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T. RADZIK and K. ORŁOWSKI (Wrocław)

A MIXED GAME OF TIMING: INVESTIGATION OF STRATEGIES

0. Introduction. The general model of the game of timing can be described in the following way. Each of two opponents, denoted by A and B, has a fixed number of actions to be taken in the time interval $\langle 0, 1\rangle$. In consequence of taking the actions the player may achieve a success. If the player takes one of his actions at moment t, $t \in \langle 0, 1\rangle$, then he achieves success with a certain probability. The probability is defined by the function $P(t)$ for player A and by the function $Q(t)$ for player B. The functions $P(t)$ and $Q(t)$ are called *accuracy functions*.

We assume that

(i) $P(t)$ and $Q(t)$ are continuous and increasing in $\langle 0, 1\rangle$,

(ii) $P(t)$ and $Q(t)$ have continuous derivatives in $(0, 1)$,

(iii) $P(0) = Q(0) = 0$ and $P(1) = Q(1) = 1$.

Furthermore, we assume that each player obtains one unit when he achieves success alone, loses one unit when his opponent succeeds alone, and in all other cases he obtains zero. Also, each player tends to maximize his payoff. The game is over at the moment when at least one of the players succeeded.

There are two types of actions, noisy and silent. If one of the players takes his noisy action and does not succeed, then his opponent receives this information. In the case of a silent action this information is not imparted. In this connexion we say that an action is of *type* w, where $w = c$ (silent) or $w = g$ (noisy). The order in which the silent and noisy actions are taken is fixed before the game. All the above assumptions are known for both players.

Let us denote by $\Gamma_{k_1,k_2,\ldots,k_r|w_1;l_1,l_2,\ldots,l_s|w_2}$ the above-described game in which player A takes successively k_r actions of type w_1, k_{r-1} actions of type $\overline{w}_1$ ($\overline{w}_1$ denotes the reverse type to w_1), k_{r-2} actions of type w_1, and so on. At the very end he takes k_1 actions of suitable type. Similarly, taking the actions by player B is described by $l_1, l_2, \ldots, l_s|w_2$.

Among the more important games of timing with more than one action taken by at least one player the following cases have been solved up to now:

(i) $\Gamma_{k_1|c;l_1|c}$ by Restrepo [3] in 1957,

(ii) $\Gamma_{k_1|g;l_1|g}$ by Fox and Kimeldorf [1] in 1969,

(iii) $\Gamma_{1,1|c;1|g}$ by Smith [4] in 1967,

(iv) $\Gamma_{k_1|c;1|g}$ by Styszyński [5] in 1974.

The problem of the general mixed game of timing, for instance of $\Gamma_{k_1|c;l_1|g}$ ($k_1 \geqslant 1, l_1 \geqslant 2$), is still open.

The authors have found the solution of the game $\Gamma_{k_1,k_2,\ldots,k_r|w;1|g}$. However, considering its extensiveness, this paper contains only the construction of the normal form of the game and the formulation of a theorem on optimal strategies. The proof of optimality is given in paper [2].

1. Notation and definitions.

$\bar{k}_r|w = (k_1, \ldots, k_r|w)$, where $r, k_1, \ldots, k_r$ are natural numbers and w is the type of action, describes the order of taking silent and noisy actions by player A who according to $\bar{k}_r(w)$ takes successively k_r actions of type w, k_{r-1} actions of type $\bar{w}$, k_{r-2} actions of type w, and so on.

$$\bar{k}_{r,s}|w = \begin{cases} (k_1, \ldots, k_{r-1}, k_r - s|w) & \text{if } 0 \leqslant s < k_r, \\ (k_1, \ldots, k_i, k_{i+1} + \ldots + k_r - s|\tilde{w}) & \\ \qquad \text{if } k_{i+2} + \ldots + k_r \leqslant s < k_{i+1} + \ldots + k_r & (0 \leqslant i \leqslant r-2), \end{cases}$$

where

$$\tilde{w} = \begin{cases} w & \text{if } r-i \text{ is odd}, \\ \bar{w} & \text{if } r-i \text{ is even}. \end{cases}$$

The meaning of the symbol $k_{r,s}|w$ is as follows: if player A takes his actions according to $\bar{k}_r|w$, then after taking his s first ones, he takes the next actions according to $k_{r,s}|w$.

$\|\bar{k}_r|w\| = k_1 + \ldots + k_r$.

$|\bar{k}_r|w|$ — the sum of those components of the vector $\bar{k}_r|w$ which denote the numbers of noisy actions.

$\bar{x}_n = (x_1, \ldots, x_n)$.

$\bar{x}_{n,s}$ — the vector which is obtained from the vector $\bar{x}_n$ by omitting its s first components.

$\Gamma_{\bar{k}_r|w}$ — the game $\Gamma_{k_1,\ldots,k_r|w;1|g}$ described in the introduction (player B has only one noisy action).

$\bar{X}_n = \{\bar{x}_n \mid 0 \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant 1\}$.

$\bar{Y} = \{y \mid 0 \leqslant y \leqslant 1\}$.

x_i — the moment of taking by player A his i-th action in the game $\Gamma_{\bar{k}_r|w}$ ($1 \leqslant i \leqslant \|\bar{k}_r|w\|$).

w_n — the type of the last action taken by A in the game $\Gamma_{\bar{k}_r|w}$, where $\|\bar{k}_r|w\| = n$.

y — the moment of taking an action by player B in the game $\Gamma_{\bar{k}_r|w}$.

D_x — the probability measure with total mass placed at the point x.

$H_{\langle a,b\rangle}$ — an arbitrary continuous probability measure with support $\langle a, b\rangle$.

$\mathscr{B}(X)$ — the σ-algebra of Borel sets of the space X.

To simplify the notation we use, if no confusion is possible, the following substitutions:

$$\bar{k}_r | w = \lambda, \quad \bar{k}_{r,1} | w = \lambda_1,$$

$$\bar{k}_r | g = \tau, \quad \bar{k}_{r,1} | g = \tau_1,$$

$$\bar{k}_r | c = \pi, \quad \bar{k}_{r,i} | c = \pi_i.$$

To make the paper more legible we write the numbers of used formulas above the symbols of relations. For instance, $\overset{(11)(20)}{>}$ means that we use formulas (11) and (20) to conclude the inequality in question.

2. Spaces of player strategies. In this section we define the sets of strategies for players and give their interpretation.

Definition 1. Every probability measure defined on $\mathscr{B}(\bar{X}_{\|\lambda\|})$ is called a *strategy of player A* in the game Γ_λ.

The set of all these strategies is denoted by $A_{\|\lambda\|}$.

Definition 2. By a *strategy of player B* in the game Γ_λ we mean an expression of the form

$$S_m = [G^m, \{G^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{G^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}],$$

where $m = |\lambda|$, the system $\nu_1, \ldots, \nu_m$ specifies the moments of taking noisy actions by player A, G^m is a fixed probability measure defined on $\mathscr{B}(\langle 0, 1\rangle)$, and $\{G^{m-i}_{\nu_1,\ldots,\nu_i}\}_{\nu_1,\ldots,\nu_i}$ $(i = 1, \ldots, m)$ is a fixed family of probability measures defined on $\mathscr{B}(\langle \nu_i, 1\rangle)$ for every system $\nu_1, \ldots, \nu_i$ such that $0 \leqslant \nu_1 \leqslant \ldots \leqslant \nu_i < 1$.

The set of all these strategies S_m is denoted by $B_{|\lambda|}$.

We interpret the strategy $F \in A_{\|\lambda\|}$ in the following way. Player A chooses a vector $\bar{x}_n$ $(n = \|\lambda\|)$ according to the measure F and takes his actions successively at the moments $x_1, \ldots, x_n$ under the condition $x_n \leqslant y$, i.e. if player B takes his action not later than at the moment of taking by player A his last action. Since we seek the optimal strategies, we assume that if $y < x_n$ and player A was not defeated at the moment y, then he takes his last action at $t = 1$ because the probability of his success equals 1.

Now we interpret the strategy $S_m \in B_{|\lambda|}$. Using the strategy S_m player B chooses before the game a moment y according to the measure G^m and takes his action at y if $y \leqslant \nu_1$ (ν_1 — the moment of taking by player A his first noisy action), whereas if $\nu_1 < y$, then player B does not take his action at y but he chooses a new moment y' according to the measure

$G_{\nu_1}^{m-1}$ and he behaves as before, and so on. We assume additionally that if player B takes no action until the moment ν_m and $w_n = g$ (the last action of player A is of type g), then he takes it at the moment 1.

Let us introduce for every $S_m \in B_m$ the following strategies:

$$s_m(y) = [D_y, \{G_{\nu_1}^{m-1}\}_{\nu_1}, \ldots, \{G_{\nu_1,\ldots,\nu_m}^{0}\}_{\nu_1,\ldots,\nu_m}], \tag{1}$$

$$S_{m-1}(\nu_1) = [G_{\nu_1}^{m-1}, \{G_{\nu_1,\nu_2}^{m-2}\}_{\nu_2}, \ldots, \{G_{\nu_1,\ldots,\nu_m}^{0}\}_{\nu_2,\ldots,\nu_m}]. \tag{2}$$

Obviously, from this definition it follows that if $S_m \in B_m$, then $s_m(y) \in B_m$ for any $y \in \langle 0, 1\rangle$ and $S_{m-1}(\nu_1) \in B_{m-1}$ for any $\nu_1 \in \langle 0, 1)$.

