ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XVI, 3 (1979)

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FACTORIZATION OF THE DISTRIBUTION FUNCTION OF WAITING TIME

1. Introduction. The main result of this paper is to show that in a sufficiently wide class of queueing systems the limiting distribution of the waiting time is the same as the distribution of the supremum of the process

(1) $$\xi(t) = \sum_{j=1}^{[t]} \xi_j, \quad t \geqslant 0,$$

where $\{\xi_k, k \geqslant 1\}$ is a stationary sequence of random variables (r. v.'s), and $[\cdot]$ denotes the entier function. For convenience we put zero for the sums having an empty set of indices. To simplify our notation, we denote by $\{a_k\}$ any sequence for which $k \geqslant 1$. If we need other sequences, then their set of indices is indicated within the sequence brackets, e.g. $\{a_k, -\infty < k < \infty\}$. By our assumptions we show that the process ξ can be any limit of a subsequence of the sequence $\{S_k\}$ in the sense of convergence in distribution in (D, d) (see [3]), where a random element S_k of (D, d) is defined by

$$S_k(t) = \sum_{j=[k-t+]+1}^{k} X_j, \quad t \geqslant 0,$$

for some sequence $\{X_k\}$ of r. v.'s. Here the symbol $[k-t_0+]$ denotes the right-hand limit of $[k-t]$ at the point $t_0 < k$ and zero for $t_0 \geqslant k$. In the paper we apply this result to the theory of queueing systems in heavy traffic.

Let $\{X_k\}$ be any sequence of r. v.'s and let

(2) $$w_k = S_k - \inf_{0 \leqslant j \leqslant k} S_j, \quad k \geqslant 0,$$

where $S_0 = 0$, $S_k = S_{k-1} + X_k$, $k \geqslant 1$, and w_k is interpreted as the waiting time of the $(k+1)$-st customer.

It has been shown in [4] that if $\{X_k, -\infty < k < \infty\}$ is a stationary and ergodic sequence of r. v.'s and $EX_1 < 0$, then

$$w_k \xrightarrow{D} w, \quad P\{w < \infty\} = 1,$$

and

$$w \overset{D}{=} \sup_{0 \leqslant j < \infty} S_j, \tag{3}$$

where $S_0 = 0$, $S_k = S_{k-1} + X_{-k}$, $k \geqslant 1$. Let us note that (3) can be written as

$$w \overset{D}{=} \sup_{0 \leqslant t < \infty} S^-(t),$$

where S^- is a random element of (D, d) defined by

$$S^-(t) = \sum_{j=1}^{[t]} X_{-j}, \quad t \geqslant 0.$$

In [2] and [5] one considers the class of queueing systems described by sequences $\{X_k\}$ defined in the following way:

Let $\zeta_i = \{X_{i,k}, k \geqslant 1\}$, $i \geqslant 1$, where $X_{i,k}$ are r. v.'s. Let the processes $\zeta_1, \zeta_2, \ldots$ be independent and identically distributed. Write the sequence of r. v.'s

$$r_i = \min\{j\colon X_{i,1} + X_{i,2} + \ldots + X_{i,j} \leqslant 0\}, \quad i \geqslant 1,$$

$$R_0 = 0, \quad R_k = R_{k-1} + r_k, \quad k \geqslant 1,$$

$$\eta_k = \max\{j\colon R_j < k\}, \quad \gamma_k = k - R_{\eta_k}, \quad k \geqslant 1,$$

$$X_k = X_{\eta_k + 1, \gamma_k}, \quad k \geqslant 1. \tag{4}$$

It is proved in [3] that if $Er_1 < \infty$ and the distribution of r_1 is non-periodic, then

$$\lim_{k\to\infty} P\{w_k \leqslant x\} = \frac{1}{Er_1}\Big(1 + \sum_{k=1}^{\infty} P\{\min_{1 \leqslant j \leqslant k} S_{1,j} > 0,\ S_{1,k} \leqslant x\}\Big),$$

where $S_{1,k} = S_{1,k-1} + X_{1,k}$, $k \geqslant 1$.

