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BREAKDOWN TIME DISTRIBUTIONS OF SYSTEMS IN SERIES

0. In the first part of this paper, due to the first author, the breakdown time distributions of systems consisting of 2, 3, 4, and 5 facilities having independent breakdown processes with exponential distribution of working and breakdown times are considered. The results of [3], where it has been proved that the properly normalized distributions of working and breakdown times tend with an infinitely increasing number of facilities to the exponential distribution, are here strengthened. It is shown in this paper that the differences between the limit distribution and the exact distributions for systems consisting of a small number of facilities are relatively large. In the second part of the paper, due to both authors, the Monte Carlo method has been used for calculating the distributions of breakdown times in systems which consists of 2 and 4 facilities with exponential working time distributions and exponential or deterministic breakdown time distributions. A comparison of the results obtained in this way with those from theory indicates that Monte Carlo methods may be with good results used in the dealt with problem. All calculations have been made by the second author on the digital computer ODRA 1003 installed in the Department of Statistics at the Higher School of Economics in Wrocław, Poland.

1. Consider n facilities in series and let $a^{(k)}(t)$ denotes the breakdown process of facility number k. Under a breakdown process of a facility we understand a zero-one valued stochastic process for which the working times (i.e. the lengths of consecutive intervals for which $a^{(k)}(t) = 1$) are identically distributed independent random variables, the breakdown times (i.e. the lengths of consecutive intervals for which $a^{(k)}(t) = 0$) are also identically distributed independent random variables, and the breakdown and working times are independent of each other.

The breakdown process of facilities connected in series is defined as the product

$$(1) \qquad a_n(t) = \prod_{k=1}^{n} a^{(k)}(t),$$

and the breakdown process of facilities connected in parallel is defined as

(2) $$\beta_n(t) = 1-\prod_{k=1}^{n}\left(1-a^{(k)}(t)\right).$$

It is worth noticing that one obtains the process $\beta_n(t)$ by interchanging the terms "working" and "breakdown" in a system of facilities in series. The breakdown time distributions found in the paper are thus working time distributions for systems in parallel.

It has been proved in [3] that if in each process $a^{(k)}(t)$ the working time has an exponential distribution with parameter λ, the breakdown time has an exponential distribution with parameter μ, and the processes $a^{(k)}(t)$ are independent then in the breakdown process $a_n(t)$ of the system the working time is exponentially distributed with parameter $n\lambda$ and the breakdown time Y_n is a random variable such that the Laplace transform $g_n^*(s)$ of its density satisfies the equation

$$\frac{1}{s+n\lambda\left(1-g_n^*(s)\right)} = \sum_{k=0}^{n}\binom{n}{k}p^{n-k}q^k\frac{1}{s+ak}, \quad a = \lambda+\mu,$$

$$p = \frac{\mu}{\lambda+\mu}, \qquad q = 1-p.$$

We have thus

(3) $$\frac{1-g_n^*(s)}{s} = \frac{\sum_{k=0}^{n-1}\binom{n-1}{k}p^{n-1-k}q^k\dfrac{1}{s+(k+1)a}}{\sum_{k=0}^{n}\binom{n}{k}p^{n-k}q^k\dfrac{s}{s+ka}}.$$

From (3) it is easy to calculate the moments of the breakdown times

(4) $$EY_n = \left.\frac{1-g_n^*(s)}{s}\right|_{s=0} = \frac{1}{n\lambda}(p^{-n}-1),$$

(5) $$EY_n^2 = 2\,\frac{EY_n-\dfrac{1-g_n^*(s)}{s}}{s} = \frac{2}{n\lambda a p^n}\sum_{k=1}^{n}\binom{n}{k}\left(\frac{q}{p}\right)^k\frac{1}{k}.$$

It has been also shown in [3] that the normalized breakdown time

(6) $$\tilde{Y}_n = Y_n/EY$$

is a random variable whose distribution tends to the exponential distribution with parameter one if n tends to infinity. The rate of convergence may be measured by the rate of convergence of the variance of $\tilde{Y}_n$ to unity. For the particular case $\lambda = \mu = 1$, thus for $p = q = 1/2$ we have

$$EY_n = \frac{2^n-1}{n},$$

$$D^2\tilde{Y}_n = \frac{n2^n \sum_{k=1}^{n} \binom{n}{k} \frac{1}{k}}{(2^n-1)^2} - 1.$$

The values of those quantities for different n are given in Table 1. From the table one can see that the exponential breakdown time of n facilities in series rapidly increases with n and also that the variance of the normalized breakdown time tends to unity at a very slow rate.