Remark. We introduce an additional simplification for the notation of certain strategies for players A and B. Namely, by $\bar{x}_n$ we understand either the vector of the moments of taking by A his actions or the strategy from the set A_n being a measure with total mass placed at the point $\bar{x}_n$. Similarly, by y we understand either the moment at which player B takes his action or such a strategy of player B in the game Γ_λ which prescribes him in the cases $w_n = c$ or $(w_n = g, y \leqslant x_n)$ taking the action at the moment y with probability 1 independently of the information obtained about the noisy actions of player A during the game, and in the case $(w_n = g, y > x_n)$ — taking the action at the moment 1.

Let us take into account the strategy S_m of player B in the game Γ_λ $(m = |\lambda|)$ and the corresponding families of strategies $\{s_m(y)\}_y$ and $\{S_{m-1}(\nu_1)\}_{\nu_1}$ given by (1) and (2). It is easy to see that every pair $[G^m, \{S_{m-1}(\nu_1)\}_{\nu_1}]$, $[G^m, \{s_m(y)\}_y]$ determines uniquely the strategy S_m. Therefore, we shall also adopt the notation

$$S_m = [G^m, \{S_{m-1}(\nu_1)\}_{\nu_1}] = [G^m, \{s_m(y)\}_y].$$

3. Payoff function and the normal form of the game. Assume that $K(\bar{x}_n; S_m|\lambda)$ be a function defined recursively on $\bar{X}_n \times B_m$ $(n = \|\lambda\|$, $m = |\lambda|)$ with the help of (1) and (2) in the following way:

$$K(\bar{x}_n;\ S_m|\lambda) = \int\limits_{Y} K\big(\bar{x}_n;\ s_m(y)|\lambda\big)dG^m(y), \tag{3}$$

$$K\big(\bar{x}_n;\ s_m(y)|\lambda\big) \tag{4}$$

$$= \begin{cases} P(x_1)+[1-P(x_1)]K\big(\bar{x}_{n,1};\ s_m(y)|\lambda_1\big) & \text{if } x_1 < y,\ w = c, \\ P(x_1)+[1-P(x_1)]K\big(\bar{x}_{n,1};\ S_{m-1}(x_1)|\lambda_1\big) & \text{if } x_1 < y,\ w = g, \\ 1-2Q(y) & \text{if } y < x_1, \\ \{1-[1-P(x_1)]^n\}[1-Q(x_1)]-[1-P(x_1)]^nQ(x_1) & \\ \qquad\qquad \text{if } x_1 = \ldots = x_n = y, & \\ 1-Q(x_1)-[1-P(x_1)]^iQ(x_1) & \\ \qquad\qquad \text{if } x_1 = \ldots = x_i = y < x_{i+1} \quad (1 \leqslant i < n), & \end{cases}$$

where

$$K(\bar{x}_{n,n};\ S_0(x_n)\,|\,\bar{k}_{r,n}|w) = -1, \qquad K(\bar{x}_{n,n};\ s_0(y)\,|\,\bar{k}_{r,n}|w) = -Q(y).$$

The function $K(\bar{x}_n;\ S_m|\lambda)$ determines the payoff for player A in the case where he adopts the strategy $\bar{x}_n \in A_{\|\lambda\|}$ and player B uses the strategy $S_m \in B_{|\lambda|}$.

Definition 3. For every λ ($\|\lambda\| = n$, $|\lambda| = m$) a strategy $S_m \in B_m$ *belongs to the class* $B_m^0(\lambda)$ if $K(\bar{x}_n;\ S_m|\lambda)$ is a measurable function of the variable $\bar{x}_n$ with respect to $\mathscr{B}(\bar{X}_n)$. We say that $B_m^0(\lambda)$ is the *class of admissible strategies* for player B in the game Γ_λ.

In the paper we omit λ in the symbol $B_m^0(\lambda)$ when this does not lead to misunderstanding.

By the construction of the payoff for player A it can be seen that the function $K(\bar{x}_n;\ S_m|\lambda)$ may not be measurable with respect to $\mathscr{B}(\bar{X}_n)$ (it is easy to give an example for $n = 2$) and the expression $\int K(\bar{x}_n;\ S_m|\lambda)\,dF$ need not exist. To avoid this we restrict the set of strategies of player B to admissible strategies. We investigate this class later, in Lemmas 5-8.

Definition 4. A function defined on $A_n \times B_m^0(\lambda)$ ($\|\lambda\| = n$, $|\lambda| = m$) by the formula

$$K(F;\ S_m|\lambda) = \int\limits_{\bar{X}_n} K(\bar{x}_n;\ S_m|\lambda)\,dF(\bar{x}_n) \tag{5}$$

is called the *payoff function* in the game Γ_λ.

Definition 5. The triplet $\langle A_{\|\lambda\|}, B_{|\lambda|}^0, K\rangle$, where $A_{\|\lambda\|}$, $B_{|\lambda|}^0$ and K are given by Definitions 1, 3 and 4, respectively, is called the *normal form of the game* Γ_λ.

4. Properties of the payoff function. In this section we construct some auxiliary strategies for player B and investigate the payoff function with respect to them. At first we give two simple lemmas.

LEMMA 1. *For any* λ *the following relation holds*:

$$K(\bar{x}_n;\ y|\lambda) \tag{6}$$
$$= \begin{cases} P(x_1)+[1-P(x_1)]K(\bar{x}_{n,1};\ y|\lambda_1) & \text{if } x_1 < y, \\ 1-2Q(y) & \text{if } y < x_1, \\ \{1-[1-P(x_1)]^n\}[1-Q(x_1)]-[1-P(x_1)]^n Q(x_1) & \text{if } x_1 = \ldots = x_n = y, \\ 1-Q(x_1)-[1-P(x_1)]^i Q(x_1) & \text{if } x_1 = \ldots = x_i = y < x_{i+1} \\ & (1 \leqslant i < n), \end{cases}$$

where

$$K(\bar{x}_{n,n};\ y\,|\,\bar{k}_{r,n}|w) = \begin{cases} -1 & \text{if } w_n = g, \\ -Q(y) & \text{if } w_n = c. \end{cases}$$

The lemma follows immediately from relations (3), (4) and from the Remark given at the end of Section 2.

LEMMA 2. *For any λ ($\|\lambda\| = n$, $|\lambda| = m$) and for any $(F, S) \in A_n \times B_m^0(\lambda)$ the payoff function satisfies*

$$-1 \leqslant K(F;\ S\,|\,\lambda) \leqslant 1.$$

This lemma is a simple conclusion of (3) and (4).

For an arbitrary probability measure T defined on $\mathscr{B}(\langle 0, 1\rangle)$ we let

$$S_m^T = [T^m, \{T_{\nu_1}^{m-1}\}_{\nu_1}, \ldots, \{T_{\nu_1,\ldots,\nu_m}^0\}_{\nu_1,\ldots,\nu_m}] \tag{7}$$

be a strategy of player B from the set B_m such that

$$T_{\nu_1,\ldots,\nu_i}^{m-i}(V) = \begin{cases} T(V) & \text{if } i = 0, \\ \dfrac{T_{\nu_1,\ldots,\nu_{i-1}}^{m-i+1}(V)}{T_{\nu_1,\ldots,\nu_{i-1}}^{m-i+1}(\nu_i, 1\rangle} & \text{if } i \geqslant 1 \text{ and } T_{\nu_1,\ldots,\nu_{i-1}}^{m-i+1}(\nu_i, 1\rangle \neq 0, \\ \dfrac{L(V)}{1-\nu_i} & \text{if } i \geqslant 1 \text{ and } T_{\nu_1,\ldots,\nu_{i-1}}^{m-i+1}(\nu_i, 1\rangle = 0 \end{cases} \tag{8}$$

for any $V \in \mathscr{B}\big((\nu_i, 1\rangle\big)$, $i = 0, 1, \ldots, m$, where $L(V)$ is the Lebesgue measure of the set V. We omit the symbols $\nu_1, \ldots, \nu_k$ if $k = 0$.

LEMMA 3. *For any λ ($\|\lambda\| = n$, $|\lambda| = m$), $\bar{x}_n \in \bar{X}_n$, and for an arbitrary probability measure T on $\mathscr{B}(\langle 0, 1\rangle)$ we have*

$$K(\bar{x}_n;\ S_m^T|\lambda) = \int K(\bar{x}_n;\ y\,|\,\lambda)\,dT(y). \tag{9}$$

Proof. For the proof we use induction with respect to the number $n = \|\lambda\|$.

Step 1. Let $n = 1$. In this case it is easy to prove the lemma.