We show in the next section that in some class of queueing systems, including those described by (4), the limiting distribution of waiting time is the same as the distribution of the supremum of a process ξ described above.

2. Factorization. Let h be the mapping of $D = D[0, \infty)$ in R defined by

$$h(x) = \sup_{0 \leqslant t < \infty} x(t), \quad x \in D.$$

LEMMA 1. *If* $x_n \in D$, $n \geqslant 1$, $x_n(0) = x(0) = 0$, *and*

$$\lim_{t\to\infty} x(t) = -\infty, \quad x_n \xrightarrow{d} x \in D,$$

then $h(x_n) \to h(x)$.

Proof. Since

$$x_n \xrightarrow{d} x,$$

there exists a sequence $\{\lambda_n\}$, $\lambda_n \in \Lambda$ (see [3]), such that, for any $a > 0$,

$$\sup_{0\leqslant t\leqslant a} \left|x_n(\lambda_n(t)) - x(t)\right| \to 0 \quad \text{as } n \to \infty$$

and

$$\sup_{0\leqslant t<\infty} |\lambda_n(t) - t| \to 0 \quad \text{as } n \to \infty.$$

Since

$$\lim_{t\to\infty} x(t) = -\infty,$$

there exists a positive number b_1 such that $x(t) < 0$ for $t > b_1$. Hence, and from the fact that $x_n \xrightarrow{d} x$, it follows that there exists a positive number b such that $x_n(t) < 0$ and $x(t) < 0$ for $t > b$.

Let us note that

$$\begin{aligned}|h(x_n) - h(x)| &= |h(x_n \circ \lambda_n) - h(x)| \\ &\leqslant \sup_{0\leqslant t<\infty} \left|x_n(\lambda_n(t)) - x(t)\right| = \left|\sup_{0\leqslant t\leqslant b} \left(x_n(\lambda_n(t)) - x(t)\right)\right| \\ &\leqslant \sup_{0\leqslant t\leqslant b} \left|x_n(\lambda_n(t)) - x(t)\right| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

COROLLARY 1. *The mapping* h *is continuous on the set*

$$A = \{x \in D\colon x(0) = 0, \lim_{t\to\infty} x(t) = -\infty\}.$$

LEMMA 2. *Let*

$$\sum_{j=1}^{n} X_j \to -\infty \text{ a.e.}$$

If $w_k \xrightarrow{D} w$ *and if there exist a subsequence* $\{k_n, n \geqslant 1\}$ *and a process* ξ *such that*

$$S_{k_n} \xrightarrow{D} \xi \quad \text{in } (D, d),$$

then $w \overset{D}{=} h(\xi)$.

Proof. Let P_k be the distribution of S_k (P_k is the probability measure on $(D, \mathscr{D})$, where $\mathscr{D}$ is the σ-field of Borel sets in (D, d)). Since $P_{k_n} \Rightarrow P$

(P is the distribution of ξ on $(D, \mathscr{D})$) and

$$\sum_{j=1}^{n} X_j \to -\infty \text{ a.e.},$$

we have $P\{A\} = 1$, where A is as in Corollary 1. The equality $w_k = h(S_k)$, $k \geqslant 1$ (see [5]), and the convergence $w_k \xrightarrow{D} w$ imply that $P_k h^{-1} \Rightarrow \mu$ (μ is the distribution of w). From the continuity of the mapping h on the set A and from the fact $P\{A\} = 1$ we obtain the convergence $P_{k_n} h^{-1} \Rightarrow Ph^{-1}$ (see Theorem 5.1 in [1]). Ph^{-1} is the probability measure on $(R, \mathscr{R})$. Consequently, we obtain $P_k h^{-1} \Rightarrow Ph^{-1}$ and $Ph^{-1} = \mu$.

THEOREM 1. *Let*

$$\sum_{j=1}^{n} X_j \to -\infty \text{ a.e.}$$

If $\{S_k\}$ *is tight in* (D, d) *and* $w_k \xrightarrow{D} w$, *then* $w \overset{D}{=} h(\xi)$ *and the process* $\xi(t)$ *is given by* (1). *The process* ξ *is any limit of a subsequence of the sequence* $\{S_k\}$.

