ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVIII, 3 (1984), p. 351-358

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φ-BRANCHING PROCESSES IN A RANDOM ENVIRONMENT

1. Introduction. Consider a population of identical particles with initial generation size Z_0. At unit moments, each particle splits independently of others according to the offspring distribution $\{p_k, k = 0, 1, \ldots\}$ into particles of the same type. Descendants of the initial generation create the 1-st generation of size Z_1 and, in general, the size of the n-th generation is the number of progenies of the Z_{n-1} particles of the $(n-1)$-st generation.

Consider a sequence of random variables $\{Z_n, n = 0, 1, \ldots\}$ on the probability space $(\Omega, \mathscr{F}, P)$ having non-negative integers as their state space and suppose that they satisfy the condition

$$\mathrm{E}(s^{Z_{n+1}} \mid Z_0, Z_1, \ldots, Z_n) = [f(s)]^{Z_n}, \quad n \geqslant 0, \ |s| \leqslant 1,$$

where $f(s)$ is the probability generating function of the offspring distribution $\{p_k, k = 0, 1, \ldots\}$. The sequence of random variables $\{Z_n, n = 0, 1, \ldots\}$ is called the *Galton-Watson process*. In [1] the following class of φ-branching processes has been defined.

Let φ be a function from the set of non-negative integers into itself for which $\varphi(0) = 0$. Consider a sequence of random variables $\{Z_n, n = 0, 1, \ldots\}$ having non-negative integers as their state space and suppose that the process satisfies the condition

$$\mathrm{E}(s^{Z_{n+1}} \mid Z_0, Z_1, \ldots, Z_n) = [f(s)]^{\varphi(Z_n)}, \quad n \geqslant 0, \ |s| \leqslant 1,$$

where $f(s)$ is the probability generating function of the offspring distribution $\{p_k, k = 0, 1, \ldots\}$. The process $\{Z_n, n = 0, 1, \ldots\}$ is called a *φ-branching process*, and φ is called a *controlling function*.

If $\varphi(k) < k$ for some $k \geqslant 1$, then $k - \varphi(k)$ particles in the generation of size k do not take part in evolution of the process. The condition $\varphi(k) > k$ means that we add $\varphi(k) - k$ particles to the population and $\varphi(k)$ particles take part in further evolution of the process. For $\varphi(k) = k$ ($k = 0, 1, \ldots$) we are concerned with the Galton-Watson process. In [1] the conditions sufficient for extinction or non-extinction of the φ-branching process were given.

More general processes were also considered, namely φ-branching processes with the random function φ (see [3]) and branching processes with the sequence of controlling functions (see [4]).

Now we define a branching process in a random environment. Let Π be a probability distribution on non-negative integers:

$$\Pi = \Big\{\bar{p} = \{p_i, i = 0, 1, \ldots\}, \sum_{i=1}^{\infty} ip_i < \infty, 0 \leqslant p_0 + p_1 < 1\Big\}.$$

Let $\{\xi_i, i = 0, 1, \ldots\}$ be a sequence of random variables such that $\xi_i\colon (\Omega, \mathscr{F}, P) \to (\Pi, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra in Π generated by the usual topology. The stationary and ergodic process $\xi = (\xi_0, \xi_1, \ldots)$ is called an *environmental sequence* and the random variable ξ_n is called an *environment of the n-th generation*. The function

$$f_{\xi_0}(s) = \sum_{k=0}^{\infty} p_k(\xi_0)s^k$$

is called a *probability generating function* associated with the environment ξ_0, and by $m(\xi_0) = f'_{\xi_0}(1)$ we denote the mean of the probability distribution associated with ξ_0.

Consider a sequence of random variables $\{Z_n, n = 0, 1, \ldots\}$ having non-negative integers as their state space and suppose that

$$\mathrm{E}\big(s^{Z_{n+1}} \,|\, \mathscr{F}_n(\xi)\big) = [f_{\xi_n}(s)]^{Z_n}, \qquad n \geqslant 0,\ |s| \leqslant 1,$$

where $\mathscr{F}_n(\xi)$ is the σ-algebra generated by the random variables $Z_0, Z_1, \ldots$ $\ldots, Z_n$ and the environmental sequence ξ. The process $\{Z_n, n = 0, 1, \ldots\}$ is called a *branching process* in a random environment.

Now we define the class of φ-branching processes in a random environment and we give the conditions sufficient for extinction or non-extinction of the process.