Step 2. Fix $n \geqslant 1$ and assume that for any $(\bar{k}_r|w) = \lambda'$ such that $\|\lambda'\| = n$, for any $\bar{x}_n \in \bar{X}_n$ and for an arbitrary probability measure T on $\mathscr{B}(\langle 0, 1\rangle)$ equality (9) holds for $\lambda = \lambda'$.

Step 3. Fix in an arbitrary way a vector $(\bar{k}_r|w) = \lambda$ such that $\|\lambda\| = n+1$, let S_m^T ($|\lambda| = m$) be given by (7), (8), and let $\bar{x}_{n+1} \in \bar{X}_{n+1}$. We show, under the inductive hypothesis, that relation (9) holds for the vector λ fixed before.

Let us put

$$s_m^T(y) = [D_y, \{T_{\nu_1}^{m-1}\}_{\nu_1}, \ldots, \{T_{\nu_1,\ldots,\nu_m}^0\}_{\nu_1,\ldots,\nu_m}],$$

$$S_{m-1}^T(\nu_1) = [T_{\nu_1}^{m-1}, \{T_{\nu_1,\nu_2}^{m-2}\}_{\nu_2}, \ldots, \{T_{\nu_1,\ldots,\nu_m}^0\}_{\nu_2,\ldots,\nu_m}].$$

Because of (7) and (8), the strategy $S_{m-1}^{T_{\nu_1}^{m-1}}$ can be represented in the form

$$S_{m-1}^{T_{\nu_1}^{m-1}} = [T'^{m-1}, \{T'^{m-2}_{\nu_2}\}_{\nu_2}, \ldots, \{T'^{0}_{\nu_2,\ldots,\nu_m}\}_{\nu_2,\ldots,\nu_m}],$$

where, for every system $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_m < 1$, we have

$$T'^{m-1} = T^{m-1}_{\nu_1}, \quad T'^{m-i}_{\nu_2,\ldots,\nu_i} = T^{m-i}_{\nu_1,\ldots,\nu_i} \ (i = 2, \ldots, m).$$

The comparison of the above to formula (7) for $T = T^{m-1}_{\nu_1}$ leads to the equality

$$S^T_{m-1}(\nu_1) = S_{m-1}^{T_{\nu_1}^{m-1}}. \tag{10}$$

At first we consider the case $w = g$, $x_1 = \ldots = x_i < x_{i+1}$ $(1 \leqslant i \leqslant n)$. If we assume that $T^m(x_1, 1\rangle \neq 0$, we can conclude, with the help of the inductive hypothesis, the following:

$$\begin{aligned}
&K(\bar{x}_{n+1};\ S^T_m|\tau) \overset{(7)(3)}{=} \int K\big(\bar{x}_{n+1};\ s^T_m(y)\,|\,\tau\big)\,dT^m(y) \\
&\overset{(4)}{=} \int\limits_{y<x_1} [1-2Q(y)]\,dT^m(y) + \int\limits_{y=x_1} \{1-Q(x_1)-[1-P(x_1)]^i Q(x_1)\}\,dT^m(y) + \\
&\quad + \int\limits_{y>x_1} \big\{P(x_1)+[1-P(x_1)]K\big(\bar{x}_{n+1,1};\ S^T_{m-1}(x_1)\,|\,\tau_1\big)\big\}\,dT^m(y) \\
&\overset{(6)(10)}{=} \int\limits_{y<x_1} K(\bar{x}_{n+1};\ y|\tau)\,dT^m(y) + \int\limits_{y=x_1} K(\bar{x}_{n+1};\ y|\tau)\,dT^m(y) + \\
&\quad + \int\limits_{y>x_1} \{P(x_1)+[1-P(x_1)]K(\bar{x}_{n+1,1};\ S_{m-1}^{T_{x_1}^{m-1}}\,|\tau_1)\}\,dT^m(y) \\
&= \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ y|\tau)\,dT^m(y) + \\
&\quad + \int\limits_{y>x_1} \{P(x_1)+[1-P(x_1)] \int\limits_{y_1>x_1} K(\bar{x}_{n+1,1};\ y_1|\tau_1)\,dT^{m-1}_{x_1}(y_1)\}\,dT^m(y) \\
&= \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ y|\tau)\,dT^m(y) + \\
&\quad + \int\limits_{y>x_1} \int\limits_{y_1>x_1} \{P(x_1)+[1-P(x_1)]K(\bar{x}_{n+1,1};\ y_1|\tau_1)\}\,dT^{m-1}_{x_1}(y_1)\,dT^m(y) \\
&\overset{(6)}{=} \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ y|\tau)\,dT^m(y) + \int\limits_{y>x_1} \int\limits_{y_1>x_1} K(\bar{x}_{n+1};\ y_1|\tau)\,dT^{m-1}_{x_1}(y_1)\,dT^m(y) \\
&\overset{(8)}{=} \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ y|\tau)\,dT^m(y) + \int\limits_{y>x_1} \int\limits_{y_1>x_1} K(\bar{x}_{n+1};\ y_1|\tau)\frac{dT^m(y_1)}{T^m(x_1, 1\rangle}\,dT^m(y) \\
&= \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ y|\tau)\,dT^m(y) + \int\limits_{y_1>x_1} K(\bar{x}_{n+1};\ y_1|\tau)\,dT^m(y_1) \\
&= \int K(\bar{x}_{n+1};\ y|\tau)\,dT^m(y) \overset{(8)}{=} \int K(\bar{x}_{n+1};\ y|\tau)\,dT(y).
\end{aligned}$$

In the case $T^m(x_1, 1\rangle = 0$, identical calculations as above imply

$$K(\bar{x}_{n+1};\; S_m^T|\tau) = \int_{y\leqslant x_1} K(\bar{x}_{n+1};\; y|\tau)dT^m(y)$$
$$= \int K(\bar{x}_{n+1};\; y|\tau)dT^m(y) = \int K(\bar{x}_{n+1};\; y|\tau)dT(y).$$

Now we consider the case $w = c$, $x_1 = \ldots = x_i < x_{i+1}$ $(1 \leqslant i \leqslant n)$. Then, analogously as in the first case, we calculate

$$\begin{aligned}
&K(\bar{x}_{n+1};\; S_m^T|\pi)\\
&= \int_{y\leqslant x_1} K(\bar{x}_{n+1};\, y|\pi)dT^m(y)+\\
&\quad + \int_{y>x_1} \{P(x_1)+[1-P(x_1)]K(\bar{x}_{n+1,1};\; s_m^T(y)|\pi_1)\}dT^m(y)\\
&= \int_{y\leqslant x_1} K(\bar{x}_{n+1};\; y|\pi)dT^m(y)+P(x_1)T^m(x_1, 1\rangle+\\
&\quad +[1-P(x_1)]\Big\{\int K\big(\bar{x}_{n+1,1};\; s_m^T(y)|\pi_1\big)dT^m(y)-\\
&\qquad\qquad - \int_{y\leqslant x_1} K\big(\bar{x}_{n+1,1};\; s_m^T(y)|\pi_1\big)dT^m(y)\Big\}\\
&\overset{(3)(4)(6)}{=} \int_{y\leqslant x_1} K(\bar{x}_{n+1};\; y|\pi)dT^m(y)+P(x_1)T^m(x_1, 1\rangle+\\
&\quad +[1-P(x_1)]\Big\{K(\bar{x}_{n+1,1};\; S_m^T|\pi_1)- \int_{y\leqslant x_1} K(\bar{x}_{n+1,1};\; y|\pi_1)dT^m(y)\Big\}\\
&= \int_{y\leqslant x_1} K(\bar{x}_{n+1};\; y|\pi)dT^m(y)+P(x_1)T^m(x_1, 1\rangle+\\
&\quad +[1-P(x_1)] \int_{y>x_1} K(\bar{x}_{n+1,1};\; y|\pi_1)dT^m(y)\\
&= \int_{y\leqslant x_1} K(\bar{x}_{n+1};y|\pi)dT^m(y)+ \int_{y>x_1} \{P(x_1)+[1-P(x_1)]K(\bar{x}_{n+1,1};y|\pi_1)\}Td^m(y)\\
&\overset{(6)}{=} \int_{y\leqslant x_1} K(\bar{x}_{n+1};\; y|\pi)dT^m(y)+ \int_{y>x_1} K(\bar{x}_{n+1};\; y|\pi)dT^m(y)\\
&= \int K(\bar{x}_{n+1};\; y|\pi)dT(y).
\end{aligned}$$

In the remaining cases, i.e. $x_1 = \ldots = x_{n+1}$, $w = c$ and $w = g$, the proof proceeds in a similar manner. Therefore, by the induction principle, Lemma 3 is true.

The result of Lemma 3 can be interpreted as follows. Using the strategy S_m^T by player B is equivalent to adopting by him the strategy y (see the Remark in Section 2) according to the measure T.

Before discussing further properties of the payoff function we introduce the following notation.