Proof. If the conditions of Theorem 1 are fulfilled, then so are those of Lemma 2. Hence $w \overset{D}{=} h(\xi)$. The trajectories of S_k are step functions and the set of points of discontinuity of S_k is equal to the set of all non-negative integer numbers which do not exceed k. Therefore, ξ is discontinuous on the set of all non-negative integer numbers. Hence the process $\xi(t)$ is given by (1). Now we show that $\{\xi_k\}$ is a stationary sequence of r. v.'s. Since the sequence $\{S_k\}$ is tight in the metric space (D, d), Prokhorov's theorem (see [1]) implies that the sequence of probability measures P_k, $k \geqslant 1$, which are the distributions of S_k, respectively, is relatively compact. Therefore, there exists a subsequence $\{k_l, l \geqslant 1\}$ such that

$$\begin{aligned}(5) \quad & \big(S_{k_l}(\bar{s}_1) - S_{k_l}(\bar{t}_1), S_{k_l}(\bar{s}_2) - S_{k_l}(\bar{t}_2), \ldots, S_{k_l}(\bar{s}_m) - S_{k_l}(\bar{t}_m)\big) \\ & \qquad \xrightarrow{D} \big(\xi(\bar{s}_1) - \xi(\bar{t}_1), \xi(\bar{s}_2) - \xi(\bar{t}_2), \ldots, \xi(\bar{s}_m) - \xi(\bar{t}_m)\big),\end{aligned}$$

where $\bar{t}_k = t_k - \varepsilon$, $\bar{s}_k = s_k + \varepsilon$, $0 < \varepsilon < 1$, and t_k, s_k for $1 \leqslant k \leqslant m$ are non-negative integers with

$$0 \leqslant t_1 < s_1 \leqslant t_2 < s_2 \leqslant \ldots \leqslant t_m < s_m.$$

Let us note that the random vector on the left-hand side of (5) has the same distribution function as the following random vector:

$$(6) \qquad \Big(\sum_{j=k_l-s_1}^{k_l-t_1} X_j, \sum_{j=k_l-s_2}^{k_l-t_2} X_j, \ldots, \sum_{j=k_l-s_m}^{k_l-t_m} X_j\Big).$$

Let $n_l = k_l - t_1$. Then the random vector in (6) has the same distribution function as

$$(7) \quad \big(S_{n_l}(\bar{s}_1 - t_1), S_{n_l}(\bar{s}_2 - t_1) - S_{n_l}(t_2 - \bar{t}_1), \ldots, S_{n_l}(\bar{s}_m - t_1) - S_{n_l}(t_m - \bar{t}_1)\big).$$

Since $\{P_k\}$ is relatively compact, there exists a subsequence $\{n_{l_r}, r \geqslant 1\}$ of $\{n_l, l \geqslant 1\}$ such that

$$S_{n_{l_r}} \xrightarrow{D} \xi \text{ in } (D, d) \quad \text{as } r \to \infty.$$

Therefore, (7) converges in distribution to

$$\big(\xi(\bar{s}_1 - t_1),\ \xi(\bar{s}_2 - t_1) - \xi(t_2 - \bar{t}_1),\ \ldots,\ \xi(\bar{s}_m - t_1) - \xi(t_m - \bar{t}_1)\big)$$

which has the same distribution function as

$$\big(\xi(s_1 - t_1),\ \xi(s_2 - t_1) - \xi(t_2 - t_1),\ \ldots,\ \xi(s_m - t_1) - \xi(t_m - t_1)\big).$$

Hence we obtain the assertion of the theorem.