2. φ-branching processes in a random environment. Let φ be a function mapping the set of non-negative integers into itself such that $\varphi(0) = 0$. Consider a sequence of random variables $\{Z_n, n = 0, 1, \ldots\}$ with non-negative integers as their state space and suppose that they satisfy the condition

$$\mathrm{E}\big(s^{Z_{n+1}} \,|\, \mathscr{F}_n(\xi)\big) = [f_{\xi_n}(s)]^{\varphi(Z_n)}, \qquad n \geqslant 0,\ |s| \leqslant 1,$$

where $\xi = (\xi_0, \xi_1, \ldots)$ is an environmental process, and $\mathscr{F}_n(\xi)$ is the σ-algebra generated by $Z_0, Z_1, \ldots, Z_n$ and ξ. The process $\{Z_n, n = 0, 1, \ldots\}$ is called a *φ-branching process* in a random environment.

If $\varphi(k) = k$ for $k = 0, 1, \ldots$, we have an ordinary branching process in a random environment. It is easy to see that a φ-branching process has not independent lines of descendants.

Let $q_m(\xi)$ be the conditional extinction probability of a φ-branching process in a random environment, i.e.

$$q_m(\xi) = P\big(\lim_n Z_n = 0 \mid Z_0 = m, \mathscr{F}(\xi)\big),$$

where $\mathscr{F}(\xi)$ is the σ-algebra generated by ξ. The process is *extinct* if $P\big(q_m(\xi) = 1\big) = 1$ for $m \geqslant 1$. If there exists $m \geqslant 1$ such that $P\big(q_m(\xi) < 1\big) = 1$, then the process is *non-extinct*.

Now we give the conditions sufficient for extinction of φ-branching processes in a random environment.

THEOREM 1. *Let $\{Z_n, n = 0, 1, \ldots\}$ be a φ-branching process in a random environment and suppose that the controlling function satisfies the condition $\varphi(k) \leqslant ak$ for $k \geqslant 0$, where a is a non-negative real number. If $\gamma = \mathrm{E}\log am(\xi_0)$ exists and $\gamma < 0$, then the process $\{Z_n, n = 0, 1, \ldots\}$ is extinct.*

Proof. It suffices to show that

$$\underline{\lim_n} \mathrm{E}\big(Z_n \mid Z_0 = m, \mathscr{F}(\xi)\big) = 0 \text{ a.e.} \tag{1}$$

because then, by Fatou's Lemma, we have the sequence of inequalities

$$0 \leqslant \mathrm{E}\big(\underline{\lim_n} Z_n \mid Z_0 = m, \mathscr{F}(\xi)\big) \leqslant \underline{\lim_n} \mathrm{E}\big(Z_n \mid Z_0 = m, \mathscr{F}(\xi)\big) = 0 \text{ a.e.}$$

and because, by the assumption that $\varphi(0) = 0$, zero is an absorbing state of the process $\{Z_n, n = 0, 1, \ldots\}$, and

$$\{\underline{\lim_n} Z_n = 0\} \subset \{\lim_n Z_n = 0\}.$$

Hence

$$P\big(\underline{\lim_n} Z_n = 0 \mid Z_0 = m, \mathscr{F}(\xi)\big) = 1 \text{ a.e.},$$

which is equivalent to $P\big(q_m(\xi) = 1\big) = 1$ for every $m \geqslant 1$.

Now we prove equality (1) showing that it follows from the inequality

$$\mathrm{E}\big(Z_n \mid Z_0 = m, \mathscr{F}(\xi)\big) \leqslant m \prod_{i=0}^{n-1} am(\xi_i) \text{ a.e.} \quad \text{for } n \geqslant 1. \tag{2}$$

It is easy to see that the right-hand side of (2) may be written in the form

$$m \exp\Big\{\sum_{i=0}^{n-1} \log am(\xi_i)\Big\}.$$

Since the random variables $\log am(\xi_i), i = 0, 1, \ldots,$ form a stationary and ergodic sequence, γ exists and $\gamma < 0$, we have

$$\lim_n \sum_{i=0}^{n-1} \log am(\xi_i) = -\infty \text{ a.e.},$$

which together with (2) implies (1).

We prove (2) by induction. For $n = 1$ we have

$$\mathrm{E}\big(Z_1 \mid Z_0 = m, \mathscr{F}(\xi)\big) = \varphi(m)m(\xi_0) \leqslant am\cdot m(\xi_0).$$

Assume that (2) is true for $n = k$. Using the properties of a controlling function φ we have

$$\mathrm{E}\big(Z_{k+1} \mid Z_0 = m, \mathscr{F}(\xi)\big) = \mathrm{E}\{\mathrm{E}[Z_{k+1} \mid \mathscr{F}_k(\xi)] \mid Z_0 = m, \mathscr{F}(\xi)\}$$
$$= m(\xi_k)\mathrm{E}\{\varphi(Z_k) \mid Z_0 = m, \mathscr{F}(\xi)\} \leqslant m(\xi_k)\mathrm{E}\{Z_k \mid Z_0 = m, \mathscr{F}(\xi)\}a,$$

which implies (2).