For a number $a \in \langle 0, 1\rangle$ and for strategies

$$S'_m = [G^m, \{G^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{G^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}],$$
$$S''_m = [H^m, \{H^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{H^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}],$$

where $S'_m, S''_m \in B_m$, we let

$$(11) \qquad [aS'_m+(1-a)S''_m] = [T^m, \{T^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{T^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}]$$

be a strategy of player B from the set B_m such that

$$(12) \qquad T^{m-i}_{\nu_1,\ldots,\nu_i}(V)$$
$$= \begin{cases} aG^m+(1-a)H^m \quad \text{if } i = 0, \\ \dfrac{a\prod\limits_{k=0}^{i-1} G^{m-k}_{\nu_1,\ldots,\nu_k}(\nu_{k+1}, 1\rangle G^{m-i}_{\nu_1,\ldots,\nu_i}(V)+(1-a)\prod\limits_{k=0}^{i-1} H^{m-k}_{\nu_1,\ldots,\nu_k}(\nu_{k+1}, 1\rangle H^{m-i}_{\nu_1,\ldots,\nu_i}(V)}{a\prod\limits_{k=0}^{i-1} G^{m-k}_{\nu_1,\ldots,\nu_k}(\nu_{k+1}, 1\rangle+(1-a)\prod\limits_{k=0}^{i-1} H^{m-k}_{\nu_1,\ldots,\nu_k}(\nu_{k+1}, 1\rangle} \\ \text{if } i \geqslant 1 \text{ and } a\prod\limits_{k=0}^{i-1} G^{m-k}_{\nu_1,\ldots,\nu_k}(\nu_{k+1}, 1\rangle+(1-a)\prod\limits_{k=0}^{i-1} H^{m-k}_{\nu_1,\ldots,\nu_k}(\nu_{k+1}, 1\rangle \neq 0, \\ \dfrac{L(V)}{1-\nu_i} \qquad \text{otherwise} \end{cases}$$

for any $V \in \mathscr{B}\big((\nu_i, 1\rangle\big)$, $i = 0, 1, \ldots, m$, where $L(V)$ is the Lebesgue measure of the set V. We omit the symbols $\nu_1, \ldots, \nu_l$ if $l = 0$.

LEMMA 4. *If $\bar{x}_n \in \bar{X}_n$, $S'_m, S''_m \in B_m$, and $a \in \langle 0, 1\rangle$, then for the vector λ satisfying $\|\lambda\| = n$ and $|\lambda| = m$ the relation*

$$(13) \qquad K\big(\bar{x}_n;\ [aS'_m+(1-a)S''_m]\,|\,\lambda\big) = aK(\bar{x}_n;\ S'_m\,|\,\lambda)+(1-a)K(\bar{x}_n;\ S''_m\,|\,\lambda)$$

is valid.

Proof. To prove the assertion we apply induction with respect to the number $n = \|\lambda\|$.

Step 1. For $n = 1$ the lemma is easily proved.

Step 2. Let a number $n \geqslant 1$ be fixed. Assume that, for any $(\bar{k}_r\,|\,w) = \lambda'$ such that $\|\lambda'\| = n$, equality (13) holds for $\lambda = \lambda'$ if $\bar{x}_n \in \bar{X}_n$, $S'_m, S''_m \in B_m$ $(|\lambda| = m)$.

Step 3. Fix in an arbitrary way a vector $(\bar{k}_r\,|\,w) = \lambda$ such that $\|\lambda\| = n+1$, let $\bar{x}_{n+1} \in \bar{X}_{n+1}$, $S'_m, S''_m \in B_m$ $(|\lambda| = m)$ and $a_1 \in \langle 0, 1\rangle$. We show, under the inductive hypothesis, that the equality

$$(14) \qquad K\big(\bar{x}_{n+1};\ [a_1 S'_m+(1-a_1)S''_m]\,|\,\lambda\big)$$
$$= a_1 K(\bar{x}_{n+1};\ S'_m\,|\,\lambda)+(1-a_1)K(\bar{x}_{n+1};\ S''_m\,|\,\lambda)$$

is valid.

Let us put

$$S'_m = [G^m, \{G^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{G^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}],$$

$$S'_{m-1}(x_1) = [G^{m-1}_{x_1}, \{G^{m-2}_{x_1,\nu_2}\}_{\nu_2}, \ldots, \{G^0_{x_1,\nu_2,\ldots,\nu_m}\}_{\nu_2,\ldots,\nu_m}],$$

$$S''_m = [H^m, \{H^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{H^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}],$$

$$S''_{m-1}(x_1) = [H^{m-1}_{x_1}, \{H^{m-2}_{x_1,\nu_2}\}_{\nu_2}, \ldots, \{H^0_{x_1,\nu_2,\ldots,\nu_m}\}_{\nu_2,\ldots,\nu_m}],$$

$$s'_m(y) = [D_y, \{G^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{G^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}],$$

$$s''_m(y) = [D_y, \{H^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{H^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}].$$

Then, by (11) and (12), we can write

$$(15) \qquad [a_1 S'_m + (1-a_1) S''_m] = [a_1 G^m + (1-a_1) H^m, \{T^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{T^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}],$$

where $T^{m-i}_{\nu_1,\ldots,\nu_i}$ $(i = 1, 2, \ldots, m)$ are defined by formula (12).

Further, we introduce the notation

$$s_{m,a_1}(y) = [D_y, \{T^{m-1}_{\nu_1}\}_{\nu_1}, \ldots, \{T^0_{\nu_1,\ldots,\nu_m}\}_{\nu_1,\ldots,\nu_m}],$$

$$S_{m-1,a_1}(x_1) = [T^{m-1}_{x_1}, \{T^{m-2}_{x_1,\nu_2}\}_{\nu_2}, \ldots, \{T^0_{x_1,\nu_2,\ldots,\nu_m}\}_{\nu_2,\ldots,\nu_m}],$$

$$a_2 = \frac{a_1 G^m(x_1, 1\rangle}{a_1 G^m(x_1, 1\rangle + (1-a_1) H^m(x_1, 1\rangle}.$$

It is easy to check with the help of (11) and (12) that

$$(16) \qquad S_{m-1,a_1}(x_1) = [a_2 S'_{m-1}(x_1) + (1-a_2) S''_{m-1}(x_1)]$$

if $a_1 G^m(x_1, 1\rangle + (1-a_1) H^m(x_1, 1\rangle \neq 0$.

Now we prove (14). First, consider the case $w = g$, $x_1 = \ldots = x_i < x_{i+1}$ $(1 \leqslant i \leqslant n)$. If $a_1 G^m(x_1, 1\rangle + (1-a_1) H^m(x_1, 1\rangle \neq 0$, then we can conclude the following:

$$\begin{aligned}
&K(\bar{x}_{n+1};\ [a_1 S'_m + (1-a_1) S''_m] \,|\, \tau) \\
&\overset{(3)(15)}{=} \int K(\bar{x}_{n+1};\ s_{m,a_1}(y) \,|\, \tau)\, d\{a_1 G^m + (1-a_1) H^m\}(y) \\
&\overset{(4)}{=} a_1 \int_{y<x_1} [1-2Q(y)]\, dG^m(y) + (1-a_1) \int_{y<x_1} [1-2Q(y)]\, dH^m(y) + \\
&\quad + a_1 \int_{y=x_1} \{1-Q(x_1) - [1-P(x_1)]^i Q(x_1)\}\, dG^m(y) + \\
&\quad + (1-a_1) \int_{y=x_1} \{1-Q(x_1) - [1-P(x_1)]^i Q(x_1)\}\, dH^m(y) + \\
&\quad + \int_{y>x_1} \{P(x_1) + [1-P(x_1)] K(\bar{x}_{n+1,1};\ S_{m-1,a_1}(x_1) \,|\, \tau_1)\}\, d\{a_1 G^m + \\
&\qquad\qquad + (1-a_1) H^m\}(y)
\end{aligned}$$

$$\overset{(4)}{=} a_1 \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ s'_m(y)\,|\,\tau)\,dG^m(y) + (1-a_1) \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ s''_m(y)\,|\,\tau)\,dH^m(y) +$$
$$+ a_1 \int\limits_{y>x_1} P(x_1)\,dG^m(y) + (1-a_1) \int\limits_{y>x_1} P(x_1)\,dH^m(y) +$$
$$+ [1-P(x_1)]K(\bar{x}_{n+1,1};\ S_{m-1,a_1}(x_1)\,|\,\tau_1)\{a_1 G^m(x_1, 1\rangle +$$
$$+ (1-a_1)H^m(x_1, 1\rangle\}.$$

But, in view of (16), the inductive hypothesis implies

$$K(\bar{x}_{n+1,1};\ S_{m-1,a_1}(x_1)\,|\,\tau_1)$$
$$= a_2 K(\bar{x}_{n+1,1};\ S'_{m-1}(x_1)\,|\,\tau_1) + (1-a_2)K(\bar{x}_{n+1,1};\ S''_{m-1}(x_1)\,|\,\tau_1),$$

whence

$$[1-P(x_1)]K(\bar{x}_{n+1,1};\ S_{m-1,a_1}(x_1)\,|\,\tau_1)\{a_1 G^m(x_1, 1\rangle + (1-a_1)H^m(x_1, 1\rangle\}$$
$$= a_1 \int\limits_{y>x_1} [1-P(x_1)]K(\bar{x}_{n+1,1};\ S'_{m-1}(x_1)\,|\,\tau_1)\,dG^m(y) +$$
$$+ (1-a_1) \int\limits_{y>x_1} [1-P(x_1)]K(\bar{x}_{n+1,1};\ S''_{m-1}(x_1)\,|\,\tau_1)\,dH^m(y).$$