TABLE 1. Expected values and variances of the normalized breakdown time

n	EY_n	$D^2\tilde{Y}_n$
1	1.000	1.000
2	1.500	1.222
3	2.333	1.367
4	3.750	1.441
5	6.200	1.461
6	10.500	1.446
7	18.143	1.412
8	31.875	1.372
9	56.778	1.331
10	102.300	1.293
15	2184.467	1.173
20	52428.750	1.120

The exact distributions of Y_n may be found from (3). For $\lambda = \mu = 1$ we calculate easily

$$\frac{1-g_n^*(s)}{s} = \frac{2\sum_{k=0}^{n-1}\binom{n-1}{k}(s+2)\dots(s+2k)\big(s+2(k+2)\big)\dots(s+2n)}{\sum_{k=0}^{n}\binom{n}{k}s(s+2)\dots\big(s+2(k-1)\big)\big(s+2(k+1)\big)\dots(s+2n)}$$

from which we obtain

$$g_2^*(s) = \frac{s+2}{s^2+4s+2} \approx \frac{s+2}{(s+0{,}5858)(s+3{,}4142)}, \tag{7}$$

$$g_3^*(s) = \frac{s^2+6s+6}{s^3+9s^2+20s+6} \approx \frac{s^2+6s+6}{(s+0{,}3542)(s+3)(s+5{,}6458)}, \tag{8}$$

$$g_4^*(s) = \frac{s^3+12s^2+38s+24}{s^4+16s^3+80s^2+128s+24} \tag{9}$$

$$\approx \frac{s^3+12s^2+38s+24}{(s+0{,}2152)(s+2{,}7056)(s+5{,}2943)(s+7{,}7848)},$$

$$g_5^*(s) = \frac{s^4+20s^3+128s^2+280s+120}{s^5+25s^4+220s^3+800s^2+1024s+120} \tag{10}$$

$$\approx \frac{s^4+20s^3+128s^2+280s+120}{(s+0{,}1299)(s+2{,}4936)(s+5)(s+7{,}5064)(s+9{,}8701)}.$$

It is known (see [2], p. 148) that if $g_n^*(s)$ is a rational function

$$g_n^*(s) = \frac{p_n(s)}{q_n(s)},$$

where $q_n(s) = (s-s_1)(s-s_2)\ldots(s-s_n)$ and $s_i \neq s_j$ for $i \neq j$, then $g_n^*(s)$ is the Laplace transform of the function

$$g_n(y) = \sum_{k=1}^{n} \frac{p_n(s_k)}{q_n'(s_k)} e^{s_k y}. \tag{11}$$

We are interested in the distributions $\tilde{g}_n(y)$ of the normalized random variable $\tilde{Y}_n$. We have, of course,

$$\tilde{g}_n^*(s) = g_n^*\left(\frac{s}{EY_n}\right).$$

For $n = 2, 3, 4, 5$ the requirements justifying (11) are satisfied, thus

$$\tilde{g}_n(y) = EY_n \sum_{k=1}^{n} \frac{p_n(s_k)}{q_n'(s_k)} \exp(s_k EY_n y). \tag{12}$$

From (7)-(10) we have after simple calculations

$$\tilde{g}_2(y) \approx 0{,}75\, e^{-0{,}8786y} + 0{,}75\, e^{-5{,}1213y}, \tag{13}$$

$$\tilde{g}_3(y) \approx 0{,}6667\, e^{-0{,}8266y} + e^{-7y} + 0{,}6667\, e^{-13{,}1734y}, \tag{14}$$

$$\tilde{g}_4(y) \approx 0{,}6410\, e^{-0{,}8071y} + 1{,}2340\, e^{-10{,}1460y} + 1{,}2340\, e^{-19{,}8540y} + 0{,}6410\, e^{-29{,}1929y}, \tag{15}$$

$$\tilde{g}_5(y) \approx 0{,}6428\, e^{-0{,}8054y} + 1{,}5210\, e^{-15{,}4601y} + 1{,}8720\, e^{-31y} + 1{,}5210\, e^{-46{,}5400y} + 0{,}6428\, e^{-61{,}1946y}. \tag{16}$$

