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OPTIMAL CONTROL IN LINEAR SYSTEMS
WITHOUT THE CONDITION OF REGULAR CONTROLLABILITY

1. Introduction. In this paper we consider a linear dynamical system described by the set of n first-order linear differential equations

$$(1) \qquad \dot{x}(t) = Ax(t) + Bv(t), \qquad v(t) \in W \text{ for } t \in [t_0, T],$$

where $A: R^n \to R^n$ and $B: R^m \to R^n$ are matrices with constant elements. The counterdomain W of the control functions (i.e. the m-dimensional convex polyhedron in the space R^m) contains the origin. The functions $v(t)$ are assumed to be piecewise continuous functions defined for $t_0 \leqslant t \leqslant T$ and to satisfy the condition $v(t+) = v(t)$. These functions form the set of admissible controls.

We may formulate the problem of time-optimal controllability in the following manner: from the whole set of the admissible controls driving the object (1) from x_0 to the origin we have to find such a one for which the driving time is the shortest one.

In considerations connected with that problem, the regular control condition is usually assumed, i.e.: for each vector q, parallel to a certain edge of the polyhedron W, the vector Bq is not an element of any proper subspace invariant with respect to the operator A. This assumption gives the sufficiency condition of the Pontryagin maximum principle for the processes controlled to the origin, and allows us to prove the theorem on the uniqueness of the optimal control, the theorem on full controllability and other important properties of the optimal processes (see [1]).

The purpose of the present paper consists in developing an appropriate theory without the condition of regular controllability. We intend to give a general characterization of the optimal control driving the system from $x_0 = x(t_0)$ to the origin.

2. Formalism. Let us introduce the function $\Psi(\cdot)$ as an arbitrary non-zero solution of the homogeneous differential equation

$$(2) \qquad \dot{\Psi}(t) = -A^T \Psi(t), \qquad t \in R.$$

For every $t \in R$ the function

$$M_{\Psi}(\cdot) = \max_{w \in W} \left(\Psi(\cdot), Bw \right)$$

is defined in a unique way.

Let $B(R^m)$ be a family of all closed, bounded, non-empty subsets of the space R^m. It is obvious (cf., e.g., [2]) that on $B(R^m)$ one can introduce the Hausdorff metric. The multivalued function $V(\cdot)$, $V: R \to B(R^m)$, which satisfies the equation

(3) $$M_{\Psi}(t) = \left(\Psi(t), BV(t) \right), \quad t \in R,$$

is called an *extremal function* corresponding to $\Psi(\cdot)$. Note that under the above assumptions the relation (3) determines $V(t)$ in a unique way for every $t \in [t_0, T]$. Moreover, $V(\cdot)$ is a piecewise constant function and the counterdomain of this function is contained in the set of faces of the polyhedron W. These properties are a consequence of the following lemmas:

LEMMA 1. *The function* $M_{\Psi}(t)$ *is continuous for* $t \in R$.

For the proof of Lemma 1 see [5].

LEMMA 2. *The extremal function* $V(\cdot)$ *corresponding to* $\Psi(\cdot)$ *is continuous on the interval* $[t_0, T]$ *with exception of at most a few isolated points.*

Proof. We suppose that for an infinite number of points $t_n \in [t_0, T]$, $n \in N$, the function $V(\cdot)$, determined by (3), is not continuous. From general properties of continuous functions in metric spaces (cf. [4]) it follows that for every t_n there exists a sequence $\{t_{ni}\}$ which satisfies the conditions

$$t_{ni} \xrightarrow{i} t_n, \quad V(t_{ni}) \xrightarrow{i} W^n \neq V(t_n).$$

Since we deal with a finite number of polyhedron faces, we can choose from the sequence $\{t_n\}$ the subsequence $\{t_n^*\}$ and appropriate faces W and W^* which have the following property for $n \in N$:

