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J. FRANZ (Dresden) and R. MAGIERA (Wrocław)

ON SEQUENTIAL PLANS FOR THE EXPONENTIAL CLASS OF PROCESSES

1. Introduction. The following sequential decision problem for stochastic processes is considered. Let $\{X_t, t \in T\}$, where $T = [0, \infty)$, be a stochastic process with values in $\mathscr{X} \subseteq R$ (R denotes the real line) and with right-continuous sample functions whose probability distributions are entirely defined by a given parameter $\vartheta \in \Theta$. Denote by τ the Markov stopping time with respect to the family $\{\mathscr{F}_t, t \in T\}$ of σ-algebras generated by the random variables X_s, $s \leqslant t$. The Markov stopping time determines a time up to which the process is observed. If values of τ and X_τ are known, we wish to estimate the value of a given function $g(\vartheta)$. The problem is to find Markov stopping times τ and estimators $f(\tau, X_\tau)$ having some optimal properties.

A sequential plan is defined by a Markov stopping time τ and the unbiased estimator $f(\tau, X_\tau)$ of the function $g(\vartheta)$. The optimality of the sequential plan is established on the basis of the inequality of Cramér-Rao type. A sequential plan for which the variance of the estimator f attains its lower bound determined by the inequality of Cramér-Rao type is called an *efficient plan.*

The class of all efficient sequential plans for processes of the exponential class is found and the characterization of sequential plans for these processes is presented by introducing some conditions on moments of Markov stopping times. It is shown that the only efficient sequential plans are those ones which are determined by the first attaining times of suitable lines on the product space $T \times \mathscr{X}$. The forms of efficient estimators and efficiently estimable functions are also determined. Moreover, it is shown that the variance of a Markov stopping time τ attains its lower bound determined by the inequality of Cramér-Rao type only in the situations where the plans corresponding to τ are efficient. Taking into account the Wald identities and the so-called higher order Wald identities which are given for the exponential class of processes it is proved that the sequential plans defined by the first attaining times of some curve satisfying an algebraic equation of degree k are completely characterized by the $k(2k+1)$ values $\mathrm{E}_\vartheta^{(r)}(\tau^s)$ $(s+r \leqslant 2k,\ s \geqslant 1)$ given at any fixed value $\vartheta \in \Theta$, provided that these derivatives exist.

Some sequential estimation problems for processes of the exponential class were also solved by applying other methods. Namely, the Bayes and minimax methods were used in paper [8] and confidence interval estimation was considered in [2].

2. Preliminaries. Let $X_T := \{X_t, t \in T\}$ be a stochastic process defined on a probability space $(\Omega, \mathscr{F}, P)$ with values in a measurable space $(\mathscr{X}, \mathscr{B})$. We assume that $\mathscr{X} \subseteq R$, where R denotes the real line, and $\mathscr{B}$ is a σ-algebra of Borel subsets of $\mathscr{X}$. By T we denote the half-line $[0, \infty)$.

The probability measure P depends on a parameter ϑ, in general unknown, and it can be interpreted as an element of the family of probability measures $\{P_\vartheta, \vartheta \in \Theta\}$. The one-dimensional probability distributions on $(\mathscr{X}, \mathscr{B})$ of the process X_T are entirely defined by a given parameter $\vartheta \in \Theta$. We assume that Θ is a finite or infinite interval of the real line, and ϑ plays the role of an unknown parameter of the process. To estimate this parameter or its function we shall use a sequential method. In consideration of optimal sequential estimation problems it is natural to assume that all measures P_ϑ, $\vartheta \in \Theta$, are mutually absolutely continuous. By $\mathrm{E}_\vartheta(\cdot)$ and $\mathrm{D}_\vartheta(\cdot)$ we denote the expected value and the variance, respectively, evaluated with respect to the measure P_ϑ. If h is any function depending on ϑ, then by h' we denote its derivative with respect to ϑ.

Let us note that if the sample functions of the process X_T are right-continuous, then for every finite (with probability 1) Markov stopping time τ the function $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$ is a random variable and is $\mathscr{F}_\tau$-measurable, where $\mathscr{F}_\tau$ denotes a σ-algebra consisting of all sets $A \in \mathscr{F}$ such that $A \cap \{\omega\colon \tau(\omega) \leqslant t\} \in \mathscr{F}_t$ for every $t \in T$ (see, e.g., [10], p. 70).

Let us quote another fact: if the sample functions of the process X_T are right-continuous and the set $C \subset T \times \mathscr{X}$ is closed, then the Markov stopping time

$$\tau(\omega) = \sup\{t'\colon (t, X_t(\omega)) \notin C,\ 0 \leqslant t \leqslant t'\}$$

is the *first attaining time* of the set C. We then have, in fact, $(\tau(\omega), X_\tau(\omega)) \in C$ and, for any $t < \tau(\omega)$, $(t, X_t(\omega)) \notin C$.