The graphs of these densities are given in Fig. 1; those of the cumulative distribution functions in Fig. 2. For the convenience of confrontation, the figures enclose also appropriate graphs of the exponential distribution. It is shown here once more that the limit distribution is rather distant from the exact distributions. Table 2 gives the numerical values of the exact cumulative distribution functions and densities of breakdown time in systems consisting of 2, 3, 4, and 5 facilities.

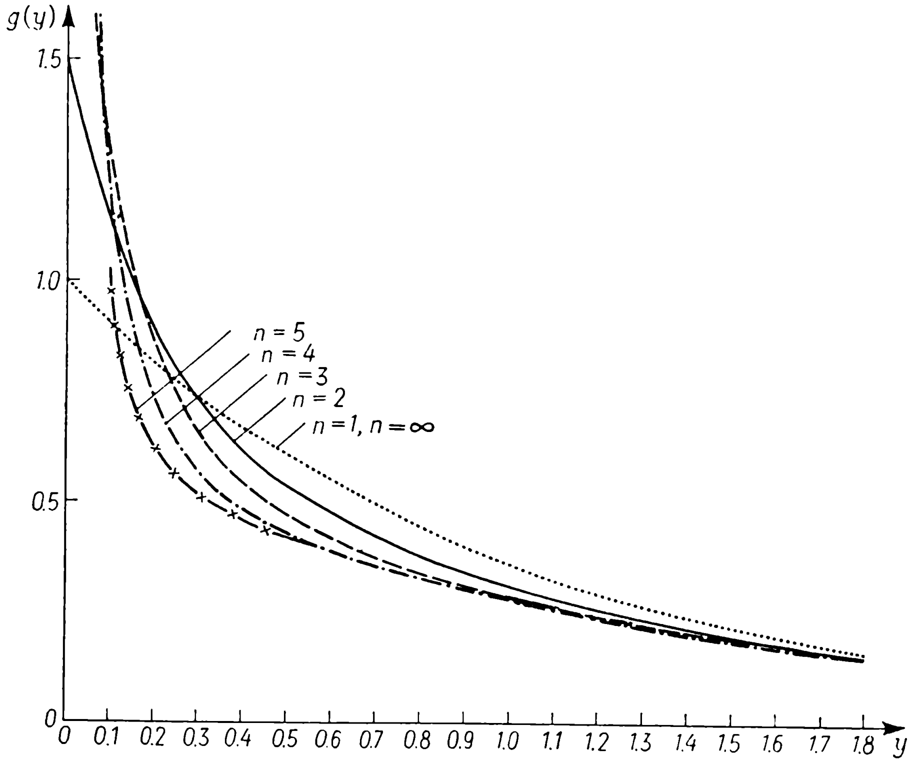

Fig. 1. Density functions of the normalized breakdown time of systems consisting of n facilities