(4) $$t_{ni}^* \xrightarrow{i} t_n^*, \quad V(t_{ni}^*) = W, \quad V(t_n^*) = W^*, \quad W \neq W^*.$$

By Lemma 1, for every t_n^* we obtain

$$\left(\Psi(t_n^*), BW^* \right) = \left(\Psi(t_n^*), BV(t_n^*) \right) = M_{\Psi}(t_n^*) = \lim_i M_{\Psi}(t_{ni}^*)$$

$$= \lim_i \left(\Psi(t_{ni}^*), BV(t_{ni}^*) \right) = \left(\Psi(t_n^*), BW \right).$$

Since the function $\Psi(\cdot)$ is analytic, for every $t \in R$ we get $\left(\Psi(t), BW^* \right)$ $= \left(\Psi(t), BW \right)$. Then, from the uniqueness of the function $V(\cdot)$ determined by formula (3) it follows that $W \subset W^*$ because

$$\max_{w \in W} \left(\Psi(t_n^*), Bw \right) = \left(\Psi(t_n^*), BW^* \right).$$

On the other hand, the equation

$$\max_{w \in W} \left(\Psi(t_{ni}^*), Bw \right) = \left(\Psi(t_{ni}^*), BW \right)$$

is fulfilled, which implies, in an analogical way as above, that $W^* \subset W$.

Hence $W^* = W$, which contradicts condition (4) and completes our proof.

The above-mentioned isolated points divide the time interval $[t_0, T]$ into a finite number of intervals (τ_{i-1}, τ_i), $i = 1, 2, ..., k$, $\tau_0 = t_0$, $\tau_k = T$. In each subinterval the function $V(\cdot)$ is constant: $V(t) = W^i$ for $t \in (\tau_{i-1}, \tau_i)$, where W^i is a face of the polyhedron W.

We emphasize that in the obtained sequence of faces W^1, W^2, ..., W^k any two neighbouring faces are disjoint, i.e. we have $W^{i-1} \cap W^i = \varnothing$ for $i = 2, 3, ..., k$.

An arbitrary one-valued function $v(\cdot)$, being a selection of the function $V(\cdot)$, is called an *extremal control* corresponding to $\Psi(\cdot)$ or, in other words, it is the control which satisfies the Pontryagin maximum principle. Note that only piecewise continuous controls have been considered.

3. Results. In this section we discuss our result via a few theorems. In the first theorem we show that the Pontryagin maximum principle is a sufficient condition of the optimality of the extremal control corresponding to $\Psi(\cdot)$ which is a selection of such an extremal function $V(\cdot)$ for which $0 \notin BW^k$. We remind that $E^\perp \left(\Psi(\cdot) \right)$ means the orthogonal complement of the space $E \left(\Psi(\cdot) \right)$ generated by $\Psi(\cdot)$.

THEOREM 1. *Let $\Psi(\cdot)$ be a non-zero solution of* (2) *and let $V(\cdot)$ be such an extremal function corresponding to $\Psi(\cdot)$ determined by* (3) *for which $0 \notin BW^k$. If $v(t)$, $t \in [t_0, T]$, is the extremal control corresponding to $\Psi(t)$, which drives the system from $x_0 = x(t_0)$ to $x_1 = x(T) \in E^\perp \left(\Psi(t) \right)$, then $v(t)$ is optimal.*

Proof. By assumption, on the interval $[t_0, T]$ we have

$$\max_{w \in W} \left(\Psi(t), Bw \right) = \left(\Psi(t), Bv(t) \right) \geqslant 0,$$

and $\left(\Psi(t), Bv(t) \right) > 0$ for $t \in [\tau_{k-1}, T]$ because $0 \notin BW^k$. The control $v(\cdot)$ determines a trajectory $x(\cdot)$ such that $x(t_0) = x_0$ and $x(T) = x_1$. We suppose now that our control $v(\cdot)$ is not optimal. Then, there exists an admissible control $\hat{v}(\cdot)$ which drives the system from x_0 at $t = t_0$ to x_1 at $t = T_1 < T$ along the trajectory $\hat{x}(\cdot)$. By the maximum principle we have

