B,x)$ for all $x\in D$ and $t\in(0,t_0]$. Now, in the interval $(0,t_0]$ we choose a sequence t_i convergent to 0. Obviously, in Definition 6 we may accept g_i to be equal to $g(A(t_i),\cdot)$, which completes the proof.

3. Characterization of an optimal starting approximation by families with the weak betweenness property. Let us assume that Φ is a continuous mapping of the subset $K\subset C(X)$ into $C(X)$ and that M is an arbitrary fixed non-empty subset of K.

The following three definitions (from [9]) will be useful in the sequel.

Definition 7. The element $p \in M$ is an *optimal starting approximation* in M for $g \in \Phi(K)$ if

$$\|g-\Phi(p)\| \leqslant \|g-\Phi(h)\| \quad \text{for all } h \in M.$$

Definition 8. The operator Φ is called *pointwise strictly monotone* at $f \in K$ if for each $h, g \in K$ we have

$$|\Phi(f)(x_0)-\Phi(h)(x_0)| < |\Phi(f)(x_0)-\Phi(g)(x_0)| \quad \text{for each } x_0 \in X,$$

where either $g(x_0) < h(x_0) \leqslant f(x_0)$ or $f(x_0) \leqslant h(x_0) < g(x_0)$.

Definition 9. The operator Φ is said to be *pointwise fixed* at $f \in K$ if $h \in K$ with $h(x_0) = f(x_0)$ for $x_0 \in X$ implies $\Phi(h)(x_0) = \Phi(f)(x_0)$.

Note that every ordered function [4] and more general transformations considered in [6] are examples of operators being pointwise strictly monotone and pointwise fixed at f, where f and K may be arbitrarily chosen. For other examples and properties of operators being pointwise strictly monotone and pointwise fixed at f see [9]. Obviously, Φ may be equal to the identity operator. Hence the theory of optimal starting approximation contains the theory of ordinary Chebyshev approximation. Moreover, note that the change of the norm $\|\cdot\|$ defined in Section 1 into the norm (see [9])

$$\|f\|_w = \max\{w(x)|f(x)|: x \in X\},$$

where $w(x) > 0$ for all $x \in X$, does not generalize our considerations. Indeed, the optimal starting approximation with the operator Φ and the norm $\|\cdot\|_w$ may be replaced by the approximation with the operator $w\Phi$ and the norm $\|\cdot\|$.

We assume throughout our discussion that the function f and the operator Φ are fixed. Denote by $D(g)$ a closed subset of X defined by

$$D(g) = \{x \in X: |\Phi(f)(x)-\Phi(g)(x)| = \|\Phi(f)-\Phi(g)\|\}.$$

THEOREM 3. *Let Φ: $K \to C(X)$ be a continuous operator. Let G be an arbitrary subset of $C(X)$ having the weak betweenness property and let $M = K \cap G$ be a non-empty relatively open subset of G. Finally, assume that Φ is pointwise strictly monotone at $f \in K \setminus M$. Then a necessary condition for $g \in M$ to be an optimal starting approximation to $\Phi(f)$ is that there exists no element $h \in G$ such that*

$$[f(x)-g(x)][h(x)-g(x)] > 0 \quad \text{for all } x \in D(g).$$

Proof. Suppose, on the contrary, that there exists a function $h \in G$ satisfying the inequality in the theorem. Hence for $x \in D(g)$ we have either

$$h(x) > g(x) \quad \text{and} \quad f(x) > g(x)$$

or

$$h(x) < g(x) \quad \text{and} \quad f(x) < g(x).$$

Since G has the weak betweenness property and M is relatively open in G, the inequalities above imply that there exist a sequence g_i of elements of G and an integer m such that $g_i(x)$ lies strictly between $f(x)$ and $g(x)$ for all $x \in D(g)$ and $i \geqslant m$, and $g_i \in M$ for all $i \geqslant m$. From the pointwise strict monotonicity of Φ at f it follows that

(1) $$|\Phi(f)(x) - \Phi(g_i)(x)| < |\Phi(f)(x) - \Phi(g)(x)| = \|\Phi(f) - \Phi(g)\|$$

for all $i \geqslant m$ and $x \in D(g)$. If $D(g) = X$, the proof is completed.