Now we give the conditions sufficient for non-extinction of a φ-branching process in a random environment.

THEOREM 2. *Let $\{Z_n, n = 0, 1, \ldots\}$ be a φ-branching process in a random environment and suppose that the controlling function satisfies the condition*

$$\text{(3)} \qquad \varphi(k) \geqslant ak \text{ for } k = 0, 1, \ldots, \quad \varphi(0) = 0,$$

where a is a positive real number. If

$$\gamma = \mathrm{E}\log am(\xi_0) > 0 \quad \text{and} \quad \mathrm{E}\big(-\log(1-f_{\xi_0}(0))\big) < \infty,$$

then $P\big(q_m(\xi) < 1\big) = 1$ for every $m \geqslant 1$.

The following lemmas are needed in the proof of Theorem 2.

LEMMA 1. *If the controlling function of a φ-branching process in a random environment satisfies* (3), *then the following inequality holds*:

$$\text{(4)} \quad \mathrm{E}\big(s^{Z_n} \mid Z_0 = m, \mathscr{F}(\xi)\big) \leqslant \Big(f^a_{\xi_0}\big(f^a_{\xi_1}(\ldots f^a_{\xi_{n-1}}(s) \ldots)\big)\Big)^m \text{ a.e.} \quad \text{for } n \geqslant 1.$$

Proof. We prove this inequality by induction. For $n = 1$ we have

$$\mathrm{E}\big(s^{Z_1} \mid Z_0 = m, \mathscr{F}(\xi)\big) = f_{\xi_0}(s)^{\varphi(m)} \leqslant \big(f^a_{\xi_0}(s)\big)^m.$$

Assume that inequality (4) is true for $n \geqslant 1$. Then

$$\mathrm{E}\big(s^{Z_{n+1}} \mid Z_0 = m, \mathscr{F}(\xi)\big) = \mathrm{E}\big\{\mathrm{E}\big(s^{Z_{n+1}} \mid \mathscr{F}_n(\xi)\big) \mid Z_0 = m, \mathscr{F}(\xi)\big\}$$
$$\leqslant \mathrm{E}\{f_{\xi_n}(s)^{\varphi(Z_n)} \mid Z_0 = m, \mathscr{F}(\xi)\}.$$

The last inequality follows from (3). This and the induction assumption imply (4) for every $n \geqslant 1$.

LEMMA 2. *If a φ-branching process $\{Z_n, n = 0, 1, \ldots\}$ in a random environment satisfies the conditions of Theorem 2, then*

$$\mathrm{E}\left(-\log\left(1-f_{\xi_0}^{a}(0)\right)\right) < \infty.$$

Proof. We consider two cases: $a \geqslant 1$ and $a < 1$.

If $a \geqslant 1$, then $1-f_{\xi_0}^{a}(0) \geqslant 1-f_{\xi_0}(0)$. Consequently, we have

$$\mathrm{E}\left(-\log\left(1-f_{\xi_0}^{a}(0)\right)\right) \leqslant \mathrm{E}\left(-\log\left(1-f_{\xi_0}(0)\right)\right) < \infty.$$

The last inequality follows from the assumptions of Lemma 2.

If $a < 1$, then $1-f_{\xi_0}^{a}(0) \geqslant a\left(1-f_{\xi_0}(0)\right)$. Thus

$$\mathrm{E}\left(-\log\left(1-f_{\xi_0}^{a}(0)\right)\right) \leqslant -\log a + \mathrm{E}\left(-\log\left(1-f_{\xi_0}(0)\right)\right) < \infty,$$

which completes the proof of Lemma 2.

LEMMA 3. *If a φ-branching process $\{Z_n, n = 0, 1, \ldots\}$ in a random environment satisfies the assumptions of Theorem 2, then*

$$\gamma_n = \mathrm{E}\left(-\log\frac{1-f_{\xi_0}^{a}\left(Y_{n-1}(T\xi)\right)}{1-Y_{n-1}(T\xi)}\right) < \infty$$

for every $n \geqslant 1$, where $Y_n(\xi) = f_{\xi_0}^{a}\left(f_{\xi_1}^{a}\left(\ldots f_{\xi_n}^{a}(0) \ldots\right)\right)$ and T is a shift operator, $T(\xi_0, \xi_1, \ldots) = (\xi_1, \xi_2, \ldots)$.