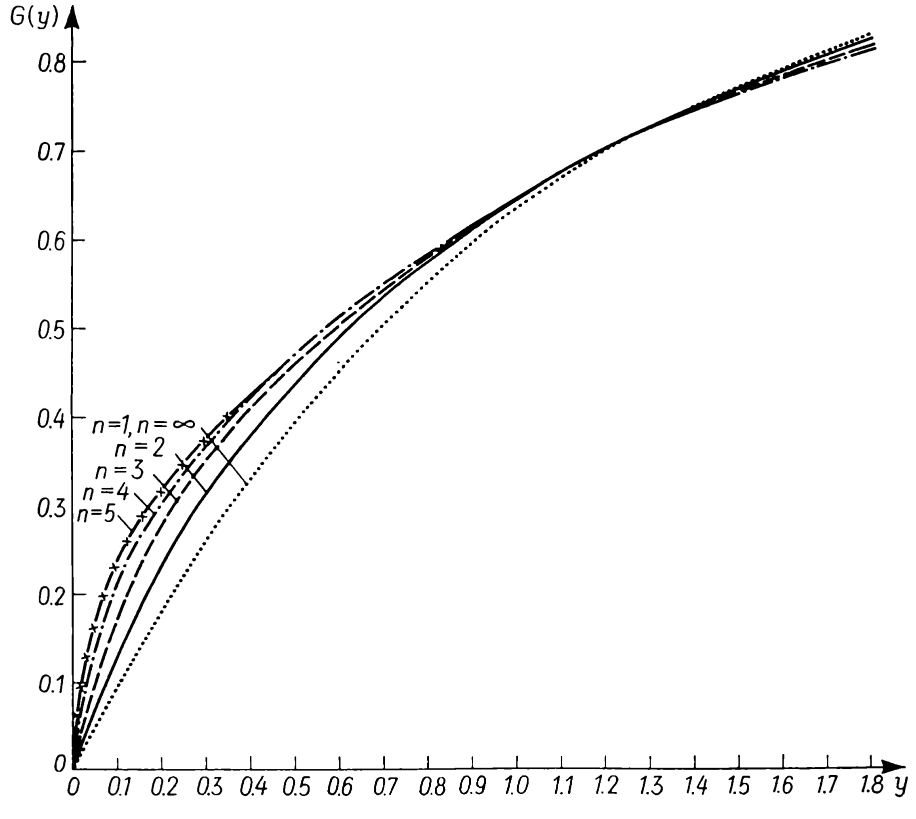

Fig. 2. Cumulative distribution functions of the normalized breakdown time of systems consisting of n facilities

y	Densities				Cumulative distribution functions			
	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
6.0	0.0038	0.0047	0.0051	0.0051	0.9956	0.9943	0.9937	0.9936
7.0	0016	0020	0023	0023	9982	9975	9972	9972
8.0	0007	0009	0010	0010	9992	9989	9988	9987
9.0	0003	0004	0004	0005	9997	9995	9994	9994
10.0	0001	0002	0002	0002	9999	9998	9998	9997

2. The investigation of system breakdown processes is a typical problem to solve with Monte Carlo methods. We have generated system breakdown processes of systems consisting of identical facilities with independent breakdown processes. The pseudo-random numbers r_n have been obtained using a process described in [1] with the first number r_0 from the interval $[0,2^{38})$ given in binary presentation as follows $r_0 = 001\ 0000\ 1011\ 1011\ 1011\ 1111\ 1010\ 0100\ 1101\ 111$. The next random number r_{n+1} is obtained as the middle 38 bits of the square of the preceding number r_n. Exponentially distributed (with parameter λ) random numbers z_n have been calculated from $z_n = -\lambda \log \frac{r_n}{2^{38}}$.

The breakdown process of n facilities, i.e. the system breakdown process, has been generated in the following way. We have assumed that at the initial moment all facilities were in working order and that their working times (till the first breakdown) were realizations of a random variable having an exponential distribution with parameter λ. The processes have then been observed in the moments of state changes; in every such moment we have registered for each facility its state and the time left till the next state change in that facility. If in the given moment for any facility the breakdown time was finished, the working time had been generated, if the working time had finished a breakdown time had been generated as a random number from the exponential distribution with parameter μ. Parallelly to generating the processes for all facilities, we have calculated the working and breakdown times of the whole system. The system's working time is counted from the moment of all facilities being in working order to the moment of the breakdown of any facility. It is easy to prove that in models of such a type the system's working time is exponentially distributed with parameter $n\lambda$. The experimental distributions obtained by simulation were statistically consistent with that theorem. The system's breakdown time is counted from the moment of the beginning of any breakdown to the moment when all facilities are again in working order. This time is nearly always a sum of time intervals between moments of change of