Otherwise, it follows from the continuity of the function $|\Phi(f) - \Phi(g_i)|$ that there exists an open set $U \supset D(g)$ such that (1) is true for all $x \in U$. Let $V = X \setminus U$. Obviously, V is a closed set. Let us put

$$\delta = \sup\{|\Phi(f)(x) - \Phi(g)(x)|:\ x \in V\}.$$

Since $V \cap D(g)$ is an empty set, we have $\|\Phi(f) - \Phi(g)\| > \delta$. From the continuity of Φ and the uniform convergence of g_i to g it follows that there exists an integer n, $n \geqslant m$, such that

$$\|\Phi(g) - \Phi(g_i)\| < \|\Phi(f) - \Phi(g)\| - \delta \quad \text{for all } i \geqslant n.$$

Hence, for all $x \in V$ and $i \geqslant n$ we obtain

$$\begin{aligned}|\Phi(f)(x) - \Phi(g_i)(x)| &\leqslant |\Phi(f)(x) - \Phi(g)(x)| + |\Phi(g)(x) - \Phi(g_i)(x)| \\ &< \delta + \|\Phi(f) - \Phi(g)\| - \delta = \|\Phi(f) - \Phi(g)\|.\end{aligned}$$

Combining this result with (1) we have

$$\|\Phi(f) - \Phi(g_i)\| < \|\Phi(f) - \Phi(g)\| \quad \text{for } i \geqslant n.$$

Hence we see that the functions g_i in M for $i \geqslant n$ are better optimal starting approximations to $\Phi(f)$ than g and this gives a contradiction.

THEOREM 4. *Let M be an arbitrary subset of K and let the operator $\Phi\colon K \to C(X)$ be pointwise strictly monotone and pointwise fixed at $f \in K \setminus M$. Then a sufficient condition for $g \in M$ to be an optimal starting approximation to $\Phi(f)$ is that there exists no element $h \in M \setminus \{g\}$ such that*

$$[f(x) - g(x)][h(x) - g(x)] \geqslant 0 \quad \textit{for all } x \in D(g).$$

Proof. Since $f \notin M$ and Φ is pointwise fixed at f, we have $f(x) \neq g(x)$ for all $x \in D(g)$. Suppose, on the contrary, that there exists an $h \in M$ such that

$$\|\Phi(f) - \Phi(h)\| < \|\Phi(f) - \Phi(g)\|.$$

Hence for all $x \in D(g)$ we have

(2) $$|\Phi(f)(x) - \Phi(h)(x)| < |\Phi(f)(x) - \Phi(g)(x)|.$$

Now, for $x \in D(g)$ either

$$f(x) > g(x) \quad \text{and} \quad h(x) \geqslant g(x)$$

or

$$f(x) < g(x) \quad \text{and} \quad h(x) \leqslant g(x).$$

On the other hand, by the pointwise monotonicity of Φ at f we have

$$|\Phi(f)(x) - \Phi(g)(x)| < |\Phi(f)(x) - \Phi(h)(x)|,$$

which contradicts (2). Combining the inequalities above for the functions f, g and h we obtain the required inequality.

The following theorem shows that in special cases the necessary condition for the function g to be an optimal starting approximation may be also the sufficient condition.

THEOREM 5. *Under the assumptions of Theorem 3 and the additional assumptions that Φ is a pointwise fixed operator at f and*

(3) $\qquad h(x) = g(x)$ *implies* $\Phi(h)(x) = \Phi(g)(x)$ *for all* $g, h \in M$,

a necessary and sufficient condition for $g \in M$ to be an optimal starting approximation to $\Phi(f)$ is that there exists no element $h \in G$ such that

$$[f(x) - g(x)][h(x) - g(x)] > 0 \quad \textit{for all } x \in D(g).$$

Proof. The assertion follows from Theorems 3 and 4 and from the fact that the equality $h(x) = g(x)$ for any $x \in D(g)$ in the proof of Theorem 4 is impossible by condition (3).

