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ON TWO THEOREMS OF BARTOSZEWICZ

1. Let $X_1, \ldots, X_n$ be independent identically distributed random variables with the probability distribution function

$$(*)\qquad F(x) = \begin{cases} 1-e^{-\lambda x} & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Let $T > 0$ be fixed and let D denote the number of those X_i for which $X_i < T$. Let

$$S = \sum_{i=1}^{D} X_{(i)} + (n-D)T,$$

where $X_{(1)} \leqslant \ldots \leqslant X_{(n)}$ are the rearrangement of the series $X_1, \ldots, X_n$ in the ascending order.

Bartoszewicz has proved the following theorem ([1], Theorem 3):

THEOREM 1. *The statistic* (D, S) *is not complete except for the trivial case* $n = 1$.

The original proof uses the method of Laplace transforms. The following proof is a probabilistic one.

Proof. Let $n \geqslant 2$ and let

$$\varphi(D, S) = \mathrm{P}(X_1 > T/2, X_2 > T/2 \mid D, S) - \mathrm{P}(X_1 > T \mid D, S).$$

Then

$$\mathrm{E}[\varphi(D, S)] = \mathrm{P}(X_1 > T/2, X_2 > T/2) - \mathrm{P}(X_1 > T) = (e^{-\lambda T/2})^2 - e^{-\lambda T} = 0.$$

On the other hand, if $S > (2n-1)T/2$, then

$$\varphi(n, S) = 1 - 0 = 1,$$

since $A = [D = n, S > (2n-1)T/2]$ implies $T/2 < X_i < T$ $(i = 1, \ldots, n)$.

Now

$$\mathrm{P}(A) > \mathrm{P}[(2n-1)T/2n < X_i < T, i = 1, \ldots, n] > 0,$$

which means that $\varphi(D, S)$ differs from zero with positive probability and has the expectation equal to zero. It follows that the statistic (D, S) is not complete.

In the case $n = 1$ the completeness follows simply from that of X_1, since (D, S) is then only a function of X_1. For the completeness of the distribution of X_1, see, e.g., [4], p. 132.

2. The proof can easily be modified to cover the two-parameter case [2].

Let now $X_1, \ldots, X_n$ be independent random variables with the common distribution function

$$F_1(x) = \begin{cases} 1-e^{-\lambda(x-a)} & \text{if } x \geqslant a, \\ 0 & \text{if } x < a. \end{cases}$$

Let $T > a$. Otherwise, we use the notation of Section 1.

In [2] Bartoszewicz has proved the following theorem ([2], Theorem 3):

THEOREM 2. *The statistic* $(X_{(1)}, D, S)$ *is not complete except for the trivial cases* $n = 1, 2$.

Here we give the following alternative proof.

Proof. It is known (see [3], Lemma 1) that $(X_{(2)}-X_{(1)}, \ldots, X_{(n)}-X_{(1)})$ is independent of $X_{(1)}$ and its distribution agrees with that of the set of ordered sample elements of a sample of size $n-1$ coming from the distribution $F(x)$ (cf. $(*)$). Rearranging the variables $X_{(2)}-X_{(1)}, \ldots, X_{(n)}-X_{(1)}$ in a random order, we obtain the independent random variables $Y_1, \ldots, Y_{n-1}$ which have the common distribution $F(x)$ and are totally independent of $X_{(1)}$.

Let $n \geqslant 3$. Now we put

$$\varphi_1(X_{(1)}, D, S) = \mathrm{P}[Y_1 > (T-X_{(1)})/2,\ Y_2 > (T-X_{(1)})/2 \mid X_{(1)}, D, S] - \mathrm{P}(Y_1 > T-X_{(1)} \mid X_{(1)}, D, S).$$

We obtain

$$\mathrm{E}[\varphi_1(X_{(1)}, D, S)] = \mathrm{E}\{\mathrm{E}[\varphi_1(X_{(1)}, D, S) \mid X_{(1)}]\} = 0$$

by using the same argument as before.

Now, let $X_{(1)} < T$ and $S > [3X_{(1)}+(2n-3)T]/2$. Then we again obtain $\varphi_1(X_{(1)}, n, S) = 1$. Since

$$\mathrm{P}\{X_{(1)} < T, D = n, S > [3X_{(1)}+(2n-3)T]/2\}$$
$$> \mathrm{P}\{X_1 < (a+T)/2, [a+(4n-1)T]/4n < X_i < T, i = 2, \ldots, n\} > 0,$$

we conclude in the same way as before that the statistic $(X_{(1)}, D, S)$ is not complete.

The exceptional cases can be dealt with similarly as in Section 1. If $n = 2$, the statistic $(X_{(1)}, X_1 + X_2)$ is complete according to Theorem 3 of [3].

References

[1] J. Bartoszewicz, *Estimation of reliability in the exponential case* (*I*), Zastosow. Matem. 14 (1974), p. 185-194.

[2] — *Estimation of reliability in the exponential case* (*II*), ibidem 14 (1975), p. 513-528.

[3] B. Epstein and M. Sobel, *Some theorems relevant to life testing from an exponential distribution*, Ann. Math. Statist. 25 (1954), p. 373-381.

[4] E. L. Lehmann, *Testing statistical hypotheses*, New York 1959.

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O DWÓCH TWIERDZENIACH BARTOSZEWICZA

STRESZCZENIE

W [1] i [2] Bartoszewicz udowodnił, że przy estymacji wykładniczej niezawodności z planem badania bez odnowy i czasem badania trwającym do ustalonej chwili, otrzymane statystyki dostatecznie nie są kompletne. W nocie podane są probabilistyczne dowody tych twierdzeń.