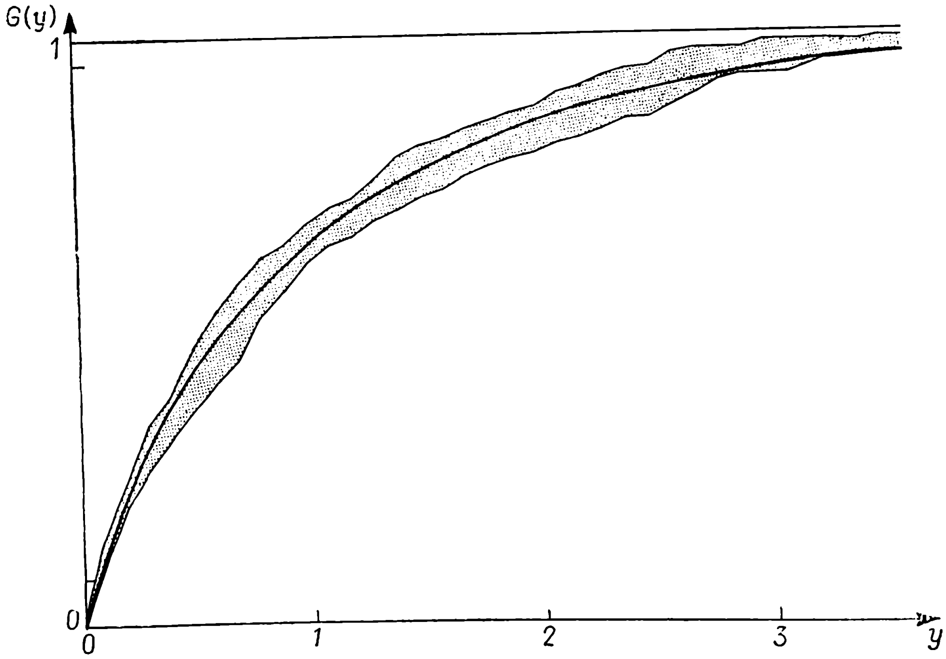

Fig. 3. Cumulative distribution function of the breakdown time of a system consisting of 2 facilities with exponential breakdown times

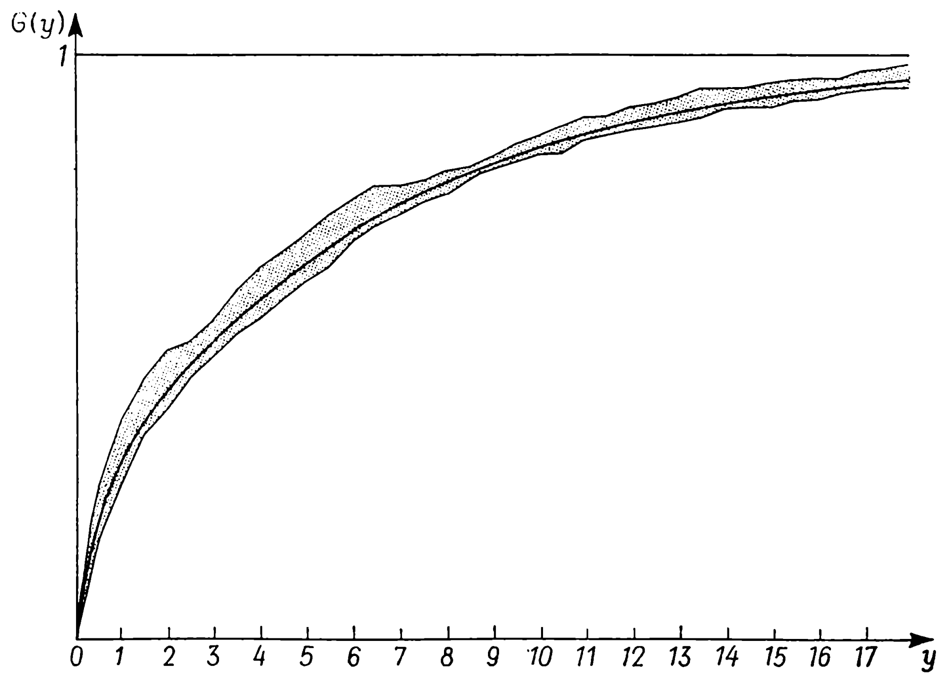

Fig. 4. Cumulative distribution function of the breakdown time of a system consisting of 4 facilities with exponential breakdown times

