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A NOTE ON AN EXTENSION OF A THEOREM OF MOROZENSKIĬ

1. Introduction. Morozenskiĭ proved the following theorem:

THEOREM (Morozenskiĭ [4]). *Let $X_1, X_2, \ldots, X_n$ $(n \geqslant 3)$ be independent and identically distributed random variables with a continuous distribution function $F(x-\theta)$ satisfying the condition*

$$\int x^2 dF(x) < \infty.$$

If, for any given $a \in (0, 1)$, there exists a uniformly most powerful (UMP) test of size a for testing H_0: $\theta = 0$ against H_1: $\theta > 0$ with a critical region of the form $\{(x_1, x_2, \ldots, x_n): \bar{x} > c(a)\}$, then F is a distribution function of a normal law.

In this paper we deduce this theorem from some well-known theorems of Bahadur [1] and Halmos and Savage [2]. Moreover, we obtain other characterizations of the normal and gamma distributions.

2. Preliminaries. In this section we recall known results which we use further.

Let $\mathfrak{X}$ be an arbitrary set and let $\mathscr{A}$ be a σ-algebra of subsets of $\mathscr{A}$. Let $\mathscr{P} = \{P_\theta: \theta \in \Omega\}$ be a family of distributions on $\mathscr{A}$. Finally, let the problem of testing a simple hypotesis H_0: $\theta = \theta_0$, $\theta_0 \in \Omega$, against a simple alternative H_1: $\theta = \theta_1$, $\theta_1 \in \Omega \setminus \{\theta_0\}$, be denoted by $T(\theta_0, \theta_1)$.

THEOREM 1 (Bahadur [1]). *If a family $R(\theta_0, \theta_1)$ of randomized (non-randomized) tests based on a statistic $T(X)$ forms an essentially complete class for the problem $T(\theta_0, \theta_1)$ in the set of randomized (non-randomized) tests, then $T(X)$ is a sufficient statistic for $\{P_{\theta_0}, P_{\theta_1}\}$.*

Remark. Theorem 1 is a particular case of a theorem proved in [1].

THEOREM 2 (Halmos and Savage [2]). *A necessary and sufficient condition that $T(X)$ be sufficient for a dominated set $\mathscr{P}$ of distributions on $\mathscr{A}$ is that $T(X)$ be sufficient for every $\{P_{\theta_0}, P_{\theta_1}\}$, where $\theta_0, \theta_1 \in \Omega$.*

THEOREM 3 (Kelker and Matthes [3]). (a) *Let $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ be independent non-degenerate random variables with distribution functions*

$F_1(x-\theta), \ldots, F_n(x-\theta)$, $-\infty < \theta < \infty$, *respectively. If* $\sum_{i=1}^{n} b_i X_i$, *where* $b_i \neq 0$, *is a sufficient statistic for* θ, *then every* X_i *is a normal variable.*

(b) *Let* $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ *be independent non-degenerate and positive random variables with distribution functions* $F_1(x/\sigma), F_2(x/\sigma), \ldots, F_n(x/\sigma)$, $\sigma > 0$, *respectively. If* $\sum_{i=1}^{n} b_i X_i$, *where* $b_i > 0$, *is a sufficient statistic, then every* X_i *has a gamma distribution.*

(c) *Let* $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ *be independent random variables with distribution functions* $F_1(x/\sigma), F_2(x/\sigma), \ldots, F_n(x/\sigma)$, $\sigma > 0$, *respectively, and let each* F_i *be absolutely continuous with respect to Lebesgue measures in a neighbourhood of the origin. Further, let at* $x = 0$ *the functions* F_i' *be non-zero and continuous. If the statistic* $\sum_{i=1}^{n} X_i^2$ *is sufficient for* σ, *then every* X_i *is a normal variable.*

Remark. Part (b) of Theorem 3 is not explicitly stated in [3], but it can be easily proved by using the Kelker-Matthes method.

THEOREM 4 (Neyman). *The following conditions are equivalent:*

(a) *for every* $\theta_0 \in \Omega$ *and every* $\alpha \in (0, 1)$ *in the class of non-randomized tests for testing* H_0: $\theta = \theta_0$ *against* H_1: $\theta \in K(\theta_0)$, $\theta_0 \notin K(\theta_0)$, *there exists a UMP-test of size* α *based on a statistic* $T(X)$;

(b) *for every* $\alpha \in (0, 1)$, *there exists a family of confidence sets* $S_\alpha(T(X)$ *at the confidence level* $1-\alpha$ *based on the statistic* $T(X)$ *which minimizes the probability* $P_\theta(\theta_0 \in \tilde{S}_\alpha(X))$ *for all* $\theta \in K(\theta_0)$ *among all level* $1-\alpha$ *families of) confidence sets* $\tilde{S}_\alpha(X)$.

3. Results. Now, we deduce some corollaries to the theorems stated in the preceding section. The first among them extends the Morozenskiĭ theorem [4].

COROLLARY 1. *Let* $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ *be independent non-degenerate random variables with distribution functions* $F_1(x-\theta), F_2(x-\theta), \ldots, F_n(x-\theta)$ $(-\infty < \theta < +\infty)$, *respectively, and let the family of distributions of* $(X_1, X_2, \ldots, X_n)$ *be dominated. If, for every pair* (θ_0, θ_1), *there exists a family of tests* $R(\theta_0, \theta_1)$ *based on the statistic* $\sum_{i=1}^{n} b_i X_i$ $(b_i \neq 0)$ *which forms an essentially complete class for the problem* $T(\theta_0, \theta_1)$, *then each* X_i *is a normal variable.*

Proof. From Theorem 1 we conclude that $\sum b_i X_i$ is sufficient for every pair $\{P_{\theta_0}, P_{\theta_1}\}$. Therefore, and from Theorem 2, the statistic $\sum_{i=1}^{n} b_i X_i$ is sufficient. Finally, applying Theorem 3a, we see that each X_i is a normal variable.

Corollary 2. *Let $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ be independent non-degenerate random variables with distribution functions $F_1(x-\theta), F_2(x-\theta), \ldots, F_n(x-\theta)$, $-\infty < \theta < \infty$, respectively, and let the family of distributions of $(X_1, \ldots, X_n)$ be dominated. Further, let $S_a\big(\sum_{i=1}^{n} b_i X_i\big)$, where $b_i \neq 0$, be a family of confidence sets at the confidence level $1-a$. If, for all $\theta > \theta_0$ and every a, the family S_a minimizes the probability $\mathrm{P}_\theta\big(\theta_0 \epsilon S_a(X_1, \ldots, X_n)\big)$ among all level $1-a$ families of confidence sets $S_a(X_1, X_2, \ldots, X_n)$, then each X_i is a normal variable.*

Proof. According to Theorem 4, an optimal family of confidence sets at every confidence level $1-a$ exists if and only if there exists, for every $a \epsilon (0,1)$ and every $\theta_0 \epsilon \Omega$, a UMP-test in the class of non-randomized tests at the level a for testing H_0: $\theta = \theta_0$ against H_1: $\theta > \theta_0$. Because the optimal families of confidence sets depend only on $\sum b_i X_i$, the UMP-tests also depend only on $\sum b_i X_i$. Hence, for every problem $T(\theta_0, \theta_1)$, there exists a family $R(\theta_0, \theta_1)$ of tests dependent on $\sum b_i X_i$ which forms an essentially complete class in the set of non-randomized tests. Now, it follows from Theorem 1 that $\sum b_i X_i$ is pairwise sufficient. Since the family of distributions of $(X_1, X_2, \ldots, X_n)$ is dominated, Theorem 2 implies that $\sum b_i X_i$ is sufficient. Thus, in view of Theorem 3a, each X_i is a normal variable.

The following corollaries can easily be proved in an analogous way:

Corollary 3. *Let $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ be independent non-degenerate positive random variables with distribution functions $F_1(x/\sigma), F_2(x/\sigma), \ldots, F_n(x/\sigma)$, $\sigma > 0$, respectively, and let the family of distributions of $(X_1, X_2, \ldots, X_n)$ be dominated. If, for every pair (σ_0, σ_1), there exists a family of tests $R(\sigma_0, \sigma_1)$ based on the statistic $\sum_{i=1}^{n} b_i X_i$ $(b_i > 0)$ which forms an essentially complete class for $T(\sigma_0, \sigma_1)$, then each X_i has a gamma distribution.*

Corollary 4. *Let $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ be independent non-degenerate positive random variables with distribution functions $F_1(x/\sigma), \ldots, F_n(x/\sigma)$, $\sigma > 0$, respectively, and let the family of distributions of $(X_1, \ldots, X_n)$ be dominated. Further, let $S_a(\sum b_i X_i)$ $(b_i > 0)$ be a family of confidence sets at the confidence level $1-a$. If, for every $a \epsilon (0,1)$ and all $\sigma > \sigma_0$, the family S_a minimizes the probability $\mathrm{P}_\sigma\big(\sigma_0 \epsilon S_a(X_1, \ldots, X_n)\big)$ among all level $1-a$ families of confidence sets $S_a(X_1, \ldots, X_n)$, then X_i $(i = 1, \ldots, n)$ have gamma distributions.*

Corollary 5. *Let $X_1, \ldots, X_n$ $(n \geqslant 2)$ be independent random variables with distribution functions $F_1(x/\sigma), \ldots, F_n(x/\sigma)$, $\sigma > 0$, respectively, satisfying the assumptions of Theorem 3c. Further, let the family of distribu-*

tions of $(X_1, \ldots, X_n)$ be dominated. If, for every pair (σ_0, σ_1), there exists a family of tests $R(\sigma_0, \sigma_1)$ based on statistic $\sum X_i^2$ which forms an essentially complete class for the problem $T(\sigma_0, \sigma_1)$, then each X_i is a normal variable.

COROLLARY 6. *Let $X_1, \ldots, X_n$ $(n \geqslant 2)$ be independent random variables with distributions functions $F_1(x/\sigma), \ldots, F_n(x/\sigma)$, $\sigma > 0$, respectively, satisfying the assumptions of Theorem* 3c *and let the family of distributions of $(X_1, \ldots, X_n)$ be dominated. Further, let $S_a\left(\sum_{i=1}^{n} X_i^2\right)$ be a family of confidence sets at the confidence level $1-a$. If, for every $a \in (0, 1)$ and all $\sigma > \sigma_0$, the family $S_a\left(\sum X_i^2\right)$ minimizes the probability $\mathrm{P}_\sigma\left(\sigma_0 \in S_a(X_1, \ldots, X_n)\right)$ among all level $1-a$ families of confidence sets $S_a(X_1, \ldots, X_n)$, then each X_i is a normal variable.*

References

[1] R. R. Bahadur, *A characterization of sufficiency*, Ann. Math. Statist. 26 (1955), p. 286-293.

[2] P. Halmos and L. Savage, *Application of the Random-Nikodym theorem to the theory of sufficient statistics*, ibidem 20 (1949), p. 225-241.

[3] D. Kelker and T. K. Matthes, *A sufficient statistics characterization of the normal distribution*, ibidem 41 (1970), p. 1086-1090.

[4] L. Yu. Morozenskiĭ (Л. Ю. Морозенский), *Характеризация нормального закона свойством оптимальности критерия основанного на выборочном среднем*, Вестник Ленингр. ун-та 13 (1971), p. 61-63.

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O PEWNYM UOGÓLNIENIU TWIERDZENIA MOROZIENSKIEGO

STRESZCZENIE

W pracy [4] Morozienskij udowodnił, że pewne optymalne własności testów charakteryzują rozkład normalny. W obecnej pracy udowodniono twierdzenie Morozienskiego dla słabszych założeń. Poza tym otrzymano inne twierdzenia, charakteryzujące rozkłady normalny i gamma przez optymalne własności testów oraz rodzin zbiorów ufności.