Condition (3) is satisfied for a large number of operators Φ. In particular, it obviously holds if Φ is the identity operator, an ordered function (see [4]) or a transformation from [6].

Definition 10. The n-dimensional subspace G of $C(X)$ is called the *Haar subspace* on X if every non-zero function in G has at most $n-1$ zeroes.

We say that x is a *simple zero* for $f \in C[a, b]$ if $f(x) = 0$ and f changes its sign at x.

Now, we prove that for some families G the sufficient condition for g to be an optimal starting approximation may also be the necessary condition. For this purpose we prove at first the following

LEMMA 1. *Let g and h be arbitrary fixed distinct elements of M and let $f \in K \setminus M$. Let G be either an n-dimensional Haar subspace or a varisolvent family on $[a, b]$. In the second case, we additionally assume that either $a \notin X$ or $b \notin X$. Let D be a closed subset of X such that $f(x) \neq g(x)$ for all $x \in D$. Then the inequality*

$$[f(x) - g(x)][h(x) - g(x)] \geqslant 0 \quad \textit{for all } x \in D$$

implies that there exists a $p \in G$ *such that*

$$[f(x)-g(x)][p(x)-g(x)] > 0 \quad \textit{for all } x \in D.$$

Proof. At first, suppose that G is a Haar subspace on $[a, b]$. Put $B = \{x \in D\colon h(x) = g(x)\}$ and suppose that this closed set is non-empty. Since M is a Haar subspace, the set B contains exactly k elements, where $1 \leqslant k \leqslant n$. Let a function $r \in G$ be defined by the interpolation conditions $r(x_i) = d_i$ for $i = 1, \ldots, n$, where, for $i \leqslant k$, $x_i \in B$ and $d_i = f(x_i)-g(x_i)$, and, for $i > k$, $x_i \in [a, b]\setminus B$ and d_i are arbitrary real numbers. Obviously, we have

$$[f(x)-g(x)][h(x)-g(x)+\lambda r(x)] > 0$$

for all $x \in B$ and arbitrary fixed $\lambda > 0$. Thus there exists an open set $U \supset B$ such that the last inequality holds for all $x \in U$. If $D \subset U$, the proof is completed. Otherwise, put $V = ([a, b]\setminus U)\cap D$ and let

$$\delta = \inf\{|h(x)-g(x)|\colon x \in V\} > 0.$$

Hence

$$[f(x)-g(x)][h(x)-g(x)+\lambda r(x)] > 0$$

for all $x \in V$ and λ such that $0 < \lambda\|r\| < \delta$. Finally, the function $p(x) = h(x)+\lambda r(x)$, where $0 < \lambda\|r\| < \delta$ has the required properties.

Now, we assume that G is a varisolvent family on $[a, b]$ and that either $a \notin X$ or $b \notin X$. Moreover, let

$$(4) \qquad [f(x)-g(A, x)][g(B, x)-g(A, x)] \geqslant 0 \quad \text{for all } x \in D$$

and let m be the degree of varisolvency of the family G at $B \in P$. Let $z_1, \ldots, z_k$ for $k < m$ be simple zeroes of the function $g(B, x)-g(A, x)$ in (a, b). Suppose that $z_i \in D$ for $i = 1, \ldots, l$, where $l \leqslant k$. From the continuity of all considered functions and inequality (4) it follows that for sufficiently small $\varepsilon > 0$ there exist sets $O_\varepsilon(z_i)$ equal to $(z_i, z_i-\varepsilon)$ or $(z_i, z_i+\varepsilon)$ such that $O_\varepsilon(z_i)\cap D = \emptyset$ for $i = 1, \ldots, l$. Let x_i, $i = 1, \ldots, m$, be distinct points of $[a, b]$ such that

$$x_i \in O_\varepsilon(z_i) \quad \text{for } i = 1, \ldots, l, \qquad x_i = z_i \quad \text{for } i = l+1, \ldots, k,$$

$$x_i \in [a, \inf\{x\colon x \in X\})\cup(\sup\{x\colon x \in X\}, b] \quad \text{for } i = k+1, \ldots, m-1,$$