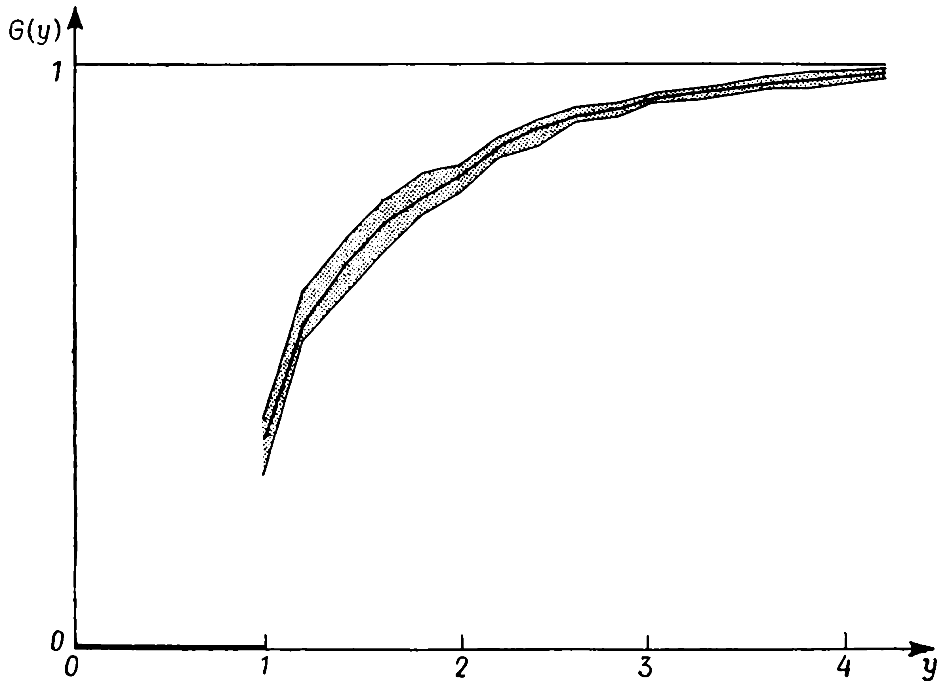

Fig. 5. Cumulative distribution function of the breakdown time of a system consisting of 2 facilities with constant breakdown times

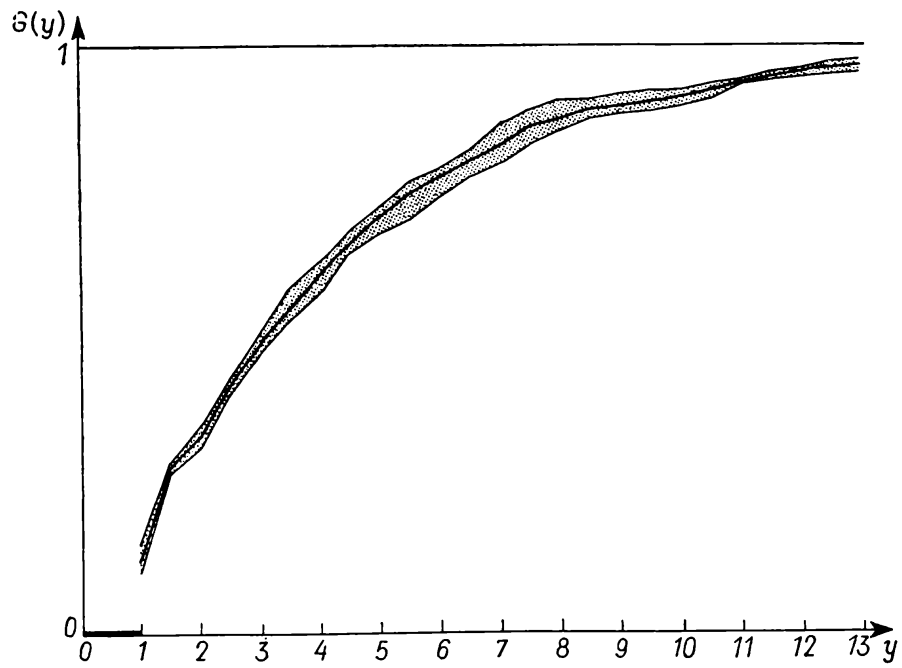

Fig. 6. Cumulative distribution function of the breakdown time of a system consisting of 4 facilities with constant breakdown times

state in one of the facilities which did not result in a state of all the facilities being in working order.