COROLLARY 2. *If the conditions of Theorem 1 are fulfilled and if X_k, $k \geqslant 1$, are independent, then ξ_k, $k \geqslant 1$, are independent and identically distributed.*

3. Tightness criterion of $\{S_k\}$.

THEOREM 2 (see [1] and [3]). *The sequence $\{P_k\}$ of probability measures is tight in $(D, \mathscr{D})$ if and only if*

(i) *for any $c, \eta > 0$ there exists $a > 0$ such that, for every $n \geqslant 1$,*

$$P_n\{x\colon \sup_{0 \leqslant t \leqslant c} |x(t)| > a\} \leqslant \eta;$$

(ii) *for any positive real numbers ε and η there exists a real number δ such that, for every $n \geqslant 1$,*

$$P_n\{x\colon w_x''(\delta) \geqslant \varepsilon\} \leqslant \eta \quad \text{and} \quad P_n\{x\colon w_x[0, \delta] \geqslant \varepsilon\} \leqslant \eta;$$

(iii) *for any positive real numbers b, ε, and η there exists a real number δ such that, for every $n \geqslant 1$,*

$$P_n\{x\colon w_x[b - \delta, b] \geqslant \varepsilon\} \leqslant \eta,$$

where

$$w_x''(\delta) = \sup_{t_1 \leqslant t \leqslant t_2,\ t_2 - t_1 \leqslant \delta} \min\{|x(t) - x(t_1)|, |x(t_2) - x(t)|\},$$

$$w_x[u, v] = \sup_{u \leqslant s \leqslant t \leqslant v} |x(t) - x(s)|.$$

THEOREM 3. *$\{S_k\}$ is tight in (D, d) if and only if for any positive number η and integer m there exists a positive number a such that, for every $k \geqslant 1$,*

$$\mathrm{P}\Big\{\sup_{k-m \leqslant j \leqslant k} \Big|\sum_{l=j}^{k} X_l\Big| \geqslant a\Big\} \leqslant \eta.$$

Proof. It is obvious that condition (i) is fulfilled if the conditions of Theorem 3 are satisfied. Therefore, we show only that (ii) and (iii) are fulfilled. But this follows from the definitions of $w_x''(\delta)$, $w_x[u, v]$ and S_k. Indeed, let us note that for $\delta < 1/2$ we have

$$\sup_{t_1 \leqslant t \leqslant t_2,\ t_2 - t_1 \leqslant 1/2} \min\{|S_k(t) - S_k(t_1)|,\ |S_k(t_2) - S_k(t)|\}$$

$$= \sup_{t_1 \leqslant t \leqslant t_2,\ t_2 - t_1 \leqslant 1/2} \min\Big\{\Big|\sum_{j=[k-t+]+1}^{[k-t_1+]} X_j\Big|,\ \Big|\sum_{j=[k-t_2+]+1}^{[k-t+]} X_j\Big|\Big\} = 0.$$

In a similar way one can show that

$$\sup_{0 \leqslant s \leqslant t \leqslant 1/2} |S_k(t) - S_k(s)| = 0 \quad \text{and} \quad \sup_{b-1/2 \leqslant s < t < b} |S_k(t) - S_k(s)| = 0.$$

COROLLARY 3. *If $\{X_k\}$ is such that for every $m \geqslant 1$ the sequence $\{|X_{k+1}| + |X_{k+2}| + \ldots + |X_{k+m}|,\ k \geqslant 1\}$ of r. v.'s is tight, then $\{S_k\}$ is tight in (D, d).*

Proof. This fact follows from

$$\sup_{k-m \leqslant j \leqslant k} |X_k + X_{k-1} + \ldots + X_j| \leqslant \sum_{j=k-m}^{k} |X_j|.$$

Let us write

$$X(n) = \{X_{k+n-1},\ k \geqslant 1\}, \qquad n \geqslant 1,$$
$$\vartheta_n = \max\{k < n\colon w_k = 0\}, \qquad n \geqslant 1,$$

where w_k is defined in (2) by $\{X_k\}$, and let

$$\gamma_n = n - \vartheta_n, \qquad n \geqslant 1.$$

THEOREM 4. *Let $\{X_k\}$ be such that for $k \leqslant n$*

$$\mathrm{P}\{X(n) \in B \mid \gamma_n = k\} = \mathrm{P}\{X(k) \in B\},$$

where B is a Borel set in R^∞. If $w_k \xrightarrow{D} w$ and $\mathrm{P}\{w < \infty = 1\}$, then $\{S_k\}$ is tight in (D, d).