Then

$$K(\bar{x}_{n+1};\ [a_1 S'_m + (1-a_1)S''_m]\,|\,\tau)$$
$$= a_1 \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ s'_m(y)\,|\,\tau)\,dG^m(y) + (1-a_1) \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ s''_m(y)\,|\,\tau)\,dH^m(y) +$$
$$+ a_1 \int\limits_{y>x_1} \{P(x_1) + [1-P(x_1)]K(\bar{x}_{n+1,1};\ S'_{m-1}(x_1)\,|\,\tau_1)\}\,dG^m(y) +$$
$$+ (1-a_1) \int\limits_{y>x_1} \{P(x_1) + [1-P(x_1)]K(\bar{x}_{n+1,1};\ S''_{m-1}(x_1)\,|\,\tau_1)\}\,dH^m(y)$$
$$\overset{(4)(3)}{=} a_1 K(\bar{x}_{n+1};\ S'_m\,|\,\tau) + (1-a_1)K(\bar{x}_{n+1};\ S''_m\,|\,\tau).$$

If $a_1 G^m(x_1, 1\rangle + (1-a_1)H^m(x_1, 1\rangle = 0$, then, concluding analogously as in the previous case, we obtain

$$K(\bar{x}_{n+1};\ [a_1 S'_m + (1-a_1)S''_m]\,|\,\tau)$$
$$= a_1 \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ s'_m(y)\,|\,\tau)\,dG^m(y) + (1-a_1) \int\limits_{y\leqslant x_1} K(\bar{x}_{n+1};\ s''_m(y)\,|\,\tau)\,dH^m(y)$$
$$\overset{(3)}{=} a_1 K(\bar{x}_{n+1};\ S'_m\,|\,\tau) + (1-a_1)K(\bar{x}_{n+1};\ S''_m\,|\,\tau).$$

Therefore, equality (14) holds in the case $w = g$, $x_1 = \ldots = x_i < x_{i+1}$ $(1 \leqslant i \leqslant n)$.

Now we consider the case $w = c$, $x_1 = \ldots = x_i < x_{i+1}$ $(1 \leqslant i \leqslant n)$. In a similar manner as before, we get

$$K\big(\bar{x}_{n+1};\ [a_1 S'_m + (1-a_1) S''_m] \,|\, \pi\big)$$
$$= a_1 \int_{y \leqslant x_1} K\big(\bar{x}_{n+1};\ s'_m(y) \,|\, \pi\big) dG^m(y) + (1-a_1) \int_{y \leqslant x_1} K\big(\bar{x}_{n+1};\ s''_m(y) \,|\pi\big) dH^m(y) +$$
$$+ a_1 \int_{y > x_1} P(x_1) dG^m(y) + (1-a_1) \int_{y > x_1} P(x_1) dH^m(y) +$$
$$+ [1 - P(x_1)] \Big\{ \int_{y > x_1} K\big(\bar{x}_{n+1,1};\ s_{m,a_1}(y) \,|\, \pi_1\big) d\{a_1 G^m + (1-a_1) H^m\}(y) \Big\}.$$

But the inductive hypothesis allows us to conclude further:

$$\int_{y > x_1} K\big(\bar{x}_{n+1,1};\ s_{m,a_1}(y) \,|\, \pi_1\big) d\{a_1 G^m + (1-a_1) H^m\}(y)$$
$$= \int K\big(\bar{x}_{n+1,1};\ s_{m,a_1}(y) \,|\, \pi_1\big) d\{a_1 G^m + (1-a_1) H^m\}(y) -$$
$$- \int_{y \leqslant x_1} K\big(\bar{x}_{n+1,1};\ s_{m,a_1}(y) \,|\, \pi_1\big) d\{a_1 G^m + (1-a_1) H^m\}(y)$$
$$\overset{(3)}{=} K\big(\bar{x}_{n+1,1};\ [a_1 S'_m + (1-a_1) S'_m] \,|\, \pi_1\big) -$$
$$- \int_{y < x_1} K\big(\bar{x}_{n+1,1};\ s_{m,a_1}(y) \,|\, \pi_1\big) d\{a_1 G^m + (1-a_1) H^m\}(y)$$
$$\overset{(4)}{=} a_1 K(\bar{x}_{n+1,1};\ S'_m \,|\, \pi_1) + (1-a_1) K(\bar{x}_{n+1,1};\ S''_m \,|\, \pi_1) -$$
$$- a_1 \int_{y \leqslant x_1} K\big(\bar{x}_{n+1,1};\ s'_m(y) \,|\, \pi_1\big) dG^m(y) -$$
$$- (1-a_1) \int_{y \leqslant x_1} K\big(\bar{x}_{n+1,1};\ s''_m(y) \,|\, \pi_1\big) dH^m(y)$$
$$\overset{(3)}{=} a_1 \int_{y > x_1} K\big(\bar{x}_{n+1,1}; s'_m(y) \,|\, \pi_1\big) dG^m(y) + (1-a_1) \int_{y > x_1} K\big(\bar{x}_{n+1,1}; s''_m(y) \,|\, \pi_1\big) dH^m(y).$$

Therefore

$$K\big(\bar{x}_{n+1};\ [a_1 S'_m + (1-a_1) S''_m] \,|\, \pi\big)$$
$$= a_1 \int_{y \leqslant x_1} K\big(\bar{x}_{n+1};\ s'_m(y) \,|\, \pi\big) dG^m(y) + (1-a_1) \int_{y \leqslant x_1} K\big(\bar{x}_{n+1}; s''_m(y) \,|\, \pi\big) dH^m(y) +$$
$$+ a_1 \int_{y > x_1} \big\{P(x_1) + [1 - P(x_1)] K\big(\bar{x}_{n+1,1};\ s'_m(y) \,|\, \pi_1\big)\big\} dG^m(y) +$$
$$+ (1-a_1) \int_{y > x_1} \big\{P(x_1) + [1 - P(x_1)] K\big(\bar{x}_{n+1,1};\ s''_m(y) \,|\, \pi_1\big)\big\} dH^m(y)$$
$$\overset{(4)}{=} a_1 \int K\big(\bar{x}_{n+1};\ s'_m(y) \,|\, \pi\big) dG^m(y) + (1-a_1) \int K\big(\bar{x}_{n+1};\ s''_m(y) \,|\, \pi\big) dH^m(y)$$
$$\overset{(3)}{=} a_1 K(\bar{x}_{n+1};\ S'_m \,|\, \pi) + (1-a_1) K(\bar{x}_{n+1};\ S''_m \,|\, \pi).$$

Summarizing, equality (14) has been proved in the case where $x_1 = \ldots = x_i < x_{i+1}$ $(1 \leqslant i \leqslant n)$.

The case $x_1 = \ldots = x_{n+1}$ is considered similarly.

Thus the proof of Lemma 4 is completed.

Now we give some lemmas characterizing the sets $B_m^0(\lambda)$ the proofs of which can be easily obtained with the help of Lemmas 1, 3, 4 and relations (3) and (4).

LEMMA 5. *For an arbitrary probability measure T on $\mathscr{B}(\langle 0,1\rangle)$ the strategy S_m^T given by* (7) *and* (8) *is admissible in the game Γ_λ if $|\lambda| = m$.*

LEMMA 6. *If the strategies S_m and S_m' for player B are admissible in the game Γ_λ, then also the strategy $[aS_m+(1-a)S_m']$ given by* (11) *and* (12) *is admissible in that game for any $a \in \langle 0,1\rangle$.*

LEMMA 7. *A strategy S_m is admissible in the game Γ_π if it is admissible in the game Γ_{π_1}.*

In the following lemma, the manner of constructing elements of the set $B_m^0(\lambda)$ is described.

LEMMA 8. *For fixed λ such that $|\lambda| = m$ $(m = 1, 2, \ldots)$ let $S_m \in B_m$ be represented in the form $S_m = [G^m, \{S_{m-1}(\nu_1)\}_{\nu_1}]$, where G^m is an arbitrary probability measure on $\mathscr{B}(\langle 0,1\rangle)$ and the family $\{S_{m-1}(\nu_1)\}_{\nu_1}$ satisfies*

(i) *for any $\nu_1 \in \langle 0,1)$*

$$S_{m-1}(\nu_1) \in B_{m-1}^0(\tau_1) \qquad \text{if } w = g,$$

$$S_{m-1}(\nu_1) \in B_{m-1}^0(\bar{k}_{r,k_r+1}\,|\,c) \qquad \text{if } w = c;$$

(ii) *in $\langle 0,1\rangle$ there exists a sequence of mutually disjoint Borel sets $\{C_i\}$ such that*

$$\bigcup_i C_i = \langle 0,1\rangle$$

and $S_{m-1}(\nu_1) = S_{m-1}(\nu_1')$ if $\nu_1, \nu_1' \in C_i$ for a certain i.