Proof. Since $\left(1-f_{\xi_0}^{a}(s)\right)/(1-s)$ is an increasing function in $s \in (0, 1)$, we have

$$am(\xi_0) \geqslant \frac{1-f_{\xi_0}^{a}\left(Y_{n-1}(T\xi)\right)}{1-Y_{n-1}(T\xi)} \geqslant 1-f_{\xi_0}^{a}(0).$$

Thus

$$(5)\qquad -\log am(\xi_0) \leqslant -\log\left(\frac{1-f_{\xi_0}^{a}\left(Y_{n-1}(T\xi)\right)}{1-Y_{n-1}(T\xi)}\right) \leqslant -\log\left(1-f_{\xi_0}^{a}(0)\right).$$

By (5) and Lemma 2 we have Lemma 3.

Proof of Theorem 2. Assume that $P\left(q_m(\xi) = 1\right) > 0$ for some $m \geqslant 1$. By Lemma 1 we have

$$(6)\qquad P\left(Z_n = 0 \mid Z_0 = m, \mathscr{F}(\xi)\right) \leqslant \left(f_{\xi_0}^{a}\left(f_{\xi_1}^{a}\left(\ldots f_{\xi_{n-1}}^{a}(0) \ldots\right)\right)\right)^m.$$

It is easy to see that the sequence $Y_n(\xi) = f_{\xi_0}^{a}\left(f_{\xi_1}^{a}\left(\ldots f_{\xi_n}^{a}(0) \ldots\right)\right)^m$ is a.e. increasing and bounded by 1, and so its limit exists a.e. Denote this limit by $Y(\xi)$. By (6) and

$$q_m(\xi) = \lim_n P\left(Z_n = 0 \mid Z_0 = m, \mathscr{F}(\xi)\right)$$

we get $P(Y(\xi) = 1) > 0$. Now, from the equality

$$Y_n(\xi) = f^a_{\xi_0}(Y_{n-1}(T\xi)) \tag{7}$$

we obtain $Y(\xi) = f^a_{\xi_0}(Y(T\xi))$. It is easy to see that the event $\{\xi, Y(\xi) = 1\}$ is shift invariant, and T is measure preserving and ergodic. Thus $P(Y(\xi) = 1) = 0$ or 1, and because $P(Y(\xi) = 1) > 0$, we have $P(Y(\xi) = 1) = 1$.

From the definition of the set Π it is clear that

$$P(Y_{n-1}(\xi) < 1 \text{ for every } n \geqslant 1) = 1.$$

Thus $0 \leqslant -\log(1 - Y_n(\xi)) < \infty$ a.e. for every $n \geqslant 0$. Let us put

$$\mu_n = \mathrm{E}\left(-\log(1 - Y_n(\xi))\right).$$

We show that $\mu_n < \infty$ for every $n \geqslant 0$. By Lemma 2, $\mu_0 < \infty$. Now from (7) we get the equality

$$-\log(1 - Y_n(\xi)) = -\log\left(\frac{1 - f^a_{\xi_0}(Y_{n-1}(T\xi))}{1 - Y_{n-1}(T\xi)}\right) - \log(1 - Y_{n-1}(T\xi)). \tag{8}$$

Integrating (8) we have

$$\mu_n = \sum_{j=1}^{n} \gamma_j + \mu_0, \quad n \geqslant 1, \tag{9}$$

which together with Lemma 3 implies $\mu_n < \infty$ for every $n \geqslant 0$.

On the other hand, $Y_{n-1}(T\xi) \uparrow 1$ a.e. Thus, by the Lebesgue monotone convergence theorem,

$$\mu_n \uparrow \infty \quad \text{and} \quad \gamma_n \downarrow \gamma = \mathrm{E}\left(-\log am(\xi_0)\right) < 0 \tag{10}$$

and, consequently,

$$\lim_n \sum_{j=1}^{n} \gamma_j = -\infty,$$

which contradicts (9) and (10).

COROLLARY 1. *If the controlling function of a φ-branching process $\{Z_n, n = 0, 1, \ldots\}$ in a random environment satisfies the condition $\varphi(k) = k$ for $k = 0, 1, \ldots$, then from Theorems* 1 *and* 2 *we get well-known results for an ordinary branching process in a random environment* (see [2]).