(5)
$$\left(\Psi(t), v(t) \right) \geqslant \left(\Psi(t), \hat{v}(t) \right), \qquad t \in [t_0, T],$$

and the following equations are also satisfied:

$$\left(\Psi(T), x(T) \right) - \left(\Psi(t_0), x(t_0) \right) = \int_{t_0}^{T} \left(\Psi(s), Bv(s) \right) ds,$$

$$\big(\varPsi(T_1),\,\hat{x}(T_1)\big)-\big(\varPsi(t_0),\,\hat{x}(t_0)\big)\;=\int\limits_{t_0}^{T_1}\big(\varPsi(s),\,Bv(s)\big)ds\,.$$

Hence

$$\int\limits_{t_0}^{T}\big(\varPsi(s),\,Bv(s)\big)ds-\int\limits_{t_0}^{T_1}\big(\varPsi(s),\,B\hat{v}(s)\big)ds\;=\;0$$

and, since $T_1 < T$, we obtain

$$(6)\qquad\int\limits_{t_0}^{T_1}\big(\varPsi(s),\,Bv(s)-B\hat{v}(s)\big)ds+\int\limits_{T_1}^{T}\big(\varPsi(s),\,Bv(s)\big)ds=0.$$

This equation cannot be satisfied because of inequality (5) and the assumption $0 \notin BW^k$. Therefore, the control $v(t)$ is optimal.

Let us consider now the optimality problem of the extremal control being a selection of such a function $V(\cdot)$ for which $0 \in BW^k$.

THEOREM 2. *Let* $\varPsi(\cdot)$ *be a non-zero solution of* (2), *let* $V(\cdot)$ *be such an extremal function corresponding to* $\varPsi(\cdot)$ *for which* $0 \in BW^k$ *and let* $v(t)$, $t \in [t_0, T]$, *be the extremal control corresponding to* $\varPsi(t)$, *driving the system from* $x_0 = x(t_0)$ *to* $x_1 = x(T) \in E^{\perp}\big(\varPsi(t)\big)$. *If the control* $\hat{v}(t)$, $t \in [t_0, T_1]$, *driving the system from* x_0 *to* x_1 *is optimal, then* $\tau_{k-1} \leqslant T_1 \leqslant T$ *and* $\hat{v}(t)$ *is the extremal control corresponding to* $\varPsi(t)$.

Proof. We have to consider only the case where $v(\cdot)$ is not an optimal control. Let us assume that $\hat{v}(t)$, $t \in [t_0, T_1]$, is the optimal control driving the system from x_0 to x_1 in time $T_1 - t_0 < T - t_0$. Then, in the same manner as in the proof of Theorem 1, one can be assured that equation (6) has to be satisfied. It is clear that $T_1 \geqslant \tau_{k-1}$; otherwise we have $\big(\varPsi(t),\,Bv(t)\big) > 0$, $t \in [T_1, \tau_{k-1}]$, which contradicts equation (6). For $T_1 \geqslant \tau_{k-1}$, in view of (6) and of the condition $0 \in BW^k$, we obtain

$$\big(\varPsi(t),\,Bv(t)-B\hat{v}(t)\big)\;=\;0,\qquad t \in [t_0, T_1].$$

Hence $\hat{v}(t)$ is the extremal control corresponding to $\varPsi(t)$.

From Theorem 2 it follows that in order to solve the optimality problem formulated by (1) we can confine ourselves to the extremal controls corresponding only to the function $\varPsi(\cdot)$.

It is clear that an arbitrary extremal control corresponding to $\varPsi(\cdot)$ and driving the system from x_0 at $t = t_0$ to the origin at $t = \tau_{k-1}$ is optimal. Let us answer now the question which of the other extremal controls corresponding to $\varPsi(\cdot)$ are also optimal.