In what follows we denote by X_T the homogeneous process with independent increments satisfying $P_\vartheta(X_0 = 0) = 1$ for all $\vartheta \in \Theta$, $\mathrm{E}_\vartheta(X_t^2) < \infty$ for each $t \in T$ and all $\vartheta \in \Theta$, and having right-continuous sample functions. We assume that the densities (with respect to the Lebesgue or counting measure) of one-dimensional probability distributions on $(\mathscr{X}, \mathscr{B})$ of the process X_T are of the form

$$p(t, x; \vartheta) = q(t, x)\exp[w_1(\vartheta)t + w_2(\vartheta)x] \tag{1}$$

$$\text{for } t > 0,\ x \in \mathscr{X},\ \text{and } \vartheta \in \Theta,$$

where $q(t, x)$ is a non-negative function defined on $T \times \mathscr{X}$, and $w_1(\vartheta)$ and $w_2(\vartheta)$ are some functions defined on Θ. Thus the process X_T belongs to the so-called *exponential class of processes* (see [3], [7], and [8]).

We assume that the functions $w_1(\vartheta)$ and $w_2(\vartheta)$ are twice continuously differentiable in the interval Θ; the derivative $w_2'(\vartheta)$ is strictly positive for all $\vartheta \in \Theta$ and the function $w_1'(\vartheta)/w_2'(\vartheta)$ is strictly decreasing in the whole interval Θ.

The expected value and the variance of the process X_T are equal to

$$\mathrm{E}_\vartheta(X_t) = -\frac{w_1'(\vartheta)}{w_2'(\vartheta)}t \quad \text{and} \quad \mathrm{D}_\vartheta(X_t) = -\frac{1}{w_2'(\vartheta)}\frac{d}{d\vartheta}\left[\frac{w_1'(\vartheta)}{w_2'(\vartheta)}\right]t,$$

respectively.

In the sequel we suppose that for the process X_T the relation

$$-\frac{w_1'(\vartheta)}{w_2'(\vartheta)} = \vartheta$$

holds for all $\vartheta \in \Theta$. Thus for this process we have

$$\mathrm{E}_\vartheta(X_t) = \vartheta t \tag{2}$$

and

$$\mathrm{D}_\vartheta(X_t) = \frac{1}{w_2'(\vartheta)}t. \tag{3}$$

If the process X_T is observed up to time t, then the random variable X_t is a sufficient statistic for the parameter ϑ. Furthermore, the pair (τ, X_τ) of $\mathscr{F}_\tau$-measurable functions is also a sufficient statistic for ϑ.

Let $U = T \times \mathscr{X}$ and let $\mathscr{U}$ be a σ-algebra of Borel subsets of U (a σ-algebra of Borel subsets of T is considered). The components of the point $u \in U$ are denoted by $t = t(u)$ and $x = x(u)$. The pair (τ, X_τ) generates, for every $\vartheta \in \Theta$, the measurable transformation

$$\mathscr{Z}\colon\ \omega \to \big(\tau(\omega), X_\tau(\omega)\big).$$

Let m_ϑ denote the distribution of $\mathscr{Z}$ on $(U, \mathscr{U})$, i.e.,

$$m_\vartheta(C) = P_\vartheta\big(\mathscr{Z}^{-1}(C)\big) = P_\vartheta\Big(\big\{\omega\colon \big(\tau(\omega), X_\tau(\omega)\big) \in C\big\}\Big) \quad \text{for every } C \in \mathscr{U}.$$

It follows from the Sudakov theorem (see [11] or [6], p. 55-59) or from Theorem 1 in [3] that on $\mathscr{U}$ there exists a σ-finite measure Q_τ independent of ϑ and such that

$$P_\vartheta\big((\tau, X_\tau) \in C\big) = \int_C \exp[w_1(\vartheta)t(u) + w_2(\vartheta)x(u)]Q_\tau(du) \tag{4}$$

for each $\vartheta \in \Theta$ and every $C \in \mathscr{U}$.