Proof. It follows from Theorem 3 that it is enough to show the validity of (i). Suppose that (i) does not hold. Then there exist numbers m and η such that for every positive integer n there exists a positive integer k_n such that

$$\mathrm{P}\Big\{\sup_{k_n - m \leqslant j \leqslant k_n} \Big|\sum_{l=j}^{k_n} X_l\Big| > n\Big\} \geqslant \eta. \tag{8}$$

Since $w_k \xrightarrow{D} w$ and $\mathrm{P}\{w < \infty\} = 1$, inequality (8) can be replaced by

$$\mathrm{P}\Big\{\inf_{k_n - m \leqslant j \leqslant k_n} (S_{k_n} - S_j) < -n\Big\} \geqslant \eta. \tag{9}$$

By the formula for the total probability and by the conditions of the theorem, the left-hand side of (9) can be written as

$$(10)\qquad \sum_{s=1}^{k_n-m} \mathrm{P}\{\inf_{k_n-m\leqslant j\leqslant k_n}(S_{k_n}-S_j) < -n \mid \gamma_{k_n-m}=s\}\mathrm{P}\{\gamma_{k_n-m}=s\}$$

$$= \sum_{s=1}^{k_n-m} \mathrm{P}\{\inf_{s\leqslant j\leqslant s+m}(S_{m+s}-S_j) < -n\}\mathrm{P}\{\gamma_{k_n-m}=s\}.$$

Let us note that for each $s \geqslant 1$

$$\mathrm{P}\{\inf_{s\leqslant j\leqslant m+s}(S_{m+s}-S_j) < -n\} \to 0 \quad \text{as } n\to\infty.$$

Hence (10) tends to zero as $n\to\infty$ which contradicts inequality (8). Thus (8) does not hold.

4. Heavy traffic. Let us consider the family of processes $\{w_k^a, k\geqslant 1\}$ with parameter $a<0$, such that $\{w_k^a, k\geqslant 1\}$ is defined for every a by some sequence $\{X_k\}$ in (2). If

$$w_k^a \xrightarrow{D} w^a, \quad \mathrm{P}\{w^a<\infty\}=1, \quad w^a \overset{D}{=} h(\xi^a),$$

$$\xi^a(t) = \sum_{j=1}^{[t]} \xi_j^a, \quad t\geqslant 0,$$

then we accept the interpretation $a = \mathrm{E}\,\xi_1^a$.

Let us define the sequence of processes $\bar{\xi}_n^a$ by

$$\bar{\xi}_n^a(t) = \frac{1}{\sqrt{n}}\sum_{j=1}^{[nt]}(\xi_j^a - a), \quad t\geqslant 0.$$

THEOREM 5. *Let the conditions of Theorem* 1 *be satisfied. If for every a there exists a σ_a such that $\sigma_a\to\sigma>0$ as $a\to 0$, and*

$$(11)\qquad \frac{1}{\sigma_a}\bar{\xi}_n^a \xrightarrow{D} \mathscr{W} \text{ in } (D, d) \quad \text{as } n\to\infty,$$

where $\mathscr{W}$ is a Wiener process in D, then the distribution of $(1/\sigma_a)|a|w^a$ converges to the negative exponential distribution.

Proof. By Theorem 1 we have

$$\xi^a(t) = \bar{\xi}^a(t) + a[t], \quad t\geqslant 0,$$

where

$$\bar{\xi}^a(t) = \sum_{j=1}^{[t]}(\xi_j^a - a), \quad t\geqslant 0.$$

Let us define the families of processes ξ_c^a and $\bar{\xi}_c^a$, $c > 0$, by

$$\xi_c^a(t) = \frac{1}{\sqrt{c}}\,\xi^a(tc), \qquad t \geqslant 0,$$

$$\bar{\xi}_c^a(t) = \frac{1}{\sqrt{c}}\,\bar{\xi}^a(tc), \qquad t \geqslant 0,$$

and the family of mappings $\varkappa_c\colon D \to D$ by

$$\varkappa_c(x)(t) = x\left(\frac{c}{[c]}\,t\right), \qquad t \geqslant 0,\ x \in D.$$

The mappings $\varkappa_c$ are $\mathscr{D}$-measurable and $\varkappa_{c_n}(x_n) \xrightarrow{d} x$ as $x_n \to x$, $x \in C[0, \infty)$ (x is continuous on $[0, \infty)$) and $c_n \to \infty$ (see [5]).