Assuming $BW^k = 0$, we infer that, for an arbitrary extremal control driving the system from x_0 to the origin along the trajectory $x(\cdot)$, the equality $x(\tau_{k-1}) = 0$ holds. Otherwise, the trajectory $x(\cdot)$, as a solution of the homogeneous differential equation $\dot{x}(t) = Ax(t)$, $t \in [\tau_{k-1}, T]$,

with the condition $x(\tau_{k-1}) \neq 0$, would never attain the origin in finite time. Therefore, in the case where $BW^k = 0$, an arbitrary extremal control corresponding to the function $\Psi(\cdot)$ drives the system from x_0 to the origin in time $\tau_{k-1} - t_0$ and is thereby an optimal one.

If $BW^k \neq 0$, then under the assumption $0 \in BW^k$ we have

$$\max_{w \in W} \big(\Psi(t), Bw\big) = \big(\Psi(t), BW^k\big) = 0 \quad \text{for } t \in [\tau_{k-1}, T],$$

whence $BW^k \subset E^{\perp}\big(\Psi(\cdot)\big)$. Note that $E^{\perp}\big(\Psi(\cdot)\big)$, as the orthogonal complement of $E\big(\Psi(\cdot)\big)$, is the subspace invariant with respect to the operator A. By the properties of invariant spaces, all trajectories, driving the system to the origin and determined by controls contained in W^k, lie in $E^{\perp}\big(\Psi(\cdot)\big)$ (cf. [3]). Therefore, the extremal control corresponding to $\Psi(\cdot)$ drives the system from x_0 to the origin along the trajectory $x(\cdot)$, $x(t) \in E^{\perp}\big(\Psi(\cdot)\big)$ for $t \in [\tau_{k-1}, T]$. Since this control, by Theorem 2, drives the system from x_0 to $x(\tau_{k-1}) \in E^{\perp}\big(\Psi(\cdot)\big)$ in the shortest time $\tau_{k-1} - t_0$, we can restrict the problem (1) for $t \in [\tau_{k-1}, T]$ to the space $E^{\perp}\big(\Psi(t)\big)$ and to the polyhedron $W_1 = W^k$, $BW_1 \subset E^{\perp}\big(\Psi(t)\big)$.

We note that $A_1 = A\big|_{E^{\perp}(\Psi(\cdot))}$. The dynamical system is described now by the equation

$$\dot{y}(t) = A_1 y(t) + Bv(t), \quad v(t) \in W_1 \text{ for } t \in [\tau_{k-1}, T],$$

where $A_1 : E^{\perp}\big(\Psi(\cdot)\big) \to E^{\perp}\big(\Psi(\cdot)\big)$ and $B : R^m \to E^{\perp}\big(\Psi(\cdot)\big)$ are matrices with constant elements.

Let $\Psi_1(\cdot)$ be a non-zero solution of the equation

$$\dot{\Psi}_1(t) = -A_1^{\mathrm{T}} \Psi_1(t), \quad t \in R,$$

and let $V_1(\cdot)$ be the extremal function corresponding to $\Psi_1(\cdot)$ determined by the equation

$$M_{\Psi_1}(t) = \big(\Psi_1(t), BV_1(t)\big), \quad t \in R.$$

The obtained function $V_1(\cdot)$ is a multivalued selection of $V(\cdot)$ and satisfies the following conditions: $V_1(t) = V(t)$ for $t \in [t_0, \tau_{k-1}]$ and $V_1(t) \subset V(t)$ for $t \in [\tau_{k-1}, T]$.

Note that $V_1(\cdot)$ is a piecewise constant function: $V_1(t) = W^i$ for $t \in (\tau_{i-1}, \tau_i)$, $i = 1, 2, \ldots, k-1$, and $V_1(t) = W_1^i$ for $t \in (\tau_{1\,i-1}, \tau_{1i})$, $i = 1, 2, \ldots, p$, $\tau_{10} = \tau_{i-1}$, $\tau_{1p} = T$, where W_1^i is a face of the polyhedron $W_1 = W^k$.