3. Equations relating moments of τ and X_τ. Let $\varphi(u; \vartheta)$ be a function defined on $U \times \Theta$. We assume that for every $\vartheta \in \Theta$ the function φ is $\mathscr{U}$-measurable and m_ϑ-integrable. Moreover, we suppose that, for almost all $u \in U$ with respect to Q_τ, $\varphi(u; \vartheta)$ is a continuously differentiable function of the variable ϑ. Let, for every $\vartheta \in \Theta$,

$$\int_U |\varphi(u; \vartheta)[x(u) - \vartheta t(u)] + \mathrm{D}_\vartheta(X_1)\varphi'(u; \vartheta)|\, m_\vartheta(du) < \infty. \tag{5}$$

Then, differentiating the function under the integral sign in the formula

$$\mathrm{E}_\vartheta[\varphi(\tau, X_\tau; \vartheta)] = \int_U \varphi(u; \vartheta)\exp[w_1(\vartheta)t(u) + w_2(\vartheta)x(u)]Q_\tau(du)$$

with respect to the parameter ϑ we obtain the equation

$$\mathrm{E}_\vartheta[(X_\tau - \vartheta\tau)\varphi(\tau, X_\tau; \vartheta)] = \mathrm{D}_\vartheta(X_1)\{\mathrm{E}'_\vartheta[\varphi(\tau, X_\tau; \vartheta)] - \mathrm{E}_\vartheta[\varphi'(\tau, X_\tau; \vartheta)]\}. \tag{6}$$

If $\varphi(\tau, X_\tau; \vartheta)$ is a polynomial in τ and X_τ of degree $p-1$ (with coefficients continuously differentiable with respect to ϑ) and if

$$\mathrm{E}_\vartheta(|X_1|^p) < \infty, \quad \mathrm{E}_\vartheta(\tau^p) < \infty \quad \text{for even } p$$

or

$$\mathrm{E}_\vartheta(|X_1|^{p+1}) < \infty, \quad \mathrm{E}_\vartheta(\tau^{p+1}) < \infty \quad \text{for odd } p,$$

then relation (5) holds and equation (6) is valid. This assertion (see [2]) is based on the result of Hall [5] and on Hölder's inequality.

Some important consequences of equation (6) are now given.

Assume that $\mathrm{E}_\vartheta(X_1^2) < \infty$ and $\mathrm{E}_\vartheta(\tau^2) < \infty$. If we put $\varphi(\tau, X_\tau; \vartheta) = 1$ into formula (6), then we obtain the first Wald identity

$$\mathrm{E}_\vartheta(X_\tau - \vartheta\tau) = 0. \tag{7}$$

If we put $\varphi(\tau, X_\tau; \vartheta) = X_\tau - \vartheta\tau$, then by (6) we have the second Wald identity

$$\mathrm{E}_\vartheta[(X_\tau - \vartheta\tau)^2] = \mathrm{D}_\vartheta(X_1)\mathrm{E}_\vartheta(\tau). \tag{8}$$

Now, let $\varphi(\tau, X_\tau; \vartheta) = \tau$. Then

(9) $$E_\vartheta[\tau(X_\tau - \vartheta\tau)] = D_\vartheta(X_1)E'_\vartheta(\tau).$$

Taking into account equalities (7), (8), and (9), we obtain

(10) $$E_\vartheta(X_\tau^2) = \vartheta^2 E_\vartheta(\tau^2) + D_\vartheta(X_1)E_\vartheta(\tau) + 2\vartheta D_\vartheta(X_1)E'_\vartheta(\tau)$$

and

(11) $$D_\vartheta(X_\tau) = \vartheta^2 D_\vartheta(\tau) + D_\vartheta(X_1)E_\vartheta(\tau) + 2\vartheta D_\vartheta(X_1)E'_\vartheta(\tau).$$

With further restrictions on X_1 and τ one obtains the higher order Wald identities and other equations relating moments of τ and X_τ. Let, for instance, $E_\vartheta(X_1^4) < \infty$ and $E_\vartheta(\tau^4) < \infty$. Then, assuming that the function $\varphi(\tau, X_\tau; \vartheta)$ is of the form $(X_\tau - \vartheta\tau)^2$, $\tau(X_\tau - \vartheta\tau)$, and $(X_\tau - \vartheta\tau)^3$, and using (9) we obtain by (6), respectively (cf. [5]):

(12) $$E_\vartheta[(X_\tau - \vartheta\tau)^3] = D_\vartheta(X_1)[3D_\vartheta(X_1)E'_\vartheta(\tau) + D'_\vartheta(X_1)E_\vartheta(\tau)],$$

(13) $$E_\vartheta[\tau(X_\tau - \vartheta\tau)^2] = D_\vartheta(X_1)[D_\vartheta(X_1)E''_\vartheta(\tau) + D'_\vartheta(X_1)E'_\vartheta(\tau) + E_\vartheta(\tau^2)],$$

(14) $$E_\vartheta[(X_\tau - \vartheta\tau)^4] = [D_\vartheta(X_1)]^2[6D_\vartheta(X_1)E''_\vartheta(\tau) + 10D'_\vartheta(X_1)E'_\vartheta(\tau) + \\ + D''_\vartheta(X_1)E_\vartheta(\tau)] + D_\vartheta(X_1)[D'_\vartheta(X_1)]^2E_\vartheta(\tau).$$

4. Sequential plans for processes of the exponential class. Let $g(\vartheta)$ be a function of the parameter ϑ, differentiable on Θ and such that its derivative $g'(\vartheta)$ is not equal to zero for any $\vartheta \in \Theta$.