Then the strategy S_m is admissible in the game Γ_λ.

Proof. Let us consider the case $w = g$, assuming $\|\lambda\| = n$. Relations (3) and (4) imply the equality

$$(17)\qquad K(\bar{x}_n;\, S_m\,|\,\tau) = \int\limits_{y\leqslant x_1} K(\bar{x}_n;\, y\,|\,\tau)\,dG^m(y) + \\ + \big\{P(x_1)+[1-P(x_1)]K\big(\bar{x}_{n,1};\, S_{m-1}(x_1)\,|\,\tau_1\big)\big\}G^m(x_1, 1\rangle.$$

Hence, from assumptions (i) and (ii) of the lemma we conclude that $K\big(\bar{x}_{n,1};\, S_{m-1}(x_1)\,|\,\tau_1\big)$ is a measurable function with respect to $\mathscr{B}(\bar{X}_n)$. The first component of the right-hand side of equality (17) is also a measurable function with respect to $\mathscr{B}(\bar{X}_n)$.

Therefore $K(\bar{x}_n;\ S_m|\tau)$ is a measurable function with respect to $\mathscr{B}(\bar{X}_n)$, which completes the proof of the lemma for $w = g$.

To prove the lemma in the case $w = c$ we reason similarly, taking into account the equality

$$(18)\qquad K(\bar{x}_n;\ S_m|\pi) = \int\limits_{y\leqslant x_{k_r+1}} K(\bar{x}_n;\ y|\pi)dG^m(y)+$$

$$+\Big\{1-\prod_{i=1}^{k_r+1}[1-P(x_i)]+\prod_{i=1}^{k_r+1}[1-P(x_i)]K\big(\bar{x}_{n,k_r+1};\ S_{m-1}(x_{k_r+1})\,\big|\,\bar{k}_{r,k_r+1}|c\big)\Big\}\times$$

$$\times G^m(x_{k_r+1},1\rangle$$

which follows from (3), (4) and (6).

LEMMA 9. *Let λ be a vector such that $\|\lambda\| = n$, $|\lambda| = m \geqslant 1$. If $F \in A_n$ and $S_m = [G^m, \{s_m(y)\}_y] \in B_m^0(\lambda)$, then*

$$(19)\qquad \int\limits_{\bar{X}_n}\Big\{\int\limits_{\bar{Y}} K\big(\bar{x}_n;\ s_m(y)|\lambda\big)dG^m(y)\Big\}\,dF(\bar{x}_n)$$

$$= \int\limits_{\bar{Y}}\Big\{\int\limits_{\bar{X}_n} K\big(\bar{x}_n;\ s_m(y)|\lambda\big)dF(\bar{x}_n)\Big\}dG^m(y).$$

Proof. According to the notation at the end of Section 2, the strategy S_m can be written in the form $S_m = [G^m, \{S_{m-1}(\nu_1)\}_{\nu_1}]$.

At first we consider the case $w = g$. Since $S_m \in B_m^0(\lambda)$, by (17) the function

$$K_1(\bar{x}_n) = P(x_1)+[1-P(x_1)]K\big(\bar{x}_{n,1};\ S_{m-1}(x_1)|\tau_1\big)$$

is measurable with respect to $\mathscr{B}(\bar{X}_n)\times\mathscr{B}(\bar{Y})$.

On the other hand, by (4) and (6), we have

$$K\big(\bar{x}_n;\ s_m(y)|\tau\big) = \begin{cases} K_1(\bar{x}_n) & \text{if } x_1 < y,\\ K(\bar{x}_n;\ y|\tau) & \text{if } x_1 \geqslant y,\end{cases}$$

whence it is easy to conclude that the function $K\big(\bar{x}_n;\ s_m(y)|\tau\big)$ is measurable with respect to $\mathscr{B}(\bar{X}_n)\times\mathscr{B}(\bar{Y})$.

In the case $w = c$, formulas (4) and (6) give

$$K\big(\bar{x}_n;\ s_m(y)|\pi\big) = \begin{cases} K(\bar{x}_n;\ y|\pi) & \text{if } y \leqslant x_{k_r+1},\\ K_2(\bar{x}_n) & \text{if } y > x_{k_r+1},\end{cases}$$

where

$$K_2(\bar{x}_n)$$

$$= 1-\prod_{i=1}^{k_r+1}[1-P(x_i)]+\prod_{i=1}^{k_r+1}[1-P(x_i)]K\big(\bar{x}_{n,k_r+1};\ S_{m-1}(x_{k_r+1})\,|\bar{k}_{r,k_r+1}|c\big).$$

Using relation (18) and the admissibility of S_m, we infer that in this case the function $K(\bar{x}_n;\ s_m(y)\,|\,\pi)$ is measurable with respect to the σ-algebra $\mathscr{B}(\bar{X}_n)\times\mathscr{B}(\bar{Y})$.

Now, in view of the boundedness of the function K (see Lemma 2), we can apply Fubini's theorem to complete the proof.

5. Construction of strategies F_λ^* and S_λ^*.

Definition 6. A point which is the infimum of a support of the marginal measure with respect to x_1 corresponding to $F \in A_n$ is called a *characteristic point of the strategy F*.

Now we define by induction with respect to the number $n = \|\lambda\|$ some strategies $F_\lambda^* \in A_{\|\lambda\|}$ and $S_\lambda^*(\varepsilon) \in B^0_{|\lambda|}(\lambda)$ $(\varepsilon > 0)$ for players A and B, respectively. The proof of optimality of these strategies is given in [2].

Step 1. Let $n = 1$. In this case we take as $F^*_{1|w}$ and $S^*_{1|w}(\varepsilon)$ the optimal strategies for players A and B in the game found in papers [1] and [5] $(w = g, c)$. Therefore, we have

$$F^*_{1|g} = D_{a_{1|g}} \quad \text{and} \quad S^*_{1|g}(\varepsilon) = [D_{a_{1|g}}, \{D_1\}_{\nu_1}]$$

for any $\varepsilon > 0$, where $P(a_{1|g})+Q(a_{1|g}) = 1$ and $0 < a_{1|g} < 1$.

In the case $w = c$, $F^*_{1|c}$ is a continuous probability measure with support $\langle a_{1|c}, 1\rangle$ whose derivative is of the form

$$\frac{dF^*_{1|c}(x_1)}{dx_1} = \frac{2Q'(x_1)}{P(x_1)Q(x_1)+P(x_1)+Q(x_1)-1}\exp\left\{-\int\limits_{a_{1|c}}^{x_1}\frac{Q'(u)[1+P(u)]du}{P(u)Q(u)+P(u)+Q(u)-1}\right\},$$

$$x_1 \in \langle a_{1|c}, 1\rangle,$$

where $a_{1|c}$ is determined uniquely by the relations

$$\int\limits_{a_{1|c}}^{1} dF^*_{1|c}(x_1) = 1 \quad \text{and} \quad 0 < a_{1|c} < 1.$$

Before defining $S^*_{1|c}(\varepsilon)$ we introduce some notation.

Let $T_{1|c}$ be a continuous probability measure with support $\langle a_{1|c}, 1\rangle$ determined by its density

$$\frac{dT_{1|c}(y)}{dy} = \frac{2l_{1|c}P'(y)}{P(y)Q(y)+P(y)+Q(y)-1}\exp\left\{\int\limits_{y}^{1}\frac{P'(u)[1+Q(u)]du}{P(u)Q(u)+P(u)+Q(u)-1}\right\},$$

$$y \in \langle a_{1|c}, 1\rangle,$$

where $l_{1|c} = a_{1|c}(1-a_{1|c})^{-1}$ and

$$a_{1|c} = \frac{P(a_{1|c})Q(a_{1|c})+P(a_{1|c})+Q(a_{1|c})-1}{1-P(a_{1|c})}.$$

Now we can describe the strategy $S^*_{1|c}(\varepsilon)$. From the analysis of the optimal strategy for player B in the game $\Gamma_{1|c}$ (see [5]) it follows that $S^*_{1|c}(\varepsilon)$ can be written in the form

$$S^*_{1|c}(\varepsilon) = [(1-a_{1|c})T_{1|c}+a_{1|c}D_1] \quad (\varepsilon > 0).$$

Moreover, in papers [1] and [5] it was shown that

$$K\big(F^*_{1|w};\ S^*_{1|w}(\varepsilon)\big|1\,|\,w\big) = 1-2Q(a_{1|w}) \quad (\varepsilon > 0). \tag{20}$$

Besides, it is easy to check that the strategy $S^*_{1|w}(\varepsilon)$ is admissible in the game $\Gamma_{1|w}$ and the characteristic point $a_{1|w}$ of $F^*_{1|w}$ satisfies the condition $0 < a_{1|w} < 1$ ($w = g, c$).