Let

$$E = \{x \in D\colon \varkappa_{c_n}(x_n) \not\to x \text{ for some } x_n \xrightarrow{d} x\}.$$

Note that

$$\varkappa_c(\bar{\xi}_{[c]}^a) = \bar{\xi}_c^a \sqrt{\frac{c}{[c]}}\,. \tag{12}$$

But $E \cap C[0, \infty) = \emptyset$ (see [5]), thus by (11), (12) and Theorem 5.5 in [1] we obtain

$$\frac{1}{\sigma_a}\,\xi_{1/a^2}^a \xrightarrow{D} \mathscr{W} - I\frac{1}{\sigma} \text{ in } (D, d) \qquad \text{as } a \to 0.$$

Hence, using Theorem 5.1 in [1] and Lemma 3.4 in [5], we have

$$\frac{1}{\sigma_a}\,|a|\,w^a \xrightarrow{D} \sup_{0 \leqslant t < \infty}\left(\mathscr{W}(t) - \frac{t}{\sigma}\right) \qquad \text{as } a \to 0.$$

But the r. v. $\sup\limits_{0 \leqslant t < \infty}\big(\mathscr{W}(t) - t/\sigma\big)$ has the negative exponential distribution.

COROLLARY 4. *Let for each parameter a the sequence $\{X_k^a, k \geqslant 1\}$ describe a sequence $\{w_k^a, k \geqslant 1\}$ such that the conditions of Theorem 1 are fulfilled. Furthermore, let for each a the r. v.'s X_k^a, $k \geqslant 1$, be independent and let*

$$\sigma_a^2 = \operatorname{Var}(\xi_1^a) \to \sigma > 0 \qquad \textit{as } a \to 0.$$

Then the limiting distribution of $(1/\sigma_a)\,|a|\,w^a$ as $a \to 0$ is a negative exponential distribution.

Similar results can be obtained for the virtual waiting time. This fact follows from the equality (see (3.1.5) in [5])

$$W(t) = h(\mathscr{L}_t), \qquad t \geqslant 0,$$

$$\mathscr{L}_t(s) = L(t) - L(t-s+), \qquad s \geqslant 0,\ L(t) = \sum_{j=1}^{N(t)} v_j - t,\ t \geqslant 0,$$

where the symbol $L(t-s_0+)$ denotes the right-hand limit of $L(t-s)$ at the point $s_0 < t$ for any fixed elementary random event and zero for $s_0 \geq t$.

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Received on 10. 1. 1977

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FAKTORYZACJA ROZKŁADU CZASU CZEKANIA

STRESZCZENIE

Głównym rezultatem pracy jest pokazanie, że w dostatecznie szerokiej klasie systemów kolejkowych graniczny rozkład czasu czekania jest taki sam jak rozkład supremum procesu

$$\xi(t) = \sum_{j=1}^{[t]} \xi_j, \quad t > 0,$$

gdzie $\{\xi_k, k \geq 1\}$ jest stacjonarnym ciągiem zmiennych losowych, a $[\cdot]$ oznacza funkcję entier. Przyjmujemy również, że suma po pustym zbiorze wskaźników jest równa zeru.

Przy naszych założeniach proces ξ jest granicą dowolnego podciągu ciągu $\{S_k, k \geq 1\}$ elementów losowych z (D, d) w sensie zbieżności według rozkładu w (D, d), gdzie

$$S_k(t) = \sum_{j=[k-t+]+1}^{k} X_j, \quad t > 0,$$

a $\{X_k, k \geq 1\}$ jest ciągiem zmiennych losowych, generującym system kolejkowy. Symbol $[k-t_0+]$ oznacza prawostronną granicę $[k-t]$ w punkcie $t_0 < k$ i zero dla $t_0 > k$.