We are concerned with only such estimators which depend on the sufficient statistic (τ, X_τ). By virtue of a suitable analogue of the Blackwell theorem [1] this fact may be entirely justified.

By a *sequential plan* we mean any pair (τ, f) consisting of a Markov stopping time τ satisfying $P_\vartheta(0 < \tau < \infty) = 1$ for all $\vartheta \in \Theta$ and an unbiased estimator f of the function $g(\vartheta)$, i.e., such that

(15) $$E_\vartheta[f(\tau, X_\tau)] = \int_U f(u)\exp[w_1(\vartheta)t(u) + w_2(\vartheta)x(u)]Q_\tau(du) = g(\vartheta)$$

for every $\vartheta \in \Theta$.

Suppose that

$$E_\vartheta(|f(\tau, X_\tau)(X_\tau - \vartheta\tau)|) < \infty \quad \text{for all } \vartheta \in \Theta.$$

By (6) we then have

(16) $$E_\vartheta[f(\tau, X_\tau)(X_\tau - \vartheta\tau)] = D_\vartheta(X_1)g'(\vartheta).$$

The following theorem gives the lower bound for the estimator variance and the form of the estimator when this bound is attained.

THEOREM 1. *For each sequential plan* (τ, f) *satisfying* $\mathrm{E}_\vartheta(f^2) < \infty$ *and* $\mathrm{E}_\vartheta(\tau^2) < \infty$ *for all* $\vartheta \in \Theta$, *the inequality*

$$\mathrm{D}_\vartheta[f(\tau, X_\tau)] \geqslant \frac{\mathrm{D}_\vartheta(X_1)[g'(\vartheta)]^2}{\mathrm{E}_\vartheta(\tau)} \tag{17}$$

holds for all $\vartheta \in \Theta$. *The equality holds at a particular value of* ϑ *if and only if*

$$f(\tau, X_\tau) = c(\vartheta)(X_\tau - \vartheta\tau) + g(\vartheta) \tag{18}$$

with probability 1, *where* $c(\vartheta) \neq 0$.

Inequality (17) is obtained from (7), (8), (16), and the Schwarz inequality. Theorem 1 is a special case of the Theorem in [7] which gives the inequality of Cramér-Rao type in a sequential case for a class of processes for which the Sudakov theorem is valid.

Let us remark that the function $f(\tau, X_\tau) = \text{const}$ with probability 1 cannot be an estimator for the parameter $\gamma = g(\vartheta)$; for, if it were, then $\mathrm{E}_\vartheta[f(\tau, X_\tau)] = \text{const} = g(\vartheta)$ for all $\vartheta \in \Theta$ which contradicts the assumption.

The following notions are used in what follows.

A sequential plan (τ, f) is said to be *efficient for* (*a given value*) ϑ if (17) becomes an equality at ϑ. The estimator f is then called *efficient at this value* ϑ, and the function g is *efficiently estimable at the point* ϑ.

A sequential plan (τ, f) is said to be *efficient* if it is efficient for each $\vartheta \in \Theta$. The estimator f is then called *efficient*, and the function g is *efficiently estimable*.

The following theorem states that in selecting efficient plans one should consider only such plans for which the measure Q_τ is accumulated on a straight line.

THEOREM 2. *If a sequential plan* (τ, f) *is efficient, then there exist constants* a_1, a_2, *not both zero, and a constant* $a_3 \neq 0$ *such that*

$$a_1\tau + a_2 X_\tau + a_3 = 0 \tag{19}$$

with probability 1 (see [7]).

A sequential plan (τ, f), where τ is equal, with probability 1, to the first attaining time of some set $C \subset U$ is called a *first attaining plan*, and C is then called a *stopping set*.

In connection with Theorem 2 we define the following types of first attaining plans.

Definition 1. A sequential plan (τ, f), where τ is equal, with probability 1, to a constant $t_0 > 0$ is called a *fixed-time plan*.

Definition 2. A sequential plan (τ, f), where τ is equal, with probability 1, to the first attaining time of the line $x(u) = x_0$ $(x_0 \neq 0)$ is called an *inverse plan*.

Definition 3. A sequential plan (τ, f), where τ is equal, with probability 1, to the first attaining time of the line

$$x(u) = at(u) - s \qquad (a \neq 0,\ s \neq 0) \tag{20}$$

is called an *oblique plan.*

The next lemma can be obtained in a similar way as that in [12] for the Poisson process.