and let x_m be a point such that $g(B, x_m) \neq g(A, x_m)$. Moreover, let y_m lie strictly between $g(B, x_m)$ and $g(A, x_m)$ and let $y_i = g(B, x_i)$ for $i = 1, \ldots, m-1$. By Definition 2, for y_m sufficiently close to $g(B, x_m)$ there exists a $C \in P$ such that $g(C, x_i) = y_i$ for $i = 1, \ldots, m$. Obviously, by this construction, assumption (4) and Lemma 7.1 from [11], p. 4, we have

$$[f(x)-g(A, x)][g(C, x)-g(A, x)] > 0 \quad \text{for all } x \in D.$$

Hence the proof of the lemma is completed.

From Lemma 1 and Theorems 3 and 4 we obtain immediately the following

THEOREM 6. *Let $\Phi: K \to C(X)$ be a continuous operator. Let G be an arbitrary Haar subspace or a varisolvent family on $[a, b]$. In the second case, we additionally assume that either $a \notin X$ or $b \notin X$. Let $M = K \cap G$ be a non-empty relatively open subset of G. Finally, assume that Φ is pointwise strictly monotone and pointwise fixed at $f \in K \setminus M$. Then a necessary and sufficient condition for $g \in M$ to be an optimal starting approximation to $\Phi(f)$ is that there exists no element $h \in G \setminus \{g\}$ such that*

$$[f(x)-g(x)][h(x)-g(x)] \geqslant 0 \quad \textit{for all } x \in D(g).$$

Now, let P and Q be spaces of algebraic polynomials defined on $[a, b]$ of degrees not greater than n and m, respectively. Denote by R the family of functions $r = p/q$, where $p \in P$, $q \in Q$ and $q(x) > 0$ for every $x \in [a, b]$. Let $r = p/q$ be a fixed irreducible element of R and let $P+rQ$ be the subspace of $C[a, b]$ such that

$$P+rQ = \{p+rq: p \in P \text{ and } q \in Q\}.$$

Then from [1], p. 162, and [12] we have

THEOREM 7. *Under the assumptions of Theorem 6 about the operator Φ a necessary and sufficient condition for $r \in M = K \cap R$ to be an optimal starting approximation to $\Phi(f) \in C(X)$ is that there exists no element $h \in P+rQ$, $h \not\equiv r$, such that*

$$[f(x)-r(x)][h(x)-r(x)] \geqslant 0 \quad \textit{for all } x \in D(g).$$

Note that the weak inequality $\geqslant$ in Theorems 6 and 7 may be replaced by the sharp inequality $>$.

4. Alternation theorems. In this section we suppose that X contains at least $n+1$ points. Now we shall prove a lemma which enables us to obtain alternation theorems from Theorems 6 and 7.

LEMMA 2. *Let $f \in C(X)$, $g \in G$, and let $D \subset X$ be a given closed set such that $f(x) \neq g(x)$ for all $x \in D$. Assume that G is either an n-dimensional Haar subspace or a varisolvent family on $[a, b]$. In the second case we additionally assume that either $a \notin X$ or $b \notin X$ and that n denotes the degree of varisolvency of the family G at g. Then there exists no function $h \in G \setminus \{g\}$ such that*

$$[f(x)-g(x)][h(x)-g(x)] \geqslant 0 \quad \textit{for each } x \in D$$

if and only if the set D contains at least $n+1$ alternation points x_i of the function $f-g$, i.e. such that $a \leqslant x_0 < x_1 < \ldots < x_n \leqslant b$ and

$$f(x_i)-g(x_i) = (-1)^i [f(x_0)-g(x_0)], \quad i = 0, \ldots, n.$$

Proof. At first, let G be an n-dimensional Haar subspace on $[a, b]$. Since a non-trivial function $h(x)-g(x) \in G$ can have at most $n-1$ variations in sign, the sufficiency of the lemma is obvious.