Now we give an example in which we apply the properties of a φ-branching process in a random environment.

3. Example. Consider a φ-branching process $\{Z_n, n = 0, 1, \ldots\}$ in which the controlment by the function φ takes place only at random moments such that $\tau_0 = 0$ and random variables $T_{n-1} = \tau_n - \tau_{n-1}$, $n \geqslant 1$, form a stationary and ergodic sequence independent of the process $\{Z_n, n = 0, 1, \ldots\}$. Assume that $P(T_0 = 0) = 0$, $\mathrm{E}(T_0) < \infty$, and the generating function $f(s)$ of the offspring distribution $\{p_k, k = 0, 1, \ldots\}$ satisfies the conditions

$$f'(1-) = m < \infty, \quad 0 \leqslant f(0) + f'(0) < 1.$$

Let $\mathscr{E}$ be the σ-algebra generated by the sequence of $\{T_n, n = 0, 1, \ldots\}$. Between the moments τ_n and τ_{n+1} the process $\{Z_n, n = 0, 1, \ldots\}$ evolves as an ordinary Galton-Watson process. We give the conditions sufficient for extinction of such a process.

THEOREM 3. *Assume that the controlling function φ of the process satisfies the condition $\varphi(k) \leqslant ak$ for $k \geqslant 0$, where a is a non-negative real number. If*

$$\gamma = \log a + \mathrm{E}(T_0)\log m < 0 \quad \textit{and} \quad f(0) > 0,$$

then

$$P(\lim_n Z_n = 0 \mid Z_0 = l, \mathscr{E}) = 1 \ \textit{a.e.} \quad \textit{for every } l \geqslant 1.$$

Proof. Consider the process Z_{τ_i} conditioned on the event $Z_0 = l$ and the σ-algebra $\mathscr{E}$. It is easy to see that

$$\mathrm{E}(s^{Z_{\tau_{i+1}}} \mid Z_0, Z_1, \ldots, Z_{\tau_i}, \mathscr{E}) = [f_{T_i}(s)]^{\varphi(Z_{\tau_i})}, \quad i \geqslant 0, \ |s| \leqslant 1.$$

Hence $\{Z_{\tau_i}, i = 0, 1, \ldots\}$ is a φ-branching process in a random environment. From the equality

$$m(T_0) = f'_{T_0}(1-) = m^{T_0}$$

it follows that the assumption

$$\mathrm{E}\log am(T_0) < 0 \tag{11}$$

is equivalent to the condition $\log a + \mathrm{E}(T_0)\log m < 0$. Now, by (11) and Theorem 1, we have

$$P(\lim_n Z_{\tau_n} = 0 \mid Z_0 = l, \mathscr{E}) = 1 \text{ a.e.}$$

Since $\tau_i \to \infty$ a.e. and zero is an absorbing state of the process $\{Z_n, n = 0, 1, \ldots\}$, we obtain

$$1 = P(\lim_i Z_{\tau_i} = 0 \mid Z_0 = l, \mathscr{E}) \leqslant P(\lim_n Z_n = 0 \mid Z_0 = l, \mathscr{E}),$$

which completes the proof.

In a similar way we can obtain the conditions sufficient for non-extinction of the process $\{Z_n, n = 0, 1, \ldots\}$.

THEOREM 4. *Assume that the controlling function* φ *of the process* $\{Z_n, n = 0, 1, \ldots\}$ *satisfies the condition* $\varphi(k) \geqslant ak$ *for* $k = 1, 2, \ldots, \varphi(0) = 0$, *where* a *is a positive real number. If*

$$\gamma = \log a + \mathrm{E}(T_0)\log m > 0 \quad \text{and} \quad \mathrm{E}\big(-\log(1-f_{T_0}(0))\big) < \infty,$$

then

$$P(\lim_n Z_n = 0 \mid Z_0 = l, \mathscr{E}) < 1 \text{ a.e.} \quad \text{for every } l \geqslant 1.$$

Proof. Since the hypothesis $\gamma = \log a + \mathrm{E}(T_0)\log m > 0$ is equivalent to $\mathrm{E}\log am(T_0) > 0$, from Theorem 2 we have

$$P(\lim_i Z_{\tau_i} = 0 \mid Z_0 = l, \mathscr{E}) < 1 \text{ a.e.}$$

Theorem 4 follows now from the inequality

$$P(\lim_n Z_n = 0 \mid Z_0 = l, \mathscr{E}) \leqslant P(\lim_i Z_{\tau_i} = 0 \mid Z_0 = l, \mathscr{E}).$$

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Received on 25. 2. 1982