LEMMA 1. *Let (τ, f) be a sequential plan satisfying $\mathrm{E}_\vartheta(\tau) < \infty$ for all $\vartheta \in \Theta$. Suppose that there exist constants a_1, a_2, not both zero, and a constant $a_3 \neq 0$ such that equality (19) holds with probability 1. Then (τ, f) is either a fixed-time plan or an inverse plan or an oblique plan.*

Lemma 1 and Theorem 2 show that the only sequential plans satisfying certain conditions necessary for a sequential plan to be efficient are the fixed-time, inverse and oblique plans. In order to show that these plans are indeed efficient one has to determine at first the moments of first attaining times τ for these plans.

Suppose that the underlying first attaining plans (τ, f) satisfy the condition $\mathrm{E}_\vartheta(\tau^2) < \infty$ for all $\vartheta \in \Theta$.

For a fixed-time plan (τ, f) we have

$$\mathrm{E}_\vartheta(\tau) = t_0 \tag{21}$$

and

$$\mathrm{D}_\vartheta(\tau) = 0. \tag{22}$$

Let (τ, f) be an inverse plan. Thus we have, with probability 1, $X_\tau = x_0$. In this case, from (7) we obtain the equality

$$\mathrm{E}_\vartheta(\tau) = \frac{x_0}{\vartheta} \tag{23}$$

and from (8) the equality

$$\vartheta^2 \mathrm{D}_\vartheta(\tau) = -\vartheta^2 [\mathrm{E}_\vartheta(\tau)]^2 + \mathrm{D}_\vartheta(X_1)\mathrm{E}_\vartheta(\tau) + 2x_0\vartheta\mathrm{E}_\vartheta(\tau) - x_0^2.$$

In view of (23) the last equality yields

$$\mathrm{D}_\vartheta(\tau) = \frac{\mathrm{D}_\vartheta(X_1)x_0}{\vartheta^3}. \tag{24}$$

For an oblique plan we have, with probability 1, $X_\tau = a\tau - s$. Then from (7) we obtain the equality

$$\mathrm{E}_\vartheta(\tau) = \frac{s}{a - \vartheta} \tag{25}$$

and from (8) the equality

$$(a-\vartheta)^2 \mathrm{D}_\vartheta(\tau) = -(a-\vartheta)^2 [\mathrm{E}_\vartheta(\tau)]^2 + \mathrm{D}_\vartheta(X_1)\mathrm{E}_\vartheta(\tau) + 2s(a-\vartheta)\mathrm{E}_\vartheta(\tau) - s^2.$$

Substituting (25) into the above equality we get

$$\mathrm{D}_\vartheta(\tau) = \frac{\mathrm{D}_\vartheta(X_1)s}{(a-\vartheta)^3}. \tag{26}$$

Let us remark that in view of $0 < \mathrm{E}_\vartheta(\tau) < \infty$ for all $\vartheta \in \Theta$ it follows from (23) and (25) that for the inverse and oblique plans the conditions $\vartheta x_0 > 0$ and $(a-\vartheta)s > 0$ for all $\vartheta \in \Theta$ must hold, respectively.

From Lemma 1 and Theorem 2, taking into account the above-given moments of the first attaining times, we obtain, in a way analogous as in [12], the following

THEOREM 3. *A sequential plan* (τ, f) *satisfying* $\mathrm{E}_\vartheta(f^2) < \infty$ *and* $\mathrm{E}_\vartheta(\tau^2) < \infty$ *for all* $\vartheta \in \Theta$ *is efficient if and only if it is either a fixed-time plan or an inverse plan or an oblique plan. For any real numbers* c_1 *and* c_2*, the following functions are the only efficiently estimable ones:*

(a) *for a fixed-time plan,*

$$g(\vartheta) = c_1\vartheta + c_2,$$

and $f(\tau, X_\vartheta) = (c_1/t_0)X_{t_0} + c_2$ *is its only efficient estimator;*

(b) *for an inverse plan,*

$$g(\vartheta) = \frac{c_1}{\vartheta} + c_2,$$

and $f(\tau, X_\tau) = (c_1/x_0)\tau + c_2$ *is its only efficient estimator;*

(c) *for an oblique plan,*

$$g(\vartheta) = \frac{c_1}{a-\vartheta} + c_2,$$

and $f(\tau, X_\tau) = (c_1/s)\tau + c_2$ *is its only efficient estimator.*

We regard two estimators as being indistinguishable and equate them if they are identical with probability 1. Two functions $g_1(\vartheta)$ and $g_2(\vartheta)$ are considered to be identical if they are identical in the appropriate interval Θ of values ϑ.