For the necessity let us suppose on the contrary that $f-g$ has exactly k ($k \leqslant n$) alternation points x_i for $i = 0, \ldots, k-1$ and that the function f_1 is a continuous extension of f on $[a, b]$. This function f_1 exists by the well-known Tietze theorem. If $k = 1$, then the proof is completed, since in every Haar subspace there exists a positive function p and we may set $h = p+g$. Otherwise, let z_i denote arbitrary fixed zeroes of f_1 in (x_{i-1}, x_i) for $i = 1, \ldots, k-1$. Moreover, let $O(z_i)$ be closed sets containing z_i such that

$$\sup\{|f_1(x)|:\ x \in O(z_i)\} < \inf\{|f_1(x)|:\ x \in D\}.$$

Obviously, we have $D \cap O(z_i) = \emptyset$. In G we choose a function r defined by the following interpolation conditions: $r(x_0) = f_1(x_0)$ and r changes its sign at $n-1$ distinct points u_i, where $u_i \in O(z_i)$ for $i = 1, \ldots, k-1$, $u_i \in O(z_1)$ for $i = k, \ldots, n-2$ and $u_{n-1} = b$ when $n-k$ is an odd number (otherwise, we assume that $u_{n-1} \in O(z_1)$). We have $f_1(x)r(x) \geqslant 0$ for all $x \in D$. Hence we may set $h = r+g$. Now, assume that G is a varisolvent family such as in the thesis of this lemma. The necessity in this case follows from the fact that G has property Z. For the sufficiency, let x_i, z_i and $O(z_i)$ be defined as above. Let u_i ($i = 1, \ldots, n$) be distinct points of $[a, b]$ such that

$$u_i \in O(z_i) \quad \text{for } i = 1, \ldots, k-1,$$

$$u_i \in [a, \inf\{x:\ x \in X\}) \cup (\sup\{x:\ x \in X\}, b] \quad \text{for } i = k+1, \ldots, n-1$$

and

$$u_n = x_0.$$

Moreover, let $v_i = g(u_i)$ for $i = 1, \ldots, n-1$ and let v_n be chosen so that

$$f_1(x_0)[v_n - g(x_0)] > 0.$$

By Definition 2, for v_n sufficiently close to $g(x_0)$ there exists a function h in G such that $h(u_i) = v_i$ for $i = 1, \ldots, n$. Obviously, by this construction and Lemma 7.1 in [11], p. 4, we have

$$[f(x)-g(x)][h(x)-g(x)] > 0 \quad \text{for all } x \in D.$$

This completes the proof.

By Lemma 2 and Theorem 6 we obtain the following generalization of Theorem 2 from [8].

THEOREM 8. *Under the assumptions of Theorem 6 a necessary and sufficient condition for $g \in M$ to be an optimal starting approximation to $\Phi(f)$ is that the set $D(g)$ contains at least $n+1$ alternation points of the function $f-g$.*

Here n denotes either the dimension of a Haar subspace G or the degree fo varisolvency of the family G at g.

By Lemma 2, Theorem 7 and [1], p. 162, we have the following theorem [5]:

THEOREM 9. *Under the assumptions of Theorem 7 a necessary and sufficient condition for an irreducible function $r = p/q \in M$ to be an optimal starting approximation to $\Phi(f)$ is that the set $D(r)$ contains at least $2+ +\max\{n+u, m+v\}$ alternation points of the function $f-r$. Here u and v denote the degrees of q and p, respectively.*

References

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NIELINIOWA OPTYMALNA APROKSYMACJA STARTOWA

STRESZCZENIE

W niniejszej pracy omówiliśmy pewną nieliniową rodzinę aproksymujących funkcji oraz sformułowaliśmy kilka twierdzeń charakteryzujących optymalną startową aproksymację przez jej elementy. Ponadto udowodniliśmy, że rodzina ta zawiera rodzinę funkcji asymptotycznie wypukłych ([7], [8]), rodzinę zdefiniowaną przez Dunhama ([2], [3]) oraz, przy dodatkowych założeniach, rodzinę funkcji lokalnie interpolujących ([10], [11]). Z tego względu niniejsza praca jest także próbą połączenia trzech różnych teorii nieliniowej aproksymacji, rozwiniętych przez J. R. Rice'a, G. Meinardusa oraz C. B. Dunhama.