Step 2. Assume that for some $n \geqslant 1$ and for any $\varepsilon > 0$ all the strategies $F^*_{\lambda'} \in A_{\|\lambda'\|}$ and $S^*_{\lambda'}(\varepsilon) \in B^0_{|\lambda'|}(\lambda')$ ($\lambda' = \bar{k}_r|w$) have been constructed for which $\|\lambda'\| = n$ and the characteristic points $a_{\lambda'}$ of $F^*_{\lambda'}$ satisfy $0 < a_{\lambda'} < 1$.

Step 3. Let us fix in an arbitrary way a vector $\bar{k}_r|w = \lambda$ such that $\|\lambda\| = n+1$. We give the definitions of F^*_λ and $S^*_\lambda(\varepsilon)$.

In the case $w = g$ we put

$$F^*_\tau(\bar{x}_{n+1}) = D_{a_\tau}(x_1)\cdot F^*_{\tau_1}(\bar{x}_{n+1,1}), \qquad S^*_\tau(\varepsilon) = [H^*_\tau(\varepsilon), \{S^*_{\tau_1}(\varepsilon/2)\}_{\nu_1}] \qquad (\varepsilon > 0),$$

where

$$Q(a_{\tau_1}) = \frac{Q(a_\tau)}{1-P(a_\tau)}, \tag{21}$$

$H^*_\tau(\varepsilon) = H_{\langle a_\tau, a_\tau+\delta_\tau(\varepsilon)\rangle}$ is any arbitrarily fixed continuous probability measure with support $\langle a_\tau, a_\tau+\delta_\tau(\varepsilon)\rangle$ and $\delta_\tau(\varepsilon)$ is an arbitrary positive function determined for $\varepsilon > 0$, which satisfies

$$\begin{cases} \lim\limits_{\varepsilon\to 0+} \delta_\tau(\varepsilon) = 0, \\ a_\tau+\delta_\tau(\varepsilon) < a_{\tau_1} \quad (\varepsilon > 0), \\ P\big(a_\tau+\delta_\tau(\varepsilon)\big) \leqslant P(a_\tau)+\varepsilon/4 \quad (\varepsilon > 0). \end{cases} \tag{22}$$

Remark. In the definition of $S^*_\tau(\varepsilon)$ we find the family of identical strategies $\{S^*_{\tau_1}(\varepsilon/2)\}_{\nu_1}$ which, according to Definition 2 and to the notation introduced in the Remark at the end of Section 2, has to be determined for all $\nu_1 \in \langle 0, 1)$. However, it is impossible because of the inequality $a_{\tau_1} < 1$, which follows from Step 2. Therefore, we shall understand that the elements of the family $\{S^*_{\tau_1}(\varepsilon/2)\}_{\nu_1}$ for $\nu_1 \in \langle 0, a_{\tau_1}\rangle$ are equal to $S^*_{\tau_1}(\varepsilon/2)$ and for $\nu_1 \in (a_{\tau_1}, 1)$ are equal to the strategy $y = 1$.

Now we consider the case $w = c$. Based on (7) and (11) we put

$$F_\pi^*(\bar{x}_{n+1}) = U_\pi(x_1)F_{\pi_1}^*(\bar{x}_{n+1,1}),$$

$$S_\pi^*(\varepsilon) = [(1-\alpha_\pi)S_{|\pi|}^{T_\pi} + \alpha_\pi S_{\pi_1}^*(\varepsilon/2)] \quad \text{for any } \varepsilon > 0,$$

where

$$\int_{a_\pi}^{a_{\pi_1}} \frac{Q(a_\pi)Q'(x_1)}{P(x_1)Q^2(x_1)}\,dx_1 = 1, \tag{23}$$

and U_π, T_π are absolutely continuous probability measures the derivatives of which are expressed by the formulas

$$\frac{dU_\pi(x_1)}{dx_1} = \frac{Q(a_\pi)Q'(x_1)}{P(x_1)Q^2(x_1)}, \quad x_1 \in \langle a_\pi, a_{\pi_1}\rangle, \tag{24}$$

$$\frac{dT_\pi(y)}{dy} = \frac{l_\pi P'(y)}{Q(y)P^2(y)}, \quad y \in \langle a_\pi, a_{\pi_1}\rangle, \tag{25}$$

where

$$l_\pi = P(a_{\pi_1})Q(a_{\pi_1})\frac{\alpha_\pi}{1-\alpha_\pi}, \tag{26}$$

$$\alpha_\pi = \frac{P(a_\pi)Q(a_\pi)}{P(a_{\pi_1})Q(a_{\pi_1})[1-P(a_\pi)]}. \tag{27}$$

Now we show that $F_\lambda^* \in A_{n+1}$ and $S_\lambda^*(\varepsilon) \in B_{|\lambda|}^0(\lambda)$ for any $\varepsilon > 0$. Let us take into account the equality

$$\frac{Q(x)}{1-P(x)} = Q(a_{\tau_1}), \tag{28}$$

where a_{τ_1} is the characteristic point of the strategy $F_{\tau_1}^*$. Since the function $f(x) = Q(x)[1-P(x)]^{-1}$ is continuous and increasing in the interval $\langle 0, 1)$ and, moreover, since

$$f(0) = 0 < Q(a_{\tau_1}), \quad f(a_{\tau_1}) = \frac{Q(a_{\tau_1})}{1-P(a_{\tau_1})} > Q(a_{\tau_1}),$$

there exists exactly one solution of equation (28), say a_τ, and we have

$$0 < a_\tau < a_{\tau_1}. \tag{29}$$

This means that equality (21) gives the unique characteristic point of the strategy F_τ^*.

Now consider the function

$$N(z) = \int_z^{a_{\pi_1}} \frac{Q'(u)\,du}{P(u)Q^2(u)} - \frac{1}{Q(z)}, \quad z \in (0, 1).$$

It is easy to show that $N(z)$ is a decreasing function in the interval $(0, a_{\pi_1}\rangle$ and

$$\lim_{z\to 0+} N(z) = +\infty, \quad N(a_{\pi_1}) = -Q^{-1}(a_{\pi_1}) < 0.$$

Since $N(z)$ is a continuous function, we conclude that the equality

$$\int_z^{a_{\pi_1}} \frac{Q(z)Q'(u)\,du}{P(u)Q^2(u)} = 1$$

has exactly one solution in the interval $(0, a_{\pi_1})$, which will be denoted by a_π.

Hence, because of (23) and (24), we have

$$0 < a_\pi < a_{\pi_1} \tag{30}$$

and

$$\int_{a_\pi}^{a_{\pi_1}} dU_\pi(x_1) = 1. \tag{31}$$

If we integrate equality (23) by parts, we get

$$\frac{1}{Q(a_\pi)} = \frac{1}{P(a_\pi)Q(a_\pi)} - \frac{1}{P(a_{\pi_1})Q(a_{\pi_1})} - \int_{a_\pi}^{a_{\pi_1}} \frac{P'(y)\,dy}{Q(y)P^2(y)},$$

whence

$$\int_{a_\pi}^{a_{\pi_1}} \frac{P'(y)\,dy}{Q(y)P^2(y)} = \frac{1-P(a_\pi)}{P(a_\pi)Q(a_\pi)} - \frac{1}{P(a_{\pi_1})Q(a_{\pi_1})}. \tag{32}$$

Since the left-hand side of equality (32) is a positive number, we have

$$a_\pi = \frac{P(a_\pi)Q(a_\pi)}{P(a_{\pi_1})Q(a_{\pi_1})[1-P(a_\pi)]} < 1.$$

Hence, by (30), we can write more precisely

$$0 < a_\pi < 1. \tag{33}$$

Now we return to relation (32) which can be transformed, by (26) and (27), to the form

$$\int_{a_\pi}^{a_{\pi_1}} \frac{P'(y)\,dy}{Q(y)P^2(y)} = \frac{1}{P(a_{\pi_1})Q(a_{\pi_1})}\,\frac{1-a_\pi}{a_\pi} \overset{(26)}{=} \frac{1}{l_\pi}.$$

Therefore we have $l_\pi > 0$ and, in view of (25),

$$\int_{a_\pi}^{a_{\pi_1}} dT_\pi(y) = 1. \tag{34}$$

Finally, from Lemmas 5-8 with the help of formulas (29)-(34) we conclude that the constructed strategies F_λ^* and $S_\lambda^*(\varepsilon)$, for the vector λ fixed in Step 3, satisfy the following conditions:

(i) the characteristic point a_λ of F_λ^* fulfils $0 < a_\lambda < a_{\lambda_1}$,

(ii) for any $\varepsilon > 0$ the strategy $S_\lambda^*(\varepsilon)$ is admissible in the game Γ_λ.

This completes the inductive definition of strategies F_λ^* and $S_\lambda^*(\varepsilon)$.

At the end of this section we give the following lemma on the above-defined strategies:

LEMMA 10. *For any $\varepsilon > 0$ and for all λ we have*

$$K(F_\lambda^*;\ S_\lambda^*(\varepsilon)\,|\,\lambda) = 1-2Q(a_\lambda). \tag{35}$$

Proof. We prove this equality by induction with respect to the number $n = \|\lambda\|$.

Step 1. For $n = 1$ the lemma is true in view of (20).