5. Some characterization of sequential plans by conditions on a Markov stopping time. The following theorem might seem to be useful in determining, for the exponential class of processes, the sequential plans for which $\mathrm{D}_\vartheta(\tau)$ is minimized. It turns out, however, that this variance attains its lower bound, detemined by the inequality of Cramér-Rao type, only in the situations where these plans are efficient.

THEOREM 4. *For each sequential plan* (τ, f) *satisfying* $\mathrm{E}_\vartheta(\tau^2) < \infty$ *for all* $\vartheta \in \Theta$ *the inequality*

$$\mathrm{D}_\vartheta(\tau) \geqslant \frac{\mathrm{D}_\vartheta(X_1)[\mathrm{E}'_\vartheta(\tau)]^2}{\mathrm{E}_\vartheta(\tau)} \tag{27}$$

holds for all $\vartheta \in \Theta$. Moreover, the sequential plan (τ, f) is efficient if and only if the equality in (27) *holds for all $\vartheta \in \Theta$.*

Proof. Let us define the variable

$$Z = \frac{E'_\vartheta(\tau)}{E_\vartheta(\tau)}(X_\tau - \vartheta\tau) - [\tau - E_\vartheta(\tau)].$$

According to formula (7) we have $E_\vartheta(Z) = 0$. Consider now the variable

$$Z^2 = \left[\frac{E'_\vartheta(\tau)}{E_\vartheta(\tau)}\right]^2 (X_\tau - \vartheta\tau)^2 + [\tau - E_\vartheta(\tau)]^2 - 2\frac{E'_\vartheta(\tau)}{E_\vartheta(\tau)}(X_\tau - \vartheta\tau)[\tau - E_\vartheta(\tau)]$$

and evaluate its expectation. In view of formulas (7)-(9) and the non-negativity of variance, we obtain

$$D_\vartheta(Z) = E_\vartheta(Z^2) = -D_\vartheta(X_1)\frac{[E'_\vartheta(\tau)]^2}{E_\vartheta(\tau)} + D_\vartheta(\tau) \geqslant 0,$$

which yields (27).

Let the equality in (27) hold for any $\vartheta \in \Theta$. Then from $E_\vartheta(Z) = 0$ and $D_\vartheta(Z) = 0$ we obtain $Z = 0$ with probability 1, and thereby

$$\left[1 + \vartheta\frac{E'_\vartheta(\tau)}{E_\vartheta(\tau)}\right]\tau - \frac{E'_\vartheta(\tau)}{E_\vartheta(\tau)}X_\tau - E_\vartheta(\tau) = 0 \tag{28}$$

with probability 1. Since the coefficients of τ and X_τ in (28) cannot vanish simultaneously for any $\vartheta \in \Theta$ and $E_\vartheta(\tau) > 0$ for all $\vartheta \in \Theta$, the measure Q_τ is accumulated on a straight line in U and it follows from Lemma 1 and Theorem 3 that the plan (τ, f) is efficient.

Suppose now that the plan (τ, f) is efficient. Then, by Theorem 3, it must be either a fixed-time plan or an inverse plan or an oblique plan. In a fixed-time plan, in view of (21) and (22) we have $D_\vartheta(\tau) = E'_\vartheta(\tau) = 0$ for all $\vartheta \in \Theta$, and the equality in (27) holds. In an inverse plan, by (23) and (24) we obtain

$$D_\vartheta(\tau) = \frac{D_\vartheta(X_1)x_0}{\vartheta^3} = \frac{D_\vartheta(X_1)[E'_\vartheta(\tau)]^2}{E_\vartheta(\tau)} \quad \text{for all } \vartheta \in \Theta,$$

and the equality in (27) also holds. In an oblique plan, formulas (25) and (26) yield

$$D_\vartheta(\tau) = \frac{D_\vartheta(X_1)s}{(a-\vartheta)^3} = \frac{D_\vartheta(X_1)[E'_\vartheta(\tau)]^2}{E_\vartheta(\tau)},$$

i.e., the equality in (27) holds for all $\vartheta \in \Theta$. Thus the proof is complete.

In an interesting manner some types of first attaining plans for processes of the exponential class are determined by very subtle conditions imposed on the first attaining times for an arbitrary fixed value of the parameter ϑ. This assertion is true, for instance, for the inverse and oblique plans.

THEOREM 5. *Let for any fixed $\vartheta \in \Theta$ and some constants a and s such that $a \neq 0$ and $(a-\vartheta)s > 0$ a first attaining plan (τ, f) satisfy the conditions*

$$E_\vartheta(\tau) = \frac{s}{a-\vartheta} \tag{29}$$

and

$$E_\vartheta(\tau^2) \leqslant \frac{D_\vartheta(X_1)s+(a-\vartheta)s^2}{(a-\vartheta)^3}. \tag{30}$$

Then (τ, f) is an oblique plan.