In the simulation we have dealt, as in section 1, with facilities having exponentially distributed (with parameter 1) working and breakdown times. Two systems have been investigated, the first one consisting of two, the second one of four facilities connected in series. Each system has been simulated 4 times, each time to the moment of completion of the 250th system breakdown.

The simulated breakdown times served as basis for constructing four empirical cumulative distribution functions for each facility. Then two functions, the minimum and the maximum ones, have been found; they are presented as boundaries of the marked region in Figs. 3 and 4. The appropriate theoretical cumulative distribution functions taken from Fig. 2 are also presented in, after a suitable change of units, Figs. 3-4.

A similar simulation has been performed for systems consisting of facilities having exponential working time distributions with parameter 1 and deterministic breakdown time (of unit length) distributions. Figs. 5 and 6 give a similar information as Figs. 3 and 4, the only difference being that instead of the theoretical distribution there is a mean empirical distribution taken from the four simulation runs.

Statistically, the theoretical and empirical graphs are the same. This allows us to claim that the Monte Carlo method is suitable for solving our problem for different distribution parameters too.

References

[1] E. D. Cashwell, C. J. Everett, *A practical manual on the Monte Carlo method for random walk problems*, Pergamon, New York 1959.

[2] G. Doetsch, *Anleitung zum praktischen Gebrauch der Laplace-Transformation*, Oldenbourg, München 1961.

[3] B. Kopociński, *Breakdown process of systems in series*, Zastosow. Matem. 9 (1967), pp. 149-160.

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ROZKŁADY CZASU AWARII W SYSTEMACH URZĄDZEŃ O UKŁADZIE SZEREGOWYM

STRESZCZENIE

W pierwszej części pracy znaleziono rozkłady czasu awarii w systemach złożonych z 2, 3, 4 i 5 urządzeń o niezależnych procesach awarii z wykładniczym rozkładem czasu pracy i czasu awarii. Uzupełniono tutaj wyniki z pracy [3], gdzie udowodniono, że rozkłady te po odpowiednim unormowaniu przy liczbie urządzeń rosnącej nieograniczenie dążą do rozkładu wykładniczego. W tej pracy pokazano, że w systemach złożonych z małej liczby urządzeń odchylenia dokładnych rozkładów od granicznego są stosunkowo duże.

W drugiej części pracy zastosowano metodę Monte Carlo do znalezienia rozkładów czasu awarii w systemach 2 i 4 urządzeń z wykładniczym rozkładem czasu pracy i z wykładniczym albo deterministycznym rozkładem czasu awarii. Porównanie wyników uzyskanych tą metodą i rozkładów teoretycznych wskazuje na przydatność metody Monte Carlo w omawianym zagadnieniu.

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РАСПРЕДЕЛЕНИЯ ВРЕМЕНИ НЕИСПРАВНОСТИ В СИСТЕМАХ УСТРОЙСТВ СОЕДИНЕННЫХ ПОСЛЕДОВАТЕЛЬНО

РЕЗЮМЕ

В первой части работы найдены распределения времени неисправности систем которые состоят из 2, 3, 4 и 5 устройств с независимыми процессами аварии, имеющих показательные распределения времен работы и аварии. Здесь продолжены исследования из работы [3], в которой было доказано, что эти распределения после соответствующей нормализации стремятся к показательному закону распределения, когда число устройств бесконечно возрастает. В настоящей работе доказано, что в системах состоящих из небольшого числа устройств отклонения точных распределений от предельного относительно велики.

Во второй части работы применен метод Монте Карло к нахождению распределений времен аварии в системах состоящих из 2 и 4 устройств с показательным распределением времени работы и с показательным или детерминистическим распределением времени аварии. Сравнение результатов полученных этим методом с теоретическими указывает на применимость метода Монте Карло в обсужденной проблеме.