In an analogical way we consider also the set BW_1^p. If $0 \notin BW_1^p$ or $BW_1^p = 0$, then, as will be proved afterwards (cf. Theorem 3), an arbitrary

extremal control corresponding to $\Psi_1(\cdot)$ and driving the system from x_0 to the origin in time $T - t_0$ is optimal.

If $0 \in BW_1^p$, then the assumptions of Theorem 2 are fulfilled in the space $E^\perp(\Psi(\cdot))$ and we have to repeat the above considerations confining ourselves to the subspace $E^\perp(\Psi_1(t))$ for $t \in [\tau_{1\,p-1}, T]$ and to the polyhedron $W_2 = W_1^p$, $BW_2 \subset E^\perp(\Psi_1(\cdot))$.

This procedure has to be finished after a finite number of steps (e.g., s), which is clear in view of the finite dimension of the space appearing in the problem determined by (1).

As a final effect we obtain the extremal function $V_s(\cdot)$ corresponding to $\Psi_s(\cdot)$ for which all one-valued piecewise selections, i.e. the extremal controls driving the system from x_0 to the origin, are optimal. The function $V_s(\cdot)$ is a selection of the extremal function $V(\cdot)$ obtained in the above-described manner. Hence we have $V_s(t) = V_{s-1}(t)$ for $t \in [t_0, \tau_{s\,r-1}]$ (here we use the notation $V_0(t) = V(t)$), $V_s(t) \subset V_{s-1}(t)$ for $t \in [\tau_{s\,r-1}, T]$ and $V_s(t) = W_s^i$ for $t \in (\tau_{s\,i-1}, \tau_{si})$, $i = 1, 2, \ldots, l$, $\tau_{s0} = \tau_{s\,r-1}$, $\tau_{sl} = T$, where W_s^i is a face of the polyhedron $W_s = W_{s-1}^r$.

Moreover, $0 \notin BW_s^l$ or $BW_s^l = 0$, which is consistent with our procedure.

THEOREM 3. *Let $\Psi(\cdot)$ be a non-zero solution of* (2), *let $V(\cdot)$ be the extremal function corresponding to $\Psi(\cdot)$ for which $0 \in BW^k$ and let $v(t)$, $t \in [t_0, T]$, be the extremal control corresponding to $\Psi(t)$, driving the system from $x_0 = x(t_0)$ to the origin. If $v(t)$ is the extremal control corresponding to $\Psi_s(t)$ for $t \in [\tau_{s\,l-1}, T]$, i.e. a selection of the extremal function $V_s(t)$ for $t \in [t_0, T]$, then it is optimal.*

Proof. We suppose that our control is not optimal. Then there exists an admissible control $\hat{v}(\cdot)$ which drives the system from $x_0 = \hat{x}(t_0)$ to the origin, $\hat{x}(T_1) = 0$, $T_1 < T$, along the trajectory $\hat{x}(\cdot)$.

Equation (6) is satisfied, which can be shown in a similar way as in the proof of Theorem 1. On the other hand, this equation cannot be satisfied under the assumption $0 \notin BW_s^l$ or $BW_s^l = 0$ and by virtue of Theorem 2. Therefore, the control $v(t)$ is optimal, which completes the proof.

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STEROWANIA OPTYMALNE W LINIOWYCH UKŁADACH
NIE SPEŁNIAJĄCYCH WARUNKU REGULARNEJ STEROWALNOŚCI

STRESZCZENIE

W pracy podano charakterystykę sterowań optymalnych w liniowych układach nie spełniających warunku regularnej sterowalności dla procesów przeprowadzających obiekt z danego położenia początkowego do położenia równowagi. Rozważając wszystkie przypadki sterowań ekstremalnych, pokazano, w których z nich zasada maksimum Pontriagina jest warunkiem dostatecznym optymalności bez założenia regularnej sterowalności.