Proof. Taking into consideration the random variable

$$Z_\tau = a\tau - X_\tau$$

we see that

$$P_\vartheta(Z_\tau = s) = P_\vartheta(X_\tau = a\tau - s).$$

Thus we must prove that $P_\vartheta(Z_\tau = s) = 1$.

By (7) and (29) we obtain

$$E_\vartheta(Z_\tau) = (a-\vartheta)E_\vartheta(\tau) = s. \tag{31}$$

Now we show that $D_\vartheta(Z_\tau) = 0$, which leads to the end of the proof. Let us evaluate at first the value

$$E_\vartheta(Z_\tau^2) = a^2 E_\vartheta(\tau^2) - 2aE_a(\tau X_\tau) + E_\vartheta(X_\tau^2). \tag{32}$$

Remark that, by virtue of (30), the first two Wald identities are valid and $E_\vartheta(Z_\tau^2) < \infty$. Taking into account (9) and (10), from (32) we obtain

$$E_\vartheta(Z_\tau^2) = (a-\vartheta)^2 E_\vartheta(\tau^2) + D_\vartheta(X_1)E_\vartheta(\tau) - 2(a-\vartheta)D_\vartheta(X_1)E'_\vartheta(\tau).$$

Substituting (29), (30), and $E'_\vartheta(\tau) = s/(a-\vartheta)^2$ into the equality above we get

$$E_\vartheta(Z_\tau^2) \leqslant s^2. \tag{33}$$

Thus, by (31) and (33), we have

$$D_\vartheta(Z_\tau) = E_\vartheta(Z_\tau^2) - [E_\vartheta(Z_\tau)]^2 \leqslant s^2 - s^2 = 0,$$

which completes the proof.

Let us remark that in the case $a = 0$ and $s = -x_0$ we obtain a theorem analogous to the above one for an inverse plan. In a particular case, for the Poisson process, it was proved in [6], p. 554.

THEOREM 6. *Let X_T be a stochastic process of the exponential class and let τ denote the first attaining time of some curve C satisfying an algebraic equation of degree k. Suppose that $\mathrm{E}_\vartheta(\tau^{2k}) < \infty$ for all $\vartheta \in \Theta$. Then the stopping set C is completely characterized by the $k(2k+1)$ values*

$$\mathrm{E}_\vartheta^{(r)}(\tau^s) \quad (s+r \leqslant 2k,\ s \geqslant 1), \tag{34}$$

given at any fixed value $\vartheta \in \Theta$, provided that all the derivatives

$$\mathrm{E}_\vartheta^{(r)}(\tau^s) = \frac{d^r}{d\vartheta^r}\mathrm{E}_\vartheta(\tau^s)$$

exist.

Proof. Let us remark that if we show that at given values (34) the equations

$$\mathrm{E}_\vartheta[F(\tau, X_\tau)] = 0 \quad \text{and} \quad \mathrm{D}_\vartheta[F(\tau, X_\tau)] = 0 \tag{35}$$

hold simultaneously for any fixed $\vartheta \in \Theta$, where $F(\tau, X_\tau)$ is a polynomial in τ and X_τ of degree k, then $F(\tau, X_\tau) = 0$ with probability 1 and the stopping set is determined by the equation $F(t, x) = 0$. In order to evaluate (35) we form the polynomials $F(\tau, X_\tau)$ and $[F(\tau, X_\tau)]^2$, the second one being of degree $2k$.

According to formula (6), for $\varphi(\tau, X_\tau; \vartheta) = \tau^s$ we obtain

$$\mathrm{E}'_\vartheta(\tau^s) = \frac{1}{\mathrm{D}_\vartheta(X_1)}\mathrm{E}_\vartheta[(X_\tau - \vartheta\tau)\tau^s]. \tag{36}$$

Thus the value $\mathrm{E}_\vartheta(X_\tau\tau^s)$ can be expressed by the values $\mathrm{E}_\vartheta(\tau^{s+1})$ and $\mathrm{E}'_\vartheta(\tau^s)$. Differentiating the last equality with respect to ϑ we have

$$\mathrm{E}''_\vartheta(\tau^s) = -\frac{\mathrm{D}'_\vartheta(X_1)}{[\mathrm{D}_\vartheta(X_1)]^2}\mathrm{E}_\vartheta[(X_\tau - \vartheta\tau)\tau^s] + \frac{1}{\mathrm{D}_\vartheta(X_1)}\mathrm{E}'_\vartheta[(X_\tau - \vartheta\tau)\tau^s]. \tag{37}$$

Moreover, assuming that $\varphi(\tau, X_\tau; \vartheta) = \tau^s(X_\tau - \vartheta\tau)$, from (6) we obtain the relation

$$\mathrm{E}'_\vartheta[(X_\tau - \vartheta\tau)\tau^s] = -\mathrm{E}_\vartheta(\tau^{s+1}) + \frac{1}{\mathrm{D}_\vartheta(X_1)}\mathrm{E}_\vartheta[(X_\tau - \vartheta\tau)^2\tau^s]. \tag{38}$$

Using (36)-(38) we can express $\mathrm{E}_\vartheta(X_\tau^2\tau^s)$ by the values $\mathrm{E}_\vartheta(\tau^{s+2})$, $\mathrm{E}_\vartheta(\tau^{s+1})$, $\mathrm{E}'_\vartheta(\tau^{s+1})$, $\mathrm{E}'_\vartheta(\tau^s)$, and $\mathrm{E}''_\vartheta(\tau^s)$. Continuing this procedure, it is easy to see that all moments $\mathrm{E}_\vartheta(X_\tau^i\tau^j)$ $(i+j \leqslant 2k, j \geqslant 1)$ can be expressed by the values $\mathrm{E}_\vartheta^{(r)}(\tau^s)$ $(s+r \leqslant 2k,\ s \geqslant 1)$. By virtue of (6) (see also its special cases (10), (12), and (14)) and the above consideration we also note that the moments $\mathrm{E}_\vartheta(X_\tau^i)$ for $i \leqslant 2k$ are determined by the values

$E_\vartheta^{(\gamma)}(\tau^s)$. Thus equations (35) are completely determined by values (34), which completes the proof.

6. Remarks on the sequential estimation for processes of the exponential class with a discrete time parameter. All notions, facts and results given in this paper can be stated for stochastic processes of the exponential class with a discrete time parameter. In this case the Markov stopping time τ takes its values in a discrete time space. The pair (τ, X_τ) is also a sufficient statistic for the parameter ϑ, and the discrete time analogue of the representation formula (4) for the two-dimensional probability distribution of (τ, X_τ) holds. Under the same assumptions as those concerning the base relation for moments (6) the analogous discrete version of this relation holds and its consequences, similarly as (7)-(14), are valid.

The process, most frequently involved in mathematical statistics, which belongs to the exponential class with a discrete time parameter is the binomial one. Some sequential estimation problems as those considered in this paper are solved for this process by de Groot in [4] and Magiera and Trybuła in [9]. Moreover, a wide discussion of sequential plans for the binomial process is contained in [6]. In paper [9] the probability distribution of the first attaining time τ for the oblique plan determined by some line of the form analogous to (20) is found and the problem of efficiency of such plans is solved. Paper [9] contains also the analogue of Theorem 5. For the case of the binomial process, the analogue of Theorem 6 is given in [6], p. 533.

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DEPT. OF MATHEMATICS
TECHNICAL UNIVERSITY
DDR-60 DRESDEN
GERMAN DEMOCRATIC REPUBLIK

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY
50-370 WROCŁAW

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J. FRANZ (Dresden) i R. MAGIERA (Wrocław)

PLANY SEKWENCYJNE DLA WYKŁADNICZEJ KLASY PROCESÓW

STRESZCZENIE

W pracy rozpatruje się problem estymacji sekwencyjnej i charakteryzacji planów sekwencyjnych dla procesów stochastycznych. W pierwszej części pracy, dla wykładniczej rodziny procesów wyznaczono klasę wszystkich efektywnych planów sekwencyjnych oraz określono postać efektywnych estymatorów i funkcji efektywnie estymowalnych. W drugiej części pracy, dla rozpatrywanej wykładniczej rodziny procesów podano charakteryzację planów sekwencyjnych poprzez warunki nałożone na momenty markowskich chwil zatrzymania. Pokazano, że wariancja chwili markowskiej τ osiąga swoje ograniczenie od dołu, wyznaczone przez nierówność typu Rao-Craméra, jedynie wtedy, gdy plany odpowiadające τ są efektywne. Na podstawie podanych w pracy tożsamości Walda wyższych rzędów dla wykładniczej rodziny procesów udowodniono, że plany sekwencyjne, określone przez chwile pierwszego osiągnięcia krzywej spełniającej równanie algebraiczne k-tego stopnia, są całkowicie określone przez $k(2k+1)$ wartości oczekiwanych i ich pochodnych $E_\vartheta^{(r)}(\tau^s)$ $(s+r \leqslant 2k,\ s \geqslant 1)$ dla pewnej ustalonej wartości parametru jednowymiarowych rozkładów procesu.