Step 2. Assume the validity of (35) for any vector $(\bar{k}_r\,|\,w) = \lambda'$ such that $\|\lambda'\| = n \geqslant 1$.

Step 3. Let us fix in an arbitrary way a vector $(\bar{k}_r\,|\,w) = \lambda$ such that $\|\lambda\| = n+1$.

At first consider the case $w = g$. Using the definitions of strategies F_τ^* and $S_\tau^*(\varepsilon)$ we get, under the inductive assumption,

$$\begin{aligned} K(F_\tau^*;\ S_\tau^*(\varepsilon)\,|\,\tau) &\overset{(5)(3)(4)}{=} P(a_\tau)+[1-P(a_\tau)]K(F_{\tau_1}^*;\ S_{\tau_1}^*(\varepsilon/2)\,|\,\tau_1) \\ &= P(a_\tau)+[1-P(a_\tau)][1-2Q(a_{\tau_1})] \overset{(21)}{=} 1-2Q(a_\tau). \end{aligned}$$

Now let us examine the case $w = c$. By (16) we have

$$\int_{a_\pi}^{y} P(x_1)\,dU_\pi(x_1) = 1-\frac{Q(a_\pi)}{Q(y)}, \qquad y \in \langle a_\pi, a_{\pi_1}\rangle. \tag{36}$$

From the definitions of F_π^* and $S_\pi^*(\varepsilon)$ we conclude, using the inductive hypothesis, the following:

$$\begin{aligned} K(F_\pi^*;\ S_\pi^*(\varepsilon)\,|\,\pi) &\overset{(5)}{=} \int K(\bar{x}_{n+1};\ S_\pi^*(\varepsilon)\,|\,\pi)\,dF_\pi^*(\bar{x}_{n+1}) \\ &\overset{(13)(9)}{=} (1-a_\pi)\iint K(\bar{x}_{n+1};\ y\,|\,\pi)\,dT_\pi(y)\,dF_\pi^*(\bar{x}_{n+1})+ \\ &\qquad + a_\pi \int K(\bar{x}_{n+1};\ S_{\pi_1}^*(\varepsilon/2)\,|\,\pi)\,dF_\pi^*(\bar{x}_{n+1}) \\ &\overset{(6)(3)(4)}{=} (1-a_\pi)\int\Big\{\int_{y>x_1} \{P(x_1)+[1-P(x_1)][1-2Q(y)]\}\,dT_\pi(y)+ \end{aligned}$$

$$+\int_{y<x_1}[1-2Q(y)]dT_\pi(y)\Big\}\,dF_\pi^*(\bar{x}_{n+1})+$$

$$+\alpha_\pi\int\{P(x_1)+[1-P(x_1)]K(\bar{x}_{n+1,1};\ S_{\pi_1}^*(\varepsilon/2)\,|\,\pi_1)\}\,dF_\pi^*(\bar{x}_{n+1})$$

$$\overset{(5)}{=}(1-\alpha_\pi)\Big\{\int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT_\pi(y)+\int_{a_\pi}^{a_{\pi_1}}\int_{x_1}^{a_{\pi_1}}2Q(y)P(x_1)dT_\pi(y)dU_\pi(x_1)\Big\}+$$

$$+\alpha_\pi\int_{a_\pi}^{a_{\pi_1}}\{P(x_1)+[1-P(x_1)]K(F_{\pi_1}^*;\ S_{\pi_1}^*(\varepsilon/2)\,|\,\pi_1)\}\,dU_\pi(x_1)$$

$$=(1-\alpha_\pi)\Big\{\int_{a_\pi}^{a_{\pi_1}}[1-2Q(y)]dT_\pi(y)+\int_{a_\pi}^{a_{\pi_1}}\int_{a_\pi}^{y}2Q(y)P(x_1)dU_\pi(x_1)dT_\pi(y)\Big\}+$$

$$+\alpha_\pi\int_{a_\pi}^{a_{\pi_1}}\{P(x_1)+[1-P(x_1)][1-2Q(a_{\pi_1})]\}\,dU_\pi(x_1)$$

$$\overset{36)(31)(34)}{=}(1-\alpha_\pi)[1-2Q(a_\pi)]+\alpha_\pi[1-2Q(a_\pi)]=1-2Q(a_\pi).$$

This completes the proof of the lemma by induction.

6. A theorem on optimal strategies. In this section we define, using the strategies F_λ^* and $S_\lambda^*(\varepsilon)$, determined in Section 5, two classes of strategies for players A and B, respectively. Then, assuming the existence of optimal strategies in these classes, we derive the necessary conditions.

Definition 7. The strategy $F\in A_n$ ($n=\|\lambda\|$) is an *element of the class* M_λ if F is defined as follows:

(i) in the case $w=g$,

$$F(\bar{x}_n)=D_{a_1}(x_1)\cdot F_{\tau_1}^*(\bar{x}_{n,1}) \tag{37}$$

for some a_1 such that $0\leqslant a_1<a_{\tau_1}$;

(ii) in the case $w=c$,

$$F(\bar{x}_n)=U_{a_2}(x_1)\cdot F_{\pi_1}^*(\bar{x}_{n,1}) \tag{38}$$

for some a_2 such that $0\leqslant a_2<a_{\pi_1}$, where U_{a_2} is a continuous probability measure with support $\langle a_2, a_{\pi_1}\rangle$.

Definition 8. The family of admissible strategies $\{S(\varepsilon)\}_{\varepsilon>0}$ in the game Γ_λ is an *element of the class* N_λ if for any $\varepsilon>0$ the strategy $S(\varepsilon)$ is defined as follows:

(i) in the case $w=g$,

$$S(\varepsilon)=[H_\tau(\varepsilon),\{S_{\tau_1}^*(\varepsilon/2)\}_{\nu_1}], \tag{39}$$

where $H_\tau = H_{\langle b_1, b_1+\delta(\varepsilon)\rangle}$, b_1 is a number independent of ε such that $0 \leqslant b_1 < a_{\tau_1}$, and $\delta(\varepsilon)$ is a fixed function determined for every $\varepsilon > 0$ such that

$$\lim_{\varepsilon \to 0+} \delta(\varepsilon) = 0, \quad 0 < \delta(\varepsilon) \leqslant a_{\tau_1} - b_1 \quad (\varepsilon > 0); \tag{40}$$

(ii) in the case $w = c$,

$$S(\varepsilon) = [(1-a) S_{|\pi|}^{T_{b_2}} + a S_{\pi_1}^*(\varepsilon/2)] \quad (\varepsilon > 0), \tag{41}$$

where b_2 is a number independent of ε such that $0 \leqslant b_2 < a_{\pi_1}$, T_{b_2} is a continuous probability measure with support $\langle b_2, a_{\pi_1}\rangle$, and a is a number independent of ε satisfying $0 \leqslant a \leqslant 1$.

THEOREM 1. *If in the classes M_λ and N_λ there exist an optimal strategy F for player A and a family $\{S(\varepsilon)\}_{\varepsilon>0}$ of ε-optimal strategies for player B in the game Γ_λ, respectively, then*

(i) $F = F_\lambda^*$;

(ii) $b_1 = a_\tau$ *for* $w = g$; *if, in addition,* $\delta(\varepsilon) = \delta_\tau(\varepsilon)$ *and* $H_\tau = H_\tau^*$ ($\delta_\tau(\varepsilon)$ *and* H_τ^* *were determined together with* $S_\tau^*(\varepsilon)$), *then* $S(\varepsilon) = S_\tau^*(\varepsilon)$;

(iii) $S(\varepsilon) = S_\pi^*(\varepsilon)$ *in the case* $w = c$;

(iv) *the value of the game* Γ_λ *equals* $v_\lambda = 1 - 2Q(a_\lambda)$.

The proofs of this theorem and of the optimality of the strategies F_λ^* and $S_\lambda^*(\varepsilon)$ are presented in paper [2].

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INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
50-370 WROCŁAW

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T. RADZIK i K. ORŁOWSKI (Wrocław)

MIESZANA GRA CZASOWA: BADANIE ZBIORÓW STRATEGII GRACZY

STRESZCZENIE

W pracy bada się model gry czasowej, w której gracz A dysponuje dowolną skończoną liczbą akcji cichych i głośnych, a gracz B — jedną akcją głośną. Funkcjami sukcesu graczy są niemalejące funkcje $P(t)$ i $Q(t)$, $t \in \langle 0, 1\rangle$. Konstruuje się zbiory strategii $\mathscr{A}$ i $\mathscr{B}$ odpowiednio dla graczy A i B oraz funkcję wypłaty K na zbiorze $\mathscr{A} \times \mathscr{B}$ w sposób uwzględniający wykorzystanie wszelkiej informacji uzyskiwanej w czasie gry. Następnie w zbiorach $\mathscr{A}$ i $\mathscr{B}$ określa się pewne działania i bada się zachowanie funkcji K względem nich.

Poza tym formułuje się twierdzenie o jednoznaczności, pozwalające znaleźć optymalne strategie graczy. Dowód tego twierdzenia i dowód optymalności znalezionych strategii są podane w